A spatial decay estimate for a visco- elastodynamic problem

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Abstract : *In this communication we establish a spatial decay estimate for the dynamic problem of the visco- elasticity of Kelvin- Voigt type similar to other known estimates for the parabolic equations.*

1. INTRODUCTION

Although the study of spatial decay estimates of several types of partial differential equations and systems has been intensive in the last three decades, not many authors have focussed their attention to hyperbolic equations (see [1,2]). The works of Flavin and Knops [3] and Flavin, Knops and Payne [4,5] are pioneering contributions for these kind of equations. In [9] Iesan and Quintanilla extend some of the results of [4,5] to elastic materials with voids.

Many authors (see [1-4]) seem to accept that spatial decay estimates obtained for the elliptic and parabolic problems of elastic wave propagation will not have analoges in the hyperbolic problems. However, this is not the case if some sort of damping effects are taken into account as shown in the recent papers [10-12]. Also, in [6], Quintanilla found spatial decay estimates for the linear hyperbolic heat equation.

In the present paper we derive the the same conclusion for visco- elastic materials of Kelvin- Voigt type. We deal with a "quasi- hyperbolic" equation of the third order, and consider a finite visco- elastic cylinder with constant cross section. Our analysis can be extended to semi- infinite cylinders with non constant cross section, but we omit these generalizations because they lead to irrelevant cumbersome calculations which affect the clarity of the exposition.

The method we use is inspired on [6] and it is based on the study of a first order differential inequality for a cross- sectional integral involving the gradient of the displacement field and its time derivative.

For related information on visco- elastic materials of Kelvin- Voigt type and, in particular, on time exponential decay of the solutions we refer the reader to [7].

In Section 2, we state the notations and basic definitions and prove one a preliminary lemma. It will be used in the Section 3, where the first spatial decay estimate is proved.

It is worth to say that a more deep study has been developed for the anti- plane shear deformations [15].

2. PRELIMINARIES

Throughout this paper a rectangular Cartesian coordinates system $0x_k (k = 1, 2, 3)$ is used. We shall employ the usual summation and differentiation conventions: Latin subscripts are understood to range over the integers (1,2,3), whereas Greek subscripts are confined to the range (2,3); summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate. We use a superposed dot to denote partial differentiation with respect to the time.

We consider a linear visco- elastic material of Kelvin- Voigt type. We assume that the body occupies a bounded cylindrical region $B = D \times [0, L]$, where D is a two- dimensional region such that its boundary ∂D allows the use of the Divergence Theorem. By $B(z, z')$ we denote the subset of points $(x_1, x_2, x_3) \in B$ such that $0 \leq z \leq x_1 \leq z' \leq L$.

The displacement vector field of a material point x will be denote by $u = (u_1, u_2, u_3)$.

We shall be concerned with the problem defined by the system of field equations described by the evolution equations

$$\rho \ddot{u}_i = \sigma_{ij,j}, \quad x \in B, t \geq 0, 2.1$$

the constitutive law for the visco- elasticity

$$\sigma_{ij} = c_{ijkl}(x) u_{k,l} + d_{ijkl}(x) \dot{u}_{k,l}, \quad x \in B, 2.2$$

the boundary conditions

$$u_i = 0, \quad on \partial D \times [0, L] \times [0, T),$$

$$u_i = 0, \quad on D(L) \times [0, T), 2.3$$

$$u_i = \tilde{u}_i(x_2, x_3, t), \quad on D(0) \times [0, T),$$

where $\tilde{u}_i$ is a prescribed function, and the initial conditions

$$u(x, 0) = u^0(x), \quad \dot{u}(x, 0) = v^0(x), \quad x \in B.2.4$$

It is known that for viscoelastic materials the following symmetry condition

$$c_{ijkl} = c_{klij} \quad x \in B, 2.5$$

holds [8]. We shall denote

$$d^1_{ijkl} = 12(d_{ijkl} + d_{klij}) \qquad and \qquad d^2_{ijkl} = 12(d_{ijkl} - d_{klij}).2.6$$

