SINGULARIDADES Y ABSORCIÓN DE ENERGÍA EN MAGNETOHIDRODINÁMICA

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Resumen

La estabilidad y la absorción de energía por resonancia de una configuración de equilibrio magnetohidrodinámico vienen dadas por las ecuaciones linealizadas de la Magnetohidrodinámica (las cuales combinan las ecuaciones de Navier-Stokes, de Maxwell y las relaciones constitutivas de estado y de Ohm en la descripción del comportamiento de un fluido magnetizado). Este es un tema ampliamente tratado en la literatura (ver por ejemplo [1], [2]) en lo que concierne a la estabilidad del equilibrio, pero no así respecto de la resolución efectiva de dichas ecuaciones linealizadas.

El principal problema radica en que para determinados valores de la frecuencia temporal, que es un parámetro de la ecuación, ésta es singular y los métodos clásicos no se pueden adaptar a su estudio. Dichos valores de la frecuencia corresponden a los modos resonantes y la absorción de energía se realiza precisamente para estos valores. En principio podrían existir familias enteras de soluciones dadas por funciones arbitrarias en la región singular (ver [3]), pero sólo una de ellas es físicamente correcta.

En la comunicación se estudia el caso de una configuración cilíndrica de plasma, mostrando analítica y numéricamente el comportamiento de la verdadera solución y la absorción de energía correspondiente. Asimismo mostramos que la utilización sin precauciones de ciertos algoritmos numéricos puede dar lugar a conclusiones erróneas.

Bibliografía

[1] Freidberg, J.P.: Ideal Magnetohydrodynamic Theory of Magnetic Fusion Systems. Rev. of Mod. Phys. **54**, 801–902 (1982).

[2] Hameiri, E.: On the Essential Spectrum of Ideal Magnetohydrodynamics. Comm. in Pure and Appl. Math. **38**, 43–66 (1985).

[3] Núñez, M. – Tristán, L.A.: A Singular Transmission Problem: Alfvenic resonance in plasmas. (1995), Preprint.

SINGULARITIES AND ENERGY ABSORPTION IN MAGNETOHYDRODYNAMICS

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Abstract: The magnetohydrodynamics equations for the case of a cylinder linearized around an equilibrium state with vertical magnetic field reduce to a single singular second order partial differential equation where the time frequency enters as a parameter. Although there could exist whole families of solutions depending on arbitrary values at the singularity, only one of them is physically correct. We analyse theoretically and numerically the problem and the physical consequences of our results.

AMS (MOS) Subject Classification: 76W05, 35C10.

§1 Introduction.

Let us consider a stationary plasma confined in a region $U \times I$, where U is an open set of $\mathbb{R}^2$, I an interval, possessing a vertical magnetic field $\mathbf{B} = \big(0, 0, B(x, y)\big)$, density ϱ and pressure p, and satisfying the state equation of polytropic gases $p = A\,\varrho^{\gamma}$. Its behaviour is ruled by the magnetohydrodynamic (MHD) equations (see for ex. [1-3]) which in this simple configuration state that the *total pressure* $p + B^2/2$ is constant through the domain filled by the plasma. Small perturbations of these magnitudes satisfy the linearized magnetohydrodynamic equations, which after a Fourier transform $z \mapsto k$ in the vertical variable and Laplace transform $t \mapsto \omega$ in time, become

$$\operatorname{div}(f \nabla p_*) + g\, p_* = \operatorname{div}(f\,\mathbf{V}) + h \tag{1}$$

where p_* is the (transformed) perturbed total pressure,

$$f = \frac{1}{\omega^2 \varrho - k^2 B^2}, \qquad g = \frac{1}{\gamma p} \frac{\omega^2 \varrho - k^2 \gamma p}{\omega^2 \varrho\Big(1 + \dfrac{B^2}{\gamma p}\Big) - k^2 B^2},$$

$$\mathbf{V} = \varrho \mathbf{u}^0 + \frac{kB}{\omega}\mathbf{b}^0, \qquad h = \frac{\dfrac{k}{\omega}\mathbf{b}^0 \cdot \nabla \mathbf{B} + \dfrac{ik\varrho}{\omega} u_3^0 + \dfrac{ik^2 B}{\omega} b_3^0 - \dfrac{i\omega\varrho}{\gamma p} p_*^0}{\omega^2 \varrho\Big(1 + \dfrac{B^2}{\gamma p}\Big) - k^2 B^2}.$$

