

## COMPATIBLE POISSON TENSORS RELATED TO BUNDLES OF LIE ALGEBRAS

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**Abstract.** We consider some recent results about the Poisson structures, arising on the co-algebra of a given Lie algebra when we have on it a structure of a bundle of Lie algebras. These tensors have applications in the study of the Hamiltonian structures of various integrable nonlinear models, among them the  $O(3)$ -chiral fields system and Landau–Lifshitz equation.

### 1. Introduction

Suppose that  $\text{Mat}(n, \mathbb{K}) \equiv \text{End}(\mathbb{K}^n)$  is the linear space of all  $n \times n$  matrices over the field  $\mathbb{K}$ , which will be either  $\mathbb{R}$  or  $\mathbb{C}$  and will be specified explicitly only if it is necessary.  $\text{Mat}(n)$  possesses a natural structure of associative algebra and as a consequence – a structure of a Lie algebra defined by the commutator  $[X, Y] = XY - YX$ , denoted by  $\mathfrak{gl}(n)$ . However, the structure of the associative algebra over  $\text{Mat}(n)$  is not unique, indeed, if we fix  $J \in \text{Mat}(n)$ , then we can define the product  $(X \circ Y)_J = XJY$  and with respect to it  $\text{Mat}(n)$  is again an associative algebra. This induces a new Lie algebra structure, defined by the bracket

$$[X, Y]_J = XJY - YJX. \quad (1)$$

Thus we obtain a family of Lie brackets, labelled by the element  $J$ . It is readily seen that we actually have a linear space of Lie brackets, since the sum of two such brackets is also a Lie bracket of the same type. The above construction can be applied even if  $X, Y, J$  are not  $n \times n$  matrices, since (1) makes sense when  $X, Y \in \text{Mat}(n, m)$  – the linear space of  $n \times m$  matrices and  $J \in \text{Mat}(m, n)$  – the linear space of  $m \times n$  matrices. According to the accepted terminology, (1) defines a linear bundle of Lie algebras. Another example is obtained if we take  $X,$

$Y$  from  $\mathfrak{so}(n)$  (the skew-symmetric matrices) and belonging to the space  $\mathfrak{sm}(n)$  of symmetric  $n \times n$  matrices. Then again we have a bundle of Lie brackets.

It is difficult to say where the above algebraic structures have been used for the first time, and in this article we shall try to outline the applications they have found recently, especially regarding the infinite dimensional integrable systems. For the finite dimensional systems, [14] provides a wide range of applications. In [17] we have tried to make a review of these algebraic structures, filling up some gaps in the theory, but we believe there are still open questions of general character. For example, see [17], the classification theorem, announced some years ago, is not proved yet, though one sees it cited, [12]. We denote a bundle of Lie brackets by  $(\mathfrak{G}, V)$ , the first space being the algebra, the second space (linear space  $V$ ) labels the brackets, that is in  $[X, Y]_J$ , we have  $X, Y \in \mathfrak{G}, J \in V$ .

In the applications one usually has the so-called closed bundles and the classification we mentioned concerns closed irreducible bundles over  $\mathbb{C}$ . In [17] it was shown that the original definition for a closed bundle, [14], can be substituted by another, requiring the closure of the space of the brackets under the ‘‘Lie derivatives’’ induced by any of the new brackets. The plan of the present paper is the following. We first introduce some class of closed bundles of Lie algebras, that is large enough, then we pass to the Poisson–Lie structures, defined on such algebras in the finite and infinite dimensional case, and finally give some applications.

## 2. Some Closed Linear Bundles of Lie Algebras

As shown in [17], all except one of the known classes of closed irreducible bundles of Lie algebras can be obtained by the following simple construction. On  $\text{Mat}(n)$  there are two natural algebraic structures, induced by the associative algebra structure: the Lie algebra structure, defined by the commutator, and the structure of commutative algebra, defined by the anti-commutator  $X_1 * X_2 \equiv X_1 X_2 + X_2 X_1$ . To distinguish between them we denote  $\text{Mat}(n)$  by  $\mathfrak{gl}(n)$  in the first case and  $\text{mat}(n)$  in the second. There exist the following natural maps:

- i) the representation  $F$  of  $\mathfrak{gl}(n)$  into  $\text{End}(\mathfrak{gl}(n))$ :  $F(X)Y = -X^t Y - YX$
- ii) the map  $G$  of  $\text{Mat}(n)$  into  $\text{End}(\text{Mat}(n))$ :  $G(X)Y = X^t Y - YX$ .