We assume that there exist two positive constant c_0, d_0 such that the inequalities

$$c_{ijkl} H_{kl} H_{ij} \geq c_0 H_{ij} H_{ij} \qquad and \qquad d_{ijkl} H_{kl} H_{ij} \geq d_0 H_{ij} H_{ij}, 2.7$$

are satisfied, for all tensor $H = H_{ij}$. Assumptions (2.7) are usual in the studies on viscoelastic materials (see [7,8]).

In case that the constitutives coefficients are bounded the assumption (2.7) and the Schwarz inequality imply the existence of three positive constants C_1 and D_1 and ω such that the inequalities

$$c_{i1kl} H_{kl} M_i \leq C_1 (H_{ij} H_{ij})^{12} (M_i M_i)^{12}, d_{i1kl} H_{kl} M_i \leq D_1 (H_{ij} H_{ij})^{12} (M_i M_i)^{12},$$

$$d^2_{ijkl} H_{kl} L_{ij} \leq \omega (c_{ijkl} H_{kl} H_{ij})^{1/2} (d^1_{ijkl} L_{kl} L_{ij})^{1/2}, 2.8$$

are satisfied, for all couple of tensors $H = H_{ij}$, $L = L_{ij}$ and vector $M = M_i$. We denote by η the first non- zero eigenvalue of the Laplacian operator in D with homogeneous Dirichlet boundary conditions. We know that the Poincare inequality

$$\eta^{-1} \int_{D(x_3)} \phi^2 da \leq \int_{D(x_3)} \phi_{,\alpha} \phi_{,\alpha} da, 2.9$$

is satisfied, for any integrable function ϕ vanishing on ∂D.

Now, we state a couple of equalities. Let u be a solution to the system (2.1), we have

$$(u_i \sigma_{ij})_{,j} = u_{i,j} \sigma_{ij} + u_i \sigma_{ij,j} = u_{i,j}[c_{ijkl} u_{k,l} + d_{ijkl} \dot{u}_{k,l}] + \rho u_i \ddot{u}_i$$

$$= c_{ijkl} u_{k,l} u_{i,j} + 12ddtd^1_{ijkl} u_{k,l} u_{i,j} + d^2_{ijkl} u_{i,j} \dot{u}_{k,l} + \rho[ddt(u_i \dot{u}_i) - \dot{u}_i \dot{u}_i].$$

After integration from 0 to t_0, we obtain

$$\int_0^{t_0} (u_i \sigma_{ij})_{,j} dt = \int_0^{t_0} [c_{ijkl} u_{k,l} u_{i,j} + d^2_{ijkl} u_{i,j} \dot{u}_{k,l} - \rho \dot{u}_i \dot{u}_i] dt$$

$$+\rho u_i \dot{u}_i + 12d_{ijkl} u_{k,l} u_{i,j} - \rho u^0_i v^0_i - 12d^1_{ijkl} u^0_{k,l} u^0_{i,j}.2.10$$

By a similar method we also obtain

$$(\dot{u}_i \sigma_{ij})_{,j} = \dot{u}_{i,j} \sigma_{ij} + \dot{u}_i \sigma_{ij,j}$$

$$= \dot{u}_{i,j}[c_{ijkl} u_{k,l} + d_{ijkl} \dot{u}_{k,l}] + \rho \dot{u}_i \ddot{u}_i$$

$$= 12ddt[c_{ijkl} u_{k,l} u_{i,j} + \rho \dot{u}_i \dot{u}_i] + d_{ijkl} \dot{u}_{i,j} \dot{u}_{k,l}.$$