$\mathbf{u}^0$, $\mathbf{b}^0$ and p_*^0 denote the perturbed velocity, magnetic field and total pressure at the instant $t = 0$. The function f becomes singular where the *local Alfvén frequency* $\pm\frac{kB}{\sqrt{\varrho}}$ matches the time frequency ω. If this happens, resonance occurs and the plasma absorbs energy at the singular set $\Gamma := \{f = \infty\}$ from any external electromagnetic wave of this frequency. This fact, as well as its applications to plasma heating (for magnetic fusion purposes) and magnetospheric oscillations, are well known for some time ago (see [4-9]). g and h may also become singular at the so-called *magnetosonic* frequencies; for independent reasons (such as the fact that the *beta* of the plasma p/B^2 is usually small) they are far away from the Alfvén frequencies, and less relevant in many plasmas, such as the incompressible ones. We will exclude them from our consideration, and assume that g and h are regular throughout the (bounded) domain U under consideration. Also we exclude stationary perturbations, so $\omega \neq 0$. Since we intend to study the behaviour induced by the singular term f, we will simplify the equation by taking $g = 0$, i.e. we consider the principal part of the differential operator.

Although for a fixed ω (1) looks a typical elliptic equation, f is singular and changes signs at both sides of the singular set Γ, so that standard methods cannot be applied. An obvious variation is to define a new space $H^1(f, U)$ as (roughly) the set of functions u such that

$$\int_U |u|^2 + \int_U |f|\,|\nabla u|^2 = \|u\|^2_{H^1(f,U)}$$

is finite. This space possesses good trace properties at the points of Γ, making possible to solve the homogeneous problem in each component V of $U \setminus \Gamma$ with arbitrary (necessarily constant) values in each component of $\partial V \cap \Gamma$ and Dirichlet conditions at $\partial V \cap \partial U$. As usual one considers the weak form of (1) and the Lax-Milgram theorem. Unfortunately, the term $\mathrm{div}(f\mathbf{V})$ is not in the dual space $H^{-1}(f, U)$ of $H^1_0(f, U)$ and further preparatory work is needed. Which one obtains is one regular solution of the Dirichlet problem coinciding in Γ with any previous given function (see [10] and the references included there).

To select among these solutions the correct one we must remember that we are dealing with the Fourier-Laplace transform

$$p_*(\omega, \mathbf{x}) = \int_0^\infty e^{i\omega t} q_*(t, \mathbf{x})\, dt$$

of the true perturbed pressure q_* as a function of time. Under very general conditions it must happen that for a real ω, $p_*(\omega, \mathbf{x})$ should be the limit of $p_*(\omega + i\varepsilon, \mathbf{x})$ when $\varepsilon \downarrow 0$: in the physicists' language, the oscillatory solution is limit of damped solutions. For $\varepsilon > 0$, f is regular throughout U and the problem is coercive; thus any Dirichlet problem has a unique solution for $\varepsilon > 0$ (see [10]).

In this paper we analyse theoretically and numerically this facts in a simple configuration.

§2 Theoretical Results.

We consider now that the plasma fills a hollow cylinder $r_1 < r < r_2$, and assume that its density ϱ and magnetic field $\mathbf{B}$ depend only on the radius r (so the same happens with p_*).

Let $F = \omega^2 \varrho - k^2 B^2$. After a Fourier transform $\theta \to n$ in the angle variable, the homogeneous problem associated to (1) becomes

$$u'' + \left(\frac{1}{r} - \frac{F'(r)}{F(r)} \right) u' - \frac{n^2}{r^2}\, u = 0\,. \tag{2}$$

At any simple zero r_0 of F, F'/F behaves as $1/(r - r_0)$. For further notational simplification, and without affecting the properties of the equation we will take $k^2 B^2 = 1$ and $\varrho(r) = r$; for $\omega^2 = 1$, then $\Gamma := \{r = 1\}$ and (2) becomes

$$u'' + \left(\frac{1}{r} - \frac{1}{r-1} \right) u' - \frac{n^2}{r^2}\, u = 0$$

which we recognise as an hypergeometric equation. Its classical theory (see [11-13]) allows us, after a cumbersome but essentially straightforward calculation, to write two independent solutions in $B(1,1)$ as