Since  $F$  is a representation, for fixed  $S$  the subspace  $\mathfrak{G}_S = \{X; F(X)S = 0\}$  is a Lie subalgebra in  $\mathfrak{gl}(n)$ . The map  $G$  possess the following interesting property

$$G(X_1 * X_2) = -F(X_1)G(X_2) - F(X_2)G(X_1) \quad (2)$$

and as a consequence the space  $V_S = \{J; G(J)S = 0\}$  is a subalgebra of the commutative algebra  $\text{mat}(n)$  containing the unity  $\mathbb{1}$ . Let us denote  $[X_1, X_2]_Y \equiv X_1 Y X_2 - X_2 Y X_1$ . For  $Y = \mathbb{1}$  this is the usual commutator. The following result is a particular case of a more general one, [17], but we shall use only it.

**Proposition 1.** *If the matrix  $S$  is nondegenerate, then  $\mathfrak{G}_S \cap V_S = \{0\}$  and  $(\mathfrak{G}_S, V_S)$  is closed linear bundle of Lie algebras with brackets defined by  $[X, Y]_M$ , where  $X, Y \in \mathfrak{G}_S, M \in V_S$ .*

Even when we restrict ourself to the bundles of the type  $(\mathfrak{G}_S, V_S)$  the class of bundles is large enough, for example, on any of the simple algebras from the classical series  $B_n = \mathfrak{so}(2n + 1), n \geq 2, C_n = \mathfrak{sp}(2n), n \geq 3,$  and  $D_n = \mathfrak{so}(2n), n \geq 4,$  one can define such structure, [17]. The construction can be also applied with slight modification to the bundle  $(\text{Mat}(p, q), \text{Mat}(q, p))$ .

We consider now the applications of the above algebraic structures for the integrable, or soliton systems, these terms being used for system of nonlinear evolution equations depending on one spacial variable  $x$  and the time  $t$ , allowing Lax representation  $[\partial_x - U, \partial_t - V] = 0$ , and such that they can be solved through some of the techniques, known in general as the **Inverse Scattering Method (ISM)**. The matrices  $U, V$  are functions on some functions  $f_i$  which in their turn depend  $x, t$  ( $f_i$  define the so-called manifold of potentials) and a parameter  $\lambda$ . Usually  $U, V$  lie in the algebra of formal Laurent series in  $\lambda$  with finite principle part and coefficients in some fixed finite-dimensional Lie algebra  $\mathfrak{G}$ . This algebra is denoted by  $\mathfrak{G} \otimes [\lambda, \lambda^{-1}]$  and referred as the jet algebra of  $\mathfrak{G}$ . (There are cases when  $U, V$  are elliptic functions on  $\lambda$  but we do not consider such pairs here). Thus the elements of the jet algebra are of the type

$$P_n = \sum_{i=-n}^{\infty} \lambda^i X_i, \quad X_i \in \mathfrak{G}. \quad (3)$$

However, see [2], there exist Lax pairs for which the matrices  $U$  and  $V$  lie in the spaces of the type  $\mathfrak{G}_S^{p_r} \equiv (\mathfrak{G}_S \otimes [\lambda, \lambda^{-1}]) p_r$ , where  $p_r$  is a fixed element from the vector space  $V_S \otimes [\lambda, \lambda^{-1}]$  – the vector space of Laurent series with finite principle part and coefficients lying in  $V_S$ . These spaces are in fact also algebras. Indeed, their elements are of the type  $P_n p_r$  where  $P_n$  is as in (3). A brief calculation shows that

$$[P_n p_r, Q_m p_r] = ([P_n, Q_m]_{p_r}) p_r. \quad (4)$$

Therefore  $\mathfrak{G}_S^{p_r}(\lambda)$  can be considered as an algebra having the same underlying space as  $\mathfrak{G}_S \otimes [\lambda, \lambda^{-1}]$  but endowed with different brackets

$$(P_n, Q_m) \mapsto [P_n, Q_m]_{p_r}. \quad (5)$$

This has been mentioned in [17] that the relevant algebraic structure used in [2] in relation with the bundle of Lax pairs for the  $O(3)$ -chiral fields system and the Landau–Lifshitz equation is that of the jet algebra  $\mathfrak{so}(4)^{(\lambda+J)}$  type. The structure (5) has been rediscovered recently in [12], where the author made use of

$p_r = A_1 - \lambda A_2$  mainly for the bundles  $(\mathfrak{so}(n), \mathfrak{sm}(n))$  and then apply the Adler–Kostant scheme to the algebra  $\mathfrak{G}_S^{Pr}(\lambda)$  in order to analyze systems that are generalizations of the Landau–Lifshitz equation. There is another application in the line of the finite dimensional case – the construction of compatible Poisson–Lie tensors.