Then, we have

$$\int_0^{t_0} (\dot{u}_i \sigma_{ij})_{,j} dt = \int_0^{t_0} d_{ijkl} \dot{u}_{k,l} \dot{u}_{i,j} dt$$

$$+12[c_{ijkl} u_{k,l} u_{i,j} + \rho \dot{u}_i \dot{u}_i] - 12[c_{ijkl} u^0_{k,l} u^0_{i,j} + \rho v^0_i v^0_i].2.11$$

Let $\tau = \max\{1, \omega^2 2c_0\}$ and $\lambda = d_0 \tau + \rho \eta$, we may consider the function

$$F(z, t_0) = \int_{D(z)} \int_0^{t_0} [(\dot{u}_i + \lambda u_i) \sigma_{ij}]_{,j} dt, forz \in [0, L]and0 \leq t_0 < T."(2.12)"$$

Lemma 2.1. The function F satisfies the inequality

$$F(z,t_0) \geq \int_{D(z)} [\lambda \int_0^{t_0} (c_0 3 u_{i,j} u_{i,j} + \tau 4 \dot{u}_{i,j} \dot{u}_{i,j}) dt + 12 [c_0 u_{i,j} u_{i,j} + \rho \tau \tau + \rho \eta \dot{u}_i \dot{u}_i]] da - R_1(z),$$

for all $0 \leq t_0 < T and 0 \leq z \leq L$, where

$$R_1(z) = d_0 \tau + \rho \eta \int_{D(z)} (\rho u_i^0 v_i^0 + 12 d^1_{ijkl} u^0_{k,l} u^0_{i,j}) da + 12 \int_{D(z)} (\rho v_i^0 v_i^0 + c_{ijkl} u^0_{k,l} u^0_{i,j}) da.$$

Proof. For all positive constant ϵ_1, we have

$$12 \rho \dot{u}_i \dot{u}_i + 12 \lambda d_0 u_{i,j} u_{i,j} + \lambda \rho u_i \dot{u}_i$$

$$= 12 (\lambda \epsilon_1 u_i + \epsilon_1^{-1} \rho \dot{u}_i)(\lambda \epsilon_1 u_i + \epsilon_1^{-1} \rho \dot{u}_i) - 12 \epsilon_1^2 \lambda^2 u_i u_i + 12 (\rho - \rho^2 \epsilon_1^{-2}) \dot{u}_i \dot{u}_i + 12 \lambda d_0 u_{i,j} u_{i,j}.$$

So that

$$\int_{D(z)} (12 \rho \dot{u}_i \dot{u}_i + 12 \lambda d_0 u_{i,j} u_{i,j} + \lambda \rho u_i \dot{u}_i) da \geq 12 \int_{D(z)} [(\rho - \rho^2 \epsilon_1^{-2}) \dot{u}_i \dot{u}_i + (\lambda d_0 - \epsilon_1^2 \lambda^2 \eta) u_{i,j} u_{i,j}] da.$$

Now, by taking $\epsilon_1^2 = d_0 \lambda \eta$, we obtain

$$\int_{D(z)} (12 \rho \dot{u}_i \dot{u}_i + 12 \lambda d_0 u_{i,j} u_{i,j} + \lambda \rho u_i \dot{u}_i) da \geq 12 \tau \rho \tau + \rho \eta \int_{D(z)} \dot{u}_i \dot{u}_i da.$$

If we apply the Poincar inequality and the arithmetic- geometric mean inequality we have that the inequality

$$\int_{D(x_3)} \Big(d_{ijkl} \dot{u}_{k,l} \dot{u}_{i,j} + \lambda (c_{ijkl} u_{k,l} u_{i,j} + d^2_{klij} u_{k,l} \dot{u}_{i,j} - \rho \dot{u}_i \dot{u}_i) \Big) da$$

$$\geq \int_{D(x_3)} \Big((d_0 - \lambda(\rho \eta + \omega \epsilon_2 2)) \dot{u}_{i,j} \dot{u}_{i,j} + \lambda (c_0 - \omega 2 \epsilon_2) u_{i,j} u_{i,j} \Big) da,$$