$$v_1(r) = r^n (1-r)^2 \sum_{k=0}^{\infty} \frac{(\alpha(n))_k (\beta(n))_k}{(3)_k} \, \frac{(1-r)^k}{k!}\,,$$

$$\begin{aligned} v_2(r) = &- \frac{2\, r^n}{(\alpha(n)-2)_2\, (\beta(n)-2)_2} + \frac{2\, r^n (r-1)}{(\alpha(n)-1)\,(\beta(n)-1)} \\ &+ r^n (1-r) \sum_{k=0}^{\infty} \frac{(\alpha(n))_k (\beta(n))_k}{(3)_k} \Big(\Psi(\alpha(n)+k) + \Psi(\beta(n)+k) \\ &\qquad - \Psi(3+k) - \Psi(1+k) \Big) \, \frac{(1-r)^{k+2}}{k!} + \log(1-r)\, v_1(r)\,, \end{aligned}$$

where $(x)_k = \Gamma(x+k)/\Gamma(x)$,

$$\alpha(n) = n + 3/2 + \sqrt{n^2 + 1/4},$$
$$\beta(n) = n + 3/2 - \sqrt{n^2 + 1/4},$$

and Ψ is the logarithmic derivative of the Γ function.

We see that

$$v_2(r) = w_2(r) + \log(1-r)v_1(r)\,,$$

where v_1 and w_2 are real valued functions. Also notice that the wronskian determinant $W(v_1, w_2) = w_2' v_1 - w_2 v_1'$ behaves as

$$W(v_1, w_2) = a(r-1) + O\,(r-1)^2$$

near 1, with $a = 4/(\alpha(n)-2)_2\,(\beta(n)-2)_2 > 0$. These function are $\mathcal{C}^1$ and, except for the term $\log(1-r)$, real valued in $[r_1, r_2]$. In the general case we would have $\log(r(\omega)-r)$, where $r(\omega)$ is the (complex) zero of F for ω near ± 1. Hence, for $\delta > 0$,

$$\log(1-(1+\delta)) = \log(1-(1-\delta)) + i\,\sigma\pi\,,$$

where $\sigma = \pm 1$, according to which branch of the logarithm is chosen. Let us study the correct determination of this imaginary part.

Lemma 1. *Let $\omega_0 = \pm 1$. Then σ is the sign of $\frac{\omega_0}{F'(1)}$ ($F'(1) = 1$ in our case).*

Proof: What we denote by u is in fact the Fourier-Laplace transform of the total pressure, hence the solution of (1) for real ω should be the limit of (damped) solutions for $\omega + i\,\varepsilon$, $\varepsilon \downarrow 0$. Let $\omega(r)$ be the (complex) frequency which makes $F(\omega(r), r) = 0$ near ($\omega_0 = \pm 1$, $r = 1$). Then

$$\frac{\partial F}{\partial \omega}\,\omega' + F' = 0\,,$$

that is, $2\omega(r)\varrho(r)\omega'(r) + F'(r) = 0$. This implies

$$\omega(r) - \omega(1) = -\,\frac{F'(\omega(1), 1)}{2\omega(1)\varrho(1)}\,(r-1) + O(r-1)^2\,;$$

thus, if $r(\omega)$ is the (complex) zero of $F(\omega, r(\omega))$ near ($\omega = \omega_0$, $r = 1$),

$$r(\omega) - r(\omega_0) = r(\omega) - 1 = -\,\frac{2\omega_0\,\varrho(1)}{F'(\omega_0, 1)}\,(\omega - \omega_0) + O(\omega - \omega_0)^2\,.$$

Thus the sign of the imaginary part of $r(\omega_0 + i\,\varepsilon)$ is the sign of $-\omega_0/F'(1)$. When positive, $r(\omega)$ lies in the lower half plane and when r goes from $1-\delta$ to $1+\delta$, $r(\omega) - r$ goes from $r(\omega) - 1 + \delta$ to $r(\omega) - 1 - \delta$ in the lower half plane, so that its argument changes from 0 to $-\pi$; otherwise, from 0 to π. Always $\sigma = \mathrm{sgn}(\omega_0/F'(1))$.

Even in the case $\varepsilon = 0$ it is possible to tackle the non-homogeneous problem by the same procedure as in the classical Sturm-Lioville theory, constructing the Green function, and one can easily checks that the solutions of this problem have the same behaviour.