### 3. Poisson–Lie Tensors Related to the Algebras $\mathfrak{G}_S$

As it is well known, a Poisson bracket over a smooth manifold  $M$  can be introduced either by some symplectic form or Poisson tensor (symplectic manifold or Poisson manifold). The symplectic case is classical, as to the Poisson tensor structure, we refer to [6]. There is a canonical way to equip the dual space  $\mathfrak{G}^*$  of a Lie algebra  $\mathfrak{G}$  with a Poisson structure, provided one can identify the spaces  $\mathfrak{G}^{**}$  and  $\mathfrak{G}$ . It amounts to the following: Let  $\mu \in \mathfrak{G}^*$  and denote by  $T_\mu(\mathfrak{G}^*)$ ,  $T_\mu^*(\mathfrak{G}^*)$  the tangent and cotangent spaces at  $\mu$ . Then

$$T_\mu(\mathfrak{G}^*) = \mathfrak{G}^*, \quad T_\mu^*(\mathfrak{G}^*) = \mathfrak{G}^{**} = \mathfrak{G} \quad (6)$$

and one can define a Poisson structure over  $\mathfrak{G}^*$  through the field of linear maps:

$$\mu \rightarrow P_\mu \in \text{Hom}(\mathfrak{G}, \mathfrak{G}^*), \quad P_\mu(X) = -\text{ad}_X^* \mu, \quad X \in \mathfrak{G}. \quad (7)$$

We shall call the tensor  $P$  the Poisson–Lie tensor. It is well known, that the Poisson–Lie tensor  $P$  can be restricted to the orbits of the coadjoint action of the corresponding group and on these orbits it becomes nondegenerate, [13], that is, the orbits are endowed with canonical symplectic structure.

If there exists a symmetric nondegenerate bilinear form  $B(X, Y)$  over  $\mathfrak{G}$ , invariant with respect to the adjoint action of  $\mathfrak{G}$  (this of course means that  $\mathfrak{G}$  is semisimple) one can identify in a canonical way  $\mathfrak{G}^*$  and  $\mathfrak{G}$  and respectively the adjoint and the coadjoint actions. Then the Poisson–Lie tensor is simply  $P_\mu(X) = \text{ad}_X \mu$  for all  $X, \mu \in \mathfrak{G}$ .

One of the characteristic properties of the soliton equations is that they are Hamiltonian with respect to different, but compatible Poisson structures [4, 7]. Two Poisson tensors  $P, Q$  are called compatible if their sum is a Poisson tensor too, and as a matter of fact we have then a two-parametric family of compatible Poisson tensors  $aP + bQ$ , with  $a, b \in \mathbb{K}$ . The linear bundles of Lie brackets provide in a natural way such compatible tensors. Indeed, if for a fixed  $S$ ,  $(\mathfrak{G}_S, V_S)$  is the closed linear bundle of Lie algebras, defined in the previous Section, we have the following:

**Proposition 2.** *On the dual space  $\mathfrak{G}_S^*$  is defined a family of compatible Poisson–Lie tensors, labelled by  $J \in V_S$*

$$q \rightarrow A_q : P_q^J(X) = - \left( \text{ad}_X^J \right)^* (q), \quad X \in \mathfrak{G}_S, q \in \mathfrak{G}_S^*. \quad (8)$$

For example, the tensors  $P_q^{\mathbb{1}}$ ,  $P_q^{J^2}$ , where  $J$  is a diagonal matrix,  $J^2$  is its square and  $\mathbb{1}$  is the identity matrix have been used in [9] to describe the bi-Hamiltonian structure of the Euler equations on the Lie algebras  $\mathfrak{so}(n)$  after identifying  $\mathfrak{so}(n)$  and  $\mathfrak{so}^*(n)$  through the trace form  $\text{Tr}(XY)$ .