holds, for all positive ϵ_2. Taking $\epsilon_2 = 3 \omega 4 c_0$, we obtain

$$\int_{D(x_3)} \Big(d_{ijkl} \dot{u}_{k,l} \dot{u}_{i,j} + \lambda (c_{ijkl} u_{k,l} u_{i,j} + d^2_{klij} u_{k,l} \dot{u}_{i,j} - \rho \dot{u}_i \dot{u}_i) \Big) da$$

$$\geq \lambda \int_{D(x_3)} \Big(13 c_0 u_{i,j} u_{i,j} + \tau 4 \dot{u}_{i,j} \dot{u}_{i,j} \Big).$$

Remark. In case that we assume zero initial conditions we obtain that $R_1(z) = 0$ for all $z \in [0, L]$.

3. A DECAY THEOREM

In this section, we state a spatial decay estimate for the solutions of the problem defined by (2.1), (2.2), (2.3) and (2.4). We define a function on the cross-sections

$$G(z, t_0) = \int_0^{t_0} \int_{D(z)} \sigma_{i1}(\dot{u}_i + \lambda u_i) dadt, forallz \geq 0andt_0 > 0.\text{"}(3.1)\text{"}$$

Using the Divergence Theorem we obtain

$$G(z + h, t_0) = G(z, t_0) + \int_z^{z+h} F(\chi, t_0) d\chi.\text{"}(3.2)\text{"}$$

Direct differentiation gives

$$\partial G \partial z(z, t_0) = F(z, t_0).$$

From the constitutive equation we have

$$G(z, t_0) = \int_0^{t_0} \int_{D(z)} (c_{i1kl} u_{k,l} + d_{i1kl} \dot{u}_{k,l})(\dot{u}_i + \lambda u_i) dadt$$

$$= \int_0^{t_0} \int_{D(z)} (c_{i1kl} u_{k,l} \dot{u}_i + \lambda c_{i1kl} u_{k,l} u_i + d_{i1kl} \dot{u}_{k,l} \dot{u}_i + \lambda d_{i1kl} \dot{u}_{k,l} u_i) dadt$$

From (2.8), Poincar and the arithmetic- geometric mean inequalities we deduce that

$$|G(z, t_0)| \leq C_1 2(\epsilon_1 + \lambda\epsilon_2) \int_0^{t_0} \int_{D(z)} u_{i,j} u_{i,j} dadt + D_1 2(\epsilon_3 + \lambda\epsilon_4) \int_0^{t_0} \int_{D(z)} \dot{u}_{i,j} \dot{u}_{i,j} dadt$$

$$+12(C_1\epsilon_1 + D_1\epsilon_3) \int_0^{t_0} \int_{D(z)} \dot{u}_i \dot{u}_i dadt + \lambda 2(C_1\epsilon_2 + D_1\epsilon_4) \int_0^{t_0} \int_{D(z)} u_i u_i dadt$$

$$\leq [C_1 2(\epsilon_1 + \lambda\epsilon_2) + \eta\lambda 2(C_1\epsilon_2 + D_1\epsilon_4)] \int_0^{t_0} \int_{D(z)} u_{i,j} u_{i,j} dadt$$

$$+[D_1 2(\epsilon_3 + \lambda\epsilon_4) + \eta 2(C_1\epsilon_1 + D_1\epsilon_3)] \int_0^{t_0} \int_{D(z)} \dot{u}_{i,j} \dot{u}_{i,j} dadt.$$

$$= 3\lambda c_0 [C_1 2(\epsilon_1 + \lambda\epsilon_2) + \eta\lambda 2(C_1\epsilon_2 + D_1\epsilon_4)] \lambda c_0 3 \int_0^{t_0} \int_{D(z)} u_{i,j} u_{i,j} dadt$$