How this fact relates to energy absorption? The increase of energy in X is given by

$$\frac{dE}{dt} = -\int_{\partial X} p_* \, \mathbf{v} \cdot \mathbf{n} \, d\sigma$$

(where $\mathbf{n}$ denotes the outer normal). It may be shown that the Fourier transforms P and V of p_* and $\mathbf{v} \cdot \mathbf{n}$ satisfy $V = (i\,\omega/F)\,P'$. Since we are dealing with a single resonant sheet, assume that p_* has only the Fourier modes $\pm k$ in z (so that $F = \omega^2 \varrho - k^2 B^2$ is fixed in the problem). We will study the integral above for a cylindrical surface $r = a$. As we know, under fairly general conditions, and with a somewhat relaxed notation,

$$\begin{aligned}
&\int_{\mathbf{T}\times\mathbb{R}} p_*(t,a,\theta,z)\,\mathbf{v}\cdot\mathbf{n}(t,a,\theta,z)\,d\theta\,dz = \\
&\qquad = \int_{\mathbb{Z}\times\mathbb{R}} \widehat{p}_*(t,a,n,k)\,(\mathbf{v}\cdot\mathbf{n})^{\vee}(t,a,n,k)\,dn\,dk \\
&\qquad = \sum_{n\in\mathbb{Z}} \big(\,(P(t,a,n,k)\,V(t,a,-n,-k) + P(t,a,n,-k)\,V(t,a,-n,k)\,\big)\,.
\end{aligned}$$

Hence the output of energy through $r = a$ is

$$-\sum_{n\in\mathbb{Z}} \big(\,(P(t,a,n,k)\,V(t,a,-n,-k) + P(t,a,n,-k)\,V(t,a,-n,k)\,\big)\,. \tag{3}$$

Assume also that the boundary conditions determine a one-dimensional space of solutions depending only on ω^2 (such as decay at infinite, $u(r_2 = \infty) = 0$, or a solid wall $u'(r_2) = 0$, etc.; most real problems are in this class). We may construct a generator as follows:

Let

$$\alpha = \lambda_1 v_1 + \lambda_2 v_2\,,$$

λ_1, λ_2 real;

$$\beta = \delta \pi \lambda_2 v_1\,,$$

with $\delta = 0$ for $r < 1$ or $\delta = 1$ for $r \geq 1$. Then

$$\begin{aligned}
P &= \lambda(1,n,l)\,(\alpha + i\,\beta)\,e^{i\,t} + \lambda(-1,n,l)\,(\alpha - i\,\beta)\,e^{-i\,t}\,, \\
V &= \frac{i}{F}\,\lambda(1,n,l)\,(\alpha' + i\,\beta')\,e^{i\,t} - \frac{i}{F}\,\lambda(-1,n,l)\,(\alpha' - i\,\beta')\,e^{-i\,t}\,.
\end{aligned}$$

Since p_* is real, $P(-\omega,-n,-k) = \overline{P(\omega,n,k)}$, so the same happens for the coefficient λ. Also dE/dt is real, so it is enough to take the real part of the terms in (3). The real part of the non-oscillatory terms is therefore

$$\frac{1}{F}\,(|\lambda(-1,n,l)|^2 + |\lambda(1,n,l)|^2)\,(\alpha'\beta - \alpha\beta')$$

for a fixed mode n (note that this corresponds to the sums in (3) not affected by e^{2it} or e^{-2it}; the time mean of these is zero and hence they are not relevant in the net energy increase). The wronskian determinant $\alpha'\beta - \alpha\beta'$ vanishes for $r < 1$, and satisfies $W' = (F'/F - 1/r)W$; hence $W = A\,F/r$ for $r \geq 1$, A a constant. One easily gets

$$A = \frac{r}{F}\,\lambda_2^2\pi(v_2'v_1 - v_2v_1') = \frac{r}{F}\,\lambda_2^2\pi\Big(w_2'v_1 - w_2v_1' + \frac{1}{r-1}\,v_1^2\Big)\,,$$

which as shown before, is positive at $r = 1$. (Recall that $F = r - 1$, $w_2'v_1 - w_2v_1' = a(r-1) + O(r-1)^2$, $a > 0$, $v_1^2 = O(r-1)^4$.)