There is another algebraic mechanism which is often used in the theory of the integrable systems in order to obtain compatible Poisson tensors. The same algebraic mechanism is also used to obtain from a *finite dimensional* tensors an *infinite dimensional* one.

**Proposition 3.** *If  $\gamma$  is a two-cocycle for the trivial action of a Lie algebra  $\mathfrak{h}$  on the field of scalars  $\mathbb{K}$ , then on  $\mathfrak{h}^*$  there exists the following two-parameter family of compatible Poisson tensors*

$$\mu \rightarrow -c_1 \text{ad}_X^* \mu - c_2 \gamma(X, \cdot), \quad X \in \mathfrak{h}, \quad c_1, c_2 \in \mathbb{K}. \quad (9)$$

We shall not comment further the finite-dimensional applications. In the theory of soliton equations the above construction is usually applied when one considers Poisson tensors of the algebra  $\mathfrak{h} = \mathfrak{G}[x]$ , which is the algebra of the smooth, fast decaying at infinity functions on the line (usually Schwartz functions) with values in some finite dimensional semisimple Lie algebra  $\mathfrak{G}$  with Killing form  $B$  (the algebraic operations are defined, of course, point-wise). Then one introduces on  $\mathfrak{G}[x]$  the bilinear form

$$\langle\langle X, Y \rangle\rangle = \int_{-\infty}^{+\infty} B(X(x), Y(x)) dx, \quad X, Y \in \mathfrak{G}[x] \quad (10)$$

and the so-called **Gel'fand–Fuchs cocycle** [11] (sometimes also called **Maurer–Cartan cocycle**, [3])  $\gamma(X, Y) = \langle\langle \partial_x X, Y \rangle\rangle$ , where  $\partial_x$  stands for the derivation operator with respect to  $x$ . The Poisson tensors obtained in this way define the Hamiltonian structure of the equations for which the matrices  $U$  and  $V$  in the Lax representation belong to the jet algebra  $\mathfrak{G} \otimes [\lambda, \lambda^{-1}]$  and have the form

$$q \rightarrow c_1 \text{ad}_X(q) + c_2 \partial_x X, \quad X \in \mathfrak{G}[x], \quad c_1, c_2 \in \mathbb{K}. \quad (11)$$

However, since one cannot actually identify  $\mathfrak{G}[x]$  and  $\mathfrak{G}[x]**$  one interprets sometimes  $X$  as a distribution. It can be seen also that the same expression gives Poisson tensor fields in the case when  $q(x)$  is not as before an element from  $\mathfrak{G}[x]$ , but an element from the bigger space  $\mathfrak{G}_0[x]$  of the smooth  $\mathfrak{G}$ -valued functions, tending fast enough to some constant value when  $|x| \rightarrow \infty$ . Finally, in the construction (9) we can always add to  $\gamma$  a **trivial cocycle**, that is a cocycle of the type  $d\beta$ , where  $\beta(X) = \langle X, \mu_0 \rangle$  and  $\mu_0$  is a constant element from  $\mathfrak{h}^*$ , and in this way to obtain a family of compatible Poisson tensors

$$\mu \rightarrow P_\mu : P_\mu(X) = -c_1 \text{ad}_X^* \mu - c_2 \gamma(X, \cdot) - \text{ad}_X^* \mu_0, \quad X \in \mathfrak{h} \quad (12)$$

with  $c_i$ ,  $i = 1, 2$  being constants. For applications of the structures (12) in the finite dimensional case see [14], as to the infinite dimensional situation, the tensors from the above family are also frequently used (though sometimes their origin is not mentioned explicitly). For example, the tensors defining the bi-Hamiltonian structure of the soliton equations in the nonlinear Schrödinger equation hierarchy are of the above type [3, 5].