$$+4\lambda\tau[D_1 2(\epsilon_3+\lambda\epsilon_4)+\eta 2(C_1\epsilon_1+D_1\epsilon_3)]\tau\lambda 4\int_0^{t_0}\int_{D(z)}\dot{u}_{i,j}\dot{u}_{i,j}dadt.$$

$$\leq K^*\int_0^{t_0}\int_{D(z)}[\lambda c_0 3u_{i,j}u_{i,j}+\tau\lambda 4\dot{u}_{i,j}\dot{u}_{i,j}]dadt,$$

where

$$K^*=\min_{\epsilon_i\in(0,\infty)}\max\Big[3\lambda c_0[C_1 2(\epsilon_1+\lambda\epsilon_2)+\eta\lambda 2(C_1\epsilon_2+D_1\epsilon_4)],4\tau\lambda[D_1 2(\epsilon_3+\lambda\epsilon_4)+\eta 2(C_1\epsilon_1+D_1\epsilon_3)]\Big],$$

for $i=1,2,3,4$.

The obtention of the constant K^* involves manipulations with a nonlinear system of algebraic equations of order greater than four. Thus, we can not expect the obtention of this constant after some elementary calculations. The natural alternative is to take an upper bound for K^*. For instance, if we choose $\epsilon_1=\epsilon_2=\epsilon_3=\epsilon_4=1$ and we denote

$$K=\max\Big[3\lambda c_0[C_1 2(1+\lambda)+\eta\lambda 2(C_1+D_1)],4\tau\lambda[D_1 2(1+\lambda)+\eta 2(C_1+D_1)]\Big],$$

we obtain

$$|G(z,t_0)|\leq K\partial G\partial z(z,t_0)+KR_1(z),3.3$$

Inequality (3.3) lead to the estimates

$$G(z,t_0)\leq K\partial G\partial z(z,t_0)+KR_1(z),$$

$$-G(z,t_0)\leq K\partial G\partial z(z,t_0)+KR_1(z).\text{"(3.4)"}$$

First inequality in (3.4) is superflous in the case that the cylinder is finite as we suppose. From the second one we get

$$-G(z,t_0)\leq -G(z_0,t_0)\exp\Big[-K(z-z_0)\Big]$$

$$+\exp\Big[-K^{-1}z\Big]\int_{z_0}^{z}\exp\Big[K^{-1}x_1\Big]R_1(x_1)dx_1,\text{"(3.5)"}$$

for all $L \geq z \geq z_0 \geq 0$. Equality (3.2) and the previous inequality imply

$$\int_z^L F(\chi, t_0) d\chi \leq \int_{z_0}^L F(\chi, t_0) d\chi \exp[-K^{-1}(z - z_0)]$$

$$+ \exp\Big[- K^{-1} z\Big] \int_{z_0}^z \exp\Big[K^{-1} x_1\Big] R_1(x_1) dx_1, \text{"(3.6)"}$$

for all $L \geq z \geq z_0 \geq 0$. Thus, we have proved the following result.

Theorem 3.1. Let u_i be a solution of the initial- boundary value problem (2.1)-(2.4) . Then the estimate (3.6) is satisfied for all $L \geq z \geq z_0 \geq 0$.

In the particular case that the initiol conditions $u^0(x)$ and $v^0(x)$ are zero, then the function R_1 is identically zero. The estiamte (3.6) can be state in the following form

$$\int_z^L F(\chi, t_0) d\chi \leq \int_{z_0}^L F(\chi, t_0) d\chi \exp[-K^{-1}(z - z_0)]$$

for all $L \geq z \geq z_0 \geq 0$.

The former results could be improved in two different points of view. First, we will obtain a similar theorem for semi- infinite visco- elastic cylinders whenever the solutions and the initial conditions satisfied some asymptotic condition similar to those imposed in [6]. The first inequality in (3.4) can be used in the analysis as a hepful tool. Second, we will obtain a similar theorem even in case that the cross- sections are not constant.

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