Hence for a set X bounded for $r = a$ and $r = b$, dE/dt is a continuous function of (a, b) whenever both are less than or greater than 1; however, if $a < 1$, $b > 1$, dE/dt jumps by a fixed amount. We have proved

Theorem 2. *Energy absorption due to alfvenic resonance takes place at the singular set.*

Notice that the fact that $\beta \neq 0$ (that is, that an imaginary part occurs at the solution after the singularity $r = 1$) is essential for the existence of this phenomenon. Any solution not taking this term in consideration is physically erroneous. This would happen if one applies directly any of a number of numerical methods of integration which tolerate the singular point, but do not get the solution analytic in ω, as we will show next.

§3 Numerical Results.

The integration of the initial value problem in a given interval (in our case $r \in [0.5, 1.5]$) is delicate because of the singularity at $r = 1$. If however, one takes $r = x + i\varepsilon$, $x \in [0.5, 1.5]$, $\varepsilon > 0$, not only the singularity disappears, but what we obtain is the correct solution in the limit $\varepsilon = 0$. This has been done in equation (2) for initial values $u(0.5) = 1$,

$u'(0.5) = 1$, $n = 1, 2$, and $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$ (see figure 1).

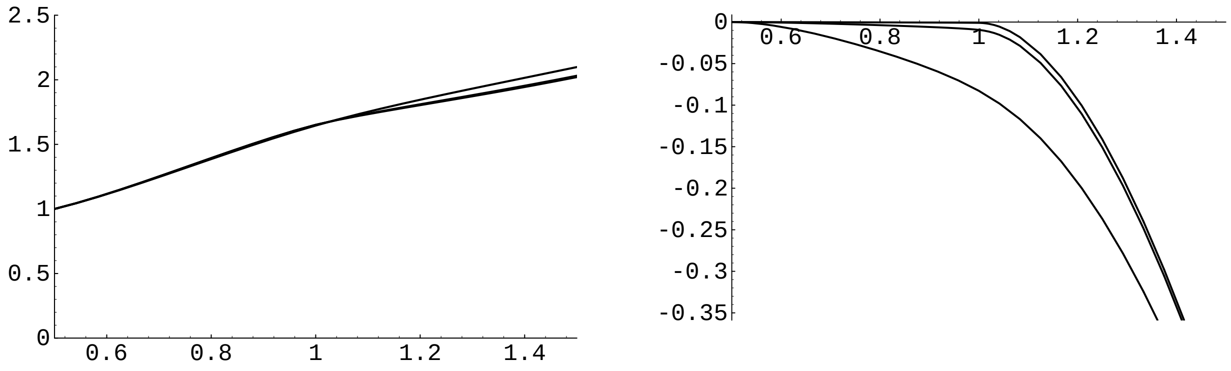

Fig. 1
Initial values problem for $n = 1$; real and imaginary part of the solutions.

Behaviour for $n = 2$ is similar, as shown in figure 2.

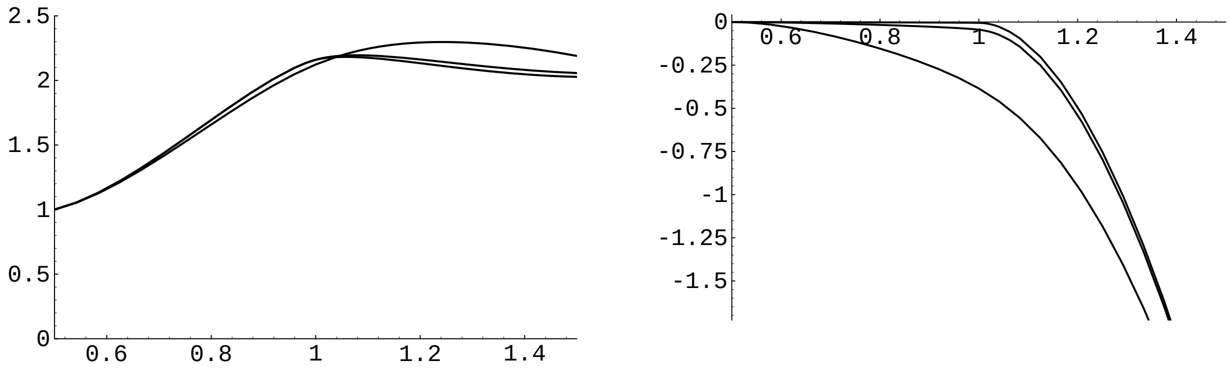

Fig. 2
Real and imaginary part of the solutions for $n = 2$.