#### 4. The Linear Bundle $(\mathfrak{so}(4), \mathfrak{sm}(4))$

Let us start by making the following remark regarding the Lie algebras  $\mathfrak{so}(n)$ . The map  $(X, Y) \mapsto [X, Y]_J$  is a two-cocycle for the adjoint representation of  $\mathfrak{so}(n)$  (with its usual structure). This cocycle is an ad-coboundary, since

$$[X, Y]_J = \text{ad}_X \alpha(Y) - \text{ad}_Y \alpha(X) - \alpha([X, Y]) \quad (13)$$

where  $\alpha \in \text{End}(\mathfrak{so}(n))$  is given by

$$\alpha(X) = \frac{1}{2}(XJ + JX) \quad (14)$$

where  $J$  is a symmetric matrix, and as usual  $\text{ad}_X(Y) \equiv [X, Y]$ . After this general remark, in what follows we shall concentrate on  $\mathfrak{so}(4)$ . It will be useful to parametrize it in such a way that the splitting  $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$  in two ideals isomorphic to  $\mathfrak{so}(3)$  be seen easily, as well as the action of  $\alpha$ . For  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^3$  we write

$$\mathbf{u} \rightarrow \{\mathbf{u}\}_I = \begin{pmatrix} 0 & u_1 & u_2 & u_3 \\ -u_1 & 0 & u_3 & -u_2 \\ -u_2 & -u_3 & 0 & u_1 \\ -u_3 & u_2 & -u_1 & 0 \end{pmatrix} \quad (15)$$

$$\mathbf{v} \rightarrow \{\mathbf{v}\}_{II} = \begin{pmatrix} 0 & v_1 & v_2 & -v_3 \\ -v_1 & 0 & v_3 & v_2 \\ -v_2 & -v_3 & 0 & -v_1 \\ v_3 & -v_2 & v_1 & 0 \end{pmatrix}. \quad (16)$$

Then each element  $A \in \mathfrak{so}(4)$  can be written in the form

$$A = \{\mathbf{u}\}_I + \{\mathbf{v}\}_{II} \quad (17)$$

and if we put

$$\mathfrak{G}_I = \left\{ \{\mathbf{u}\}_I; \mathbf{u} \in \mathbb{C}^3 \right\}, \quad \mathfrak{G}_{II} = \left\{ \{\mathbf{u}\}_{II}; \mathbf{u} \in \mathbb{C}^3 \right\} \quad (18)$$

then the algebras  $\mathfrak{G}_{I,II}$  are isomorphic to  $\mathfrak{so}(3)$  and are ideals in  $\mathfrak{so}(4)$ . As a consequence  $[\mathfrak{G}_I, \mathfrak{G}_{II}] = 0$ . Since  $J$  is symmetric, it can be diagonalized and since

when  $J$  proportional to  $\mathbb{1}$  gives the usual bracket, we can take  $J$  to be traceless. In what follows, we shall assume that

$$J = \text{diag}(-j_1 - j_2 + j_3, -j_1 + j_2 - j_3, j_1 - j_2 - j_3, j_1 + j_2 + j_3) \quad (19)$$

where  $j_s$  are numbers. The following useful properties hold

**Proposition 4.** *Let  $J$  be the diagonal matrix, introduced above. Then the map  $\alpha$ , defined in (14) interchanges the two  $\mathfrak{so}(3)$  subalgebras of  $\mathfrak{so}(4)$ . More precisely*

$$\alpha(\{\mathbf{u}\}_I) = -\{K\mathbf{u}\}_{II}, \quad \alpha(\{\mathbf{u}\}_{II}) = -\{K\mathbf{u}\}_I \quad (20)$$

where  $K = \text{diag}(j_1, j_2, j_3)$  and  $K\mathbf{a}$  denotes a vector, such that  $(K\mathbf{a})_s = j_s a_s$ .

The Lie algebra  $\mathfrak{so}(4)$  is quite exceptional as in this case one can define nondegenerate inner product which is simultaneously invariant under all Lie algebra structures in the bundle  $(\mathfrak{so}(4), \mathfrak{sm}(4))$  [17]. This, of course, can happen because  $\mathfrak{so}(4)$  is semisimple, but not simple – otherwise all invariant inner products are proportional to the Killing form  $B(X, Y)$ . It is constructed in the following way. On  $\mathfrak{so}(4)$  define the linear map  $T : \mathfrak{so}(4) \rightarrow \mathfrak{so}(4)$ , such that  $T(X_I + X_{II}) = X_I - X_{II}$ , where  $X_I \in \mathfrak{G}_I$ ,  $X_{II} \in \mathfrak{G}_{II}$ , and the bilinear form  $B_T(X, Y) \equiv B(X, T(Y))$ . One can prove that

**Proposition 5.**  *$B_T$  is invariant, non-degenerate bilinear form with respect to the adjoint action of  $\mathfrak{so}(4)_J$  and  $\alpha$  is skew-symmetric with respect to  $B_T$ .*

The constructions outlined in the previous section, using in the Gel'fand cocycle the form  $B_T$  instead of the Killing form  $B$  yields

**Proposition 6.** *Over the manifold  $\mathfrak{so}(4)_0[x]$  there exists a seven-parameter family of compatible Poisson tensor fields*

$$A \rightarrow P_A^{(c_1, c_2, c_3; J)} = c_1 \text{ad}_A + c_2 \text{ad}_A^J + c_3 \partial_x, \quad A \in \mathfrak{so}(4)_0[x]. \quad (21)$$

According to our knowledge, these Poisson structures have been used for the first time in [16] to describe the Hamiltonian properties of the  $O(3)$ -chiral fields system hierarchy. We will show how they can be applied to describe the Hamiltonian properties the hierarchy of the Landau–Lifshitz equation, obtained via polynomial bundle (the LLp hierarchy).

## 5. The P-N Structure for the LL Hierarchy Obtained Through Polynomial Bundle

Suppose  $M$  is a manifold, endowed with two compatible Poisson structures and denote the corresponding tensors by  $P$  and  $Q$ . The compatibility means that  $P + Q$  is also a Poisson structure. As mentioned, the existence of different, but compatible Poisson structures, for which the same equation is Hamiltonian, is one of the

characteristic features of the integrable equations. If in addition one of the tensors, for example  $Q$ , is invertible, then the pair  $(Q, \Lambda^*)$ , ( $\Lambda \equiv Q^{-1} \circ P$ ) endows the manifold with a special geometric structure, the so-called **Poisson–Nijenhuis structure** (P-N structure).

As it is well-known, the P-N structure is responsible for the infinite Abelian algebra of symmetries one has for the soliton equations [7]. In what follows we are going to find such structures on some specific manifold. In the above context (that is when a compatible Poisson structure exists and one can define  $\Lambda = Q^{-1} \circ P$ ), we call  $\Lambda$  a **recursion operator**.

The tensors one would like to invert are not always invertible. But if in the compatible pair  $(P, Q)$  the Poisson tensor  $Q$  is not invertible, one avoids the problem by restricting  $Q$  to some integral leaf of the distribution  $m \mapsto \text{im}(Q_m)$ . As it is known,  $Q$  will be an invertible Poisson tensor on such a leaf (this holds for the finite-dimensional case, and with some caution one can do it usually in the infinite-dimensional case, too). But even if  $Q$  is easily restricted, it might not be the case for  $P$ , since restrictions of the Poisson tensors on a given submanifold are not always possible. However, there are sufficient conditions for a submanifold in a Poisson manifold to be a Poisson manifold, too. They are described in a theorem, proved in general form in [10], and in a simpler version in [8], where also various applications to the soliton equation theory are given. We use the theorem in the following form

**Theorem 1.** *Let  $M$  be a Poisson manifold and  $N \subset M$  be a submanifold. Let  $i$  be the inclusion map of  $N$  into  $M$  and let  $X_P^*(N)_m$  be the subspace of covectors  $\alpha$  at  $m \in N$ , such that  $P_m(\alpha) \in [\text{di}]_m(T_m(N))$ . Let  $T_m^\perp(N)$  be the annihilator of  $\text{im}[\text{di}]_m$ . Let  $X_P^*(N)_m + T_m^\perp(N)_m = T_m^*(M)$  and  $X_P^*(N)_m \cap T_m^\perp(N)_m \subset \ker(P_m)$ . Then there exists unique Poisson tensor  $\bar{P}$  on  $N$ , such that  $P_m = [\text{di}]_m \circ \bar{P}_m \circ [\text{di}]_m^*$ .*

The theorem shows how to construct the restriction of  $P$  and has been used in [16] to find the recursion operators for the  $O(3)$ -chiral fields hierarchy. We use it now to find the recursion operators for the Landau–Lifshitz hierarchy of integrable equation. This is a new result and we shall outline it only in brief, the details will be published elsewhere. We need some definitions first.