Observe that the behaviour of the imaginary part is as predicted: zero up to the singularity $r = 1$, and then a nontrivial function. By contrast, direct integration of (1) with $\varepsilon = 0$, even if the algorithm somewhat surpasses the critical point at $r = 1$, will never generate an imaginary part for a real equation with real initial value. We have checked that for $\varepsilon = 0$; the real part is correctly obtained but the imaginary part of u is always zero, as one would expect.

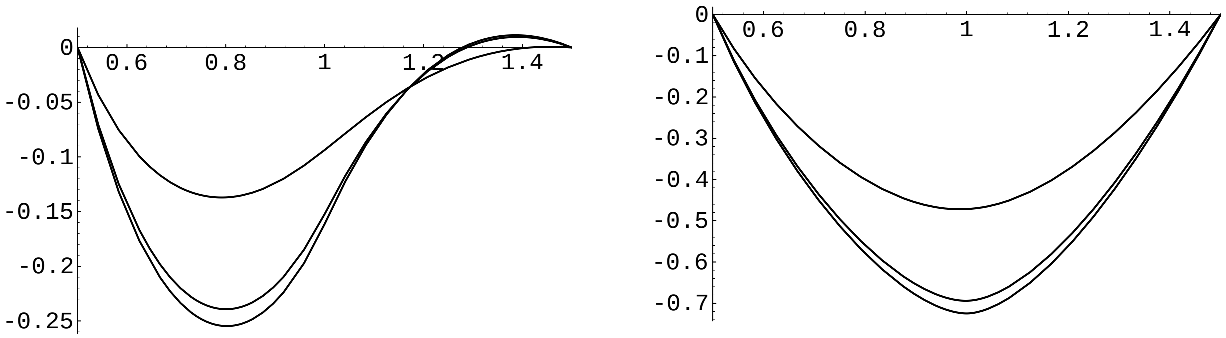

Fig. 3

Boundary value problems for $n = 2$; real and imaginary parts of the solution.

As for the boundary problem, the same method may be used. The regular boundary value problems obtained by considering $r = x + i\,\varepsilon$, $x \in [0.5, 1.5]$, with the conditions $u(0.5) = 0 = u(1.5)$ and $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$, show the convergence of the solutions to the correct physical solution. For $n = 2$, we have the results of the figure 3.

These graphics have been obtained by using the shooting method with Mathematica, which allows a high degree of interactivity and is specially good at dealing with singularities. If we execute this method setting directly $\varepsilon = 0$, we still get an answer, but a completely wrong one (fig. 4).

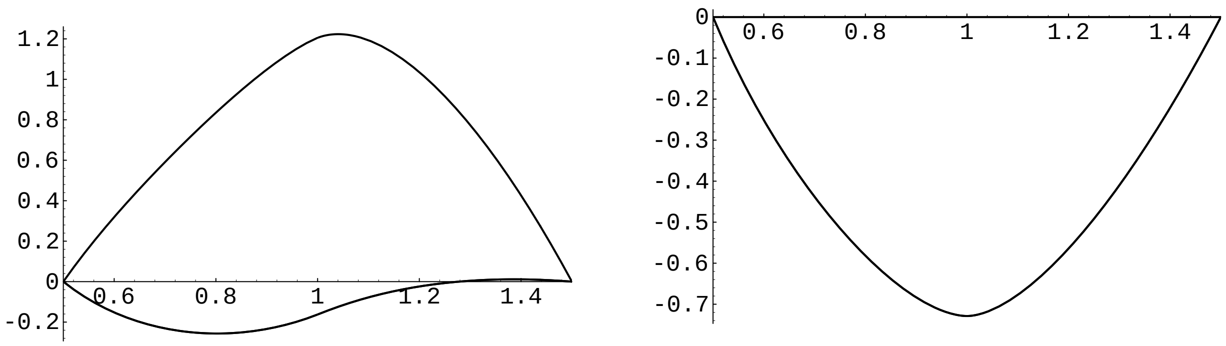

Fig. 4

The wrong answer for $\varepsilon = 0$.

In general, the behaviour of this integrator for $\varepsilon = 0$ is erratic. For $n = 1$, one gets the correct answer for some try values of $u'(0.5)$ and a wrong one for others. For $n = 2$ it is always false.