Suppose  $\mathbf{S}(x, t) \in \mathbb{R}^3$  is a smooth vector field taking values on the unit sphere  $\mathbb{S}^2 = \{\mathbf{S}; \|\mathbf{S}\|^2 = 1\}$  and tending fast enough to some limit value  $\mathbf{S}_0$  when  $|x| \rightarrow \infty$ , that is all the derivatives of  $\mathbf{S}(x)$  go to zero when  $|x| \rightarrow \infty$ . Usually  $\mathbf{S}_0$  is assumed to be  $(0, 0, 1)$  which we assume too. We require a condition that ensures the above, but it is stronger, we assume that  $\mathbf{S}(x)$  is a real-valued function taking its values on  $\mathbb{S}^2$ , such that the components of  $\mathbf{S}(x) - \mathbf{S}_0$  are a Schwartz (type) functions on the line. We write the above conditions for short as

$$(sw) \lim \mathbf{S}(x) = \mathbf{S}_0 = (0, 0, 1). \quad (22)$$



Two notorious infinite dimensional integrable dynamical systems are known to exist on the space  $M_S$  of functions  $\mathbf{S}(x)$ , satisfying the above conditions, which attracted considerable attention in the past decades:

1. The **Heisenberg ferromagnet equation** (HF)

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx}. \quad (23)$$

It is known that it is gauge equivalent to the **nonlinear Schrödinger equation** (NLS) [18, 5, 3] – another notorious completely integrable system.

2. The **Landau–Lifshitz equation** (LL)

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx} + \mathbf{S} \times R(\mathbf{S}) \quad (24)$$

where  $R = \text{diag}(r_1, r_2, r_3)$ ,  $r_i \geq 0$  and the vector field  $R(\mathbf{S})$  has components  $r_s S_s$ .

The LL equation describes waves in ferromagnet chains and it is one of the infinite dimensional completely integrable systems (cf. [3]), found after the discovery of the ISM. Is it also related to some finite-dimensional integrable systems [15].

For LL there exists a Lax pair, belonging to hierarchies of pairs lying in  $\mathfrak{so}(4)^{(\lambda+J)}$  [2], if the values of  $j_s$  are fixed to be  $j_s = i\sqrt{r_s}$ , where  $r_s$  are as in (24).

The corresponding evolution equations are

$$\begin{aligned} A_t &= (B_n)_x - \frac{1}{2}(AJB_n - B_nJA) = \frac{1}{2}[A, B_{n+1}] \\ A &= \{\mathbf{S}\}_I, \quad B_n = \{\mathbf{b}_n\}_I + \{\mathbf{c}_n\}_{II} \end{aligned} \quad (25)$$

where the quantities  $B_n$  satisfy the following infinite system of equations – the LLp chain system

$$\begin{aligned} [A, B_0] &= 0 \\ [A, B_{n+1}] &= 2(B_n)_x - (AJB_n - B_nJA), \quad n = 0, 1, \dots \end{aligned} \quad (26)$$

The equations (25) are evolution equations, defined on the set of matrices of the type  $A = \{\mathbf{S}\}_I$ , where  $\mathbf{S}(x) \in M_S$ . We denote this space by  $M_S^I$  – this is our manifold of potentials. The hierarchy (25) is the Landau–Lifshitz hierarchy of evolution equations, but as we need to distinguish hierarchies obtained through different bundles we call it LLp hierarchy. Of course, all the relations (25), (26) can be written in terms of the vector fields  $\mathbf{S}$ ,  $\mathbf{b}_n$  and  $\mathbf{c}_n$  [2]. In this case they take simpler form, but unfortunately the algebraic structures remain hidden.

Now, looking at the LLp hierarchy of evolution equations, cf. (25) and (26), and also at the tensor fields  $P_A^{(c_1, c_2, c_3; J)}$  one sees that they can be cast into the following two equivalent forms

$$A_t = P_A(B_{n+1}) = Q_A(B_n), \quad n = 0, 1, \dots \quad (27)$$

where in order to avoid the complicated indices we have put  $Q = P^{(\frac{1}{2}, 0, 0; J)}$  and  $P = P^{(0, -\frac{1}{2}, 1; J)}$ . These two tensors are compatible, which is the starting point of our considerations. We would like to restrict them to the space of potentials  $M_S^I$ . The first tensor can be immediately restricted and since its form remains unchanged we denote it by the same letter. It defines the so-called first Hamiltonian structure on  $M_S$ . We skip the details about the restriction of  $P$  and give only the final result