Finally, to emphasise the fact that one cannot choose the value of u at $r = 1$ arbitrarily and then solve the boundary problem at both intervals $[0.5, 1]$, $[1, 1.5]$, we have set $u(1) = 10$ and found the solutions for $n = 2$ for a slightly attenuated frequency $\omega = 1 + 0.1\,i$ (fig. 5). The solutions are obviously not differentiable. This shows that the problem should not be treated as an equation with a singular boundary; one cannot impose arbitrary Dirichlet conditions at $r = 1$. Those are determined by the original problem in a complex setting.

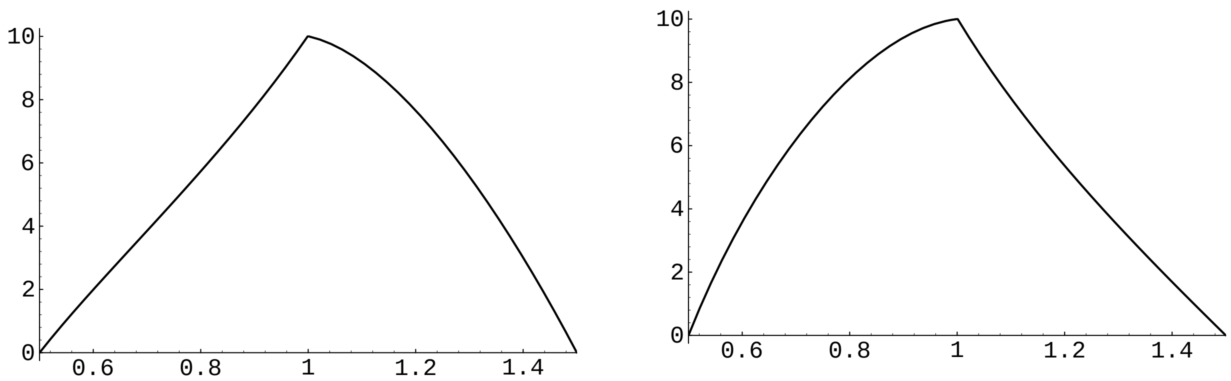

Fig. 5
The solutions for $u(1) = 10$; real and imaginary part.

Acknowledgements: This work has been partially supported by Junta de Castilla y León under project VA57/94.

References.

[1] **Bateman, G.,** MHD Instabilities, *M.I.T. Press*, Cambridge(MA) (1980).

[2] **Freidberg, J.P.,** Ideal Magnetohydrodynamic Theory of magnetic fusion systems, *Rev. of Mod. Phys.* **54**, 801–902 (1982).

[3] **Hameiri, E.,** On the Essential Spectrum of Ideal Magnetohydrodynamics, *Comm. in Pure and Appl. Math.* **38**, 43–66 (1985).

[4] **Chen, L. & Hasegawa, A.,** Theory of Long-Period Magnetic Pulsations 1: Steady State Excitation of Field Line Resonance, *J. of Geoph. Res.* **79**, 1024–1037 (1974).

[5] **Sedlacek, Z.,** Electrostatic Oscillations in Cold Inhomogeneous Plasma, *J. of Plasma Phys.* **5**, 239–263 (1971).

[6] **Tataronis, J.A.,** Energy Absorption in the Continuous Spectrum of Ideal Magnetohydrodynamics, *J. of Plasma Phys.* **13**, 87–105 (1975).

[7] **Chen, L. & Hasegawa, A.,** Plasma heating by spatial resonance of Alfvén wave. *Phys. of Fluids* **17** (1974), pp. 1399-1403.

[8] **Chen, L. & Hasegawa, A.** Steady state excitation of field line resonance. *J. of Geoph. Res.* **79** (1974), pp. 1024-1037.
[9] **Kivelson, M.G. & Southwood, D.J.,** Resonant ULF waves: a new interpretation. *Geoph. Res. Lett.* **12** (1985), pp. 49-52.
[10] **Núñez, M. & Tristán, L.A.,** A singular transmission problem: alfvenic resonance in Magnetohydrodynamics. (preprint, 1995).
[11] **Olver, F.W.J.,** Asymptotics and Special Functions, *Academic Press*, (1974).
[12] **Henrici, P.,** Applied and computational complex analysis. *John Wiley and Sons*, New York, 1976.
[13] **Whittaker, E.T. & Watson, G.N.,** A course of modern analysis. *Cambridge Univ. Press*, Cambridge, 1927.