**Proposition 7.** *On the manifold of potentials  $M_S^I$  there exists a restriction  $\bar{P}$  of the Poisson tensor  $P$  having the form*

$$\begin{aligned} \bar{P}_A(X^*) &= \pi(\partial_x(X^*)) - \frac{1}{8}A_x \int_{\pm\infty}^x B_T(X^*(y), A_y(y)) dy \\ &\quad - \frac{1}{4}[A, (\alpha \circ \mathbf{A}_{\pm}^{-1} \circ \alpha)([A, X^*])] \end{aligned} \quad (28)$$

where  $X^* \in T_A^*(M_S^I)$ ,  $\pi$  denotes the projection in  $\mathfrak{so}(4)$  onto the subspace, orthogonal to  $A$  and  $\alpha$  was introduced in (14).

The operator  $\mathbf{A}$ , whose inverse appears in the above formula, is defined as

$$\mathbf{A}(Y^*) = \partial_x Y^* - \frac{1}{2}[\alpha(A), Y^*] \quad (29)$$

and its domain is the set of functions  $Y^*(x) = \{\mathbf{c}(x)\}_{II}$  where  $(sw) \lim \mathbf{c}(x) = c_0(0, 0, 1)$ ,  $c_0$  being some constant. If  $\mathbf{A}(X^*) = Y^*$ , then  $X^* = \mathbf{A}^{-1}(Y^*)$  is specified if we fix the asymptotic of  $X^*$  at  $+\infty$  ( $-\infty$ ), which explains the subscript  $\pm$ . The description of this operator, the fact that its inverse is well-defined, and that the functions applied to  $\mathbf{A}_+$  and  $\mathbf{A}_-$  gives the same result are topics that are out of the scope of the present paper.

Since on the tangent space  $T_A(M_S^I)$  the operator  $Q_A$  is invertible, we are able to calculate the recursion operator  $\Lambda_{\pm}^L = Q_A^{-1} \circ \bar{P}_A$

$$\begin{aligned} \Lambda_{\pm}^L(X^*) &= -\frac{1}{2}[A, \partial_x(X^*)] + \frac{1}{16}[A, A_x] \int_{\pm\infty}^x B_T(X^*(y), A_y) dy \\ &\quad - \frac{1}{2}(\pi \circ \alpha \circ \mathbf{A}_{\pm}^{-1} \circ \alpha \circ \text{ad}_A)(X^*) \\ &= \Lambda_{\pm}(X^*) - \frac{1}{2}(\pi \circ \alpha \circ \mathbf{A}_{\pm}^{-1} \circ \alpha \circ \text{ad}_A)(X^*) \end{aligned} \quad (30)$$

where  $X^*$  satisfies  $B_T(X^*(x), A(x)) = 0$ . The operators

$$\Lambda_{\pm}(X^*) = -\frac{1}{2}[A, \partial_x(X^*)] + \frac{1}{16}[A, A_x] \int_{\pm\infty}^x B_T(X^*(y), A_y) dy \quad (31)$$

are the well-known recursion operators for the HF equation [5, 1]. To see this one must put  $X^* = \{\mathbf{b}\}_I$  and express  $\Lambda_{\pm}(X^*)$  through  $\mathbf{b}(x)$ . Note that  $\Lambda_{\pm}^L$  reduces to  $\Lambda_{\pm}$  when  $K = 0$ . Now the general theory of compatible Poisson tensors and P-N manifolds yields

**Proposition 8.** *The pair  $(Q, N^L = (\Lambda^L)^*)$  endows the manifold of potentials  $M_S^I$  (or  $M_S$ ) with a P-N structure.*

**Corollary 1.** *Suppose the elements of  $B_n \in \mathfrak{so}(4)_0[x]$  from the LLp hierarchy, (25) are written in the form  $B_n(x) = F_n(x) + G_n(x)$ , where  $F_n(x) \in \mathfrak{G}_I$ ,  $G_n(x) \in \mathfrak{G}_{II}$ . Then  $\pi(F_n)$  can be interpreted as closed forms on  $M_S^I$ , that are in involution with respect to both the Hamiltonian structures  $Q$  and  $\bar{P}$  and they can be obtained recursively*

$$\pi(F_{n+1}) = \Lambda_{\pm}^L(\pi(F_n)) \quad (32)$$

while the equations of the LLp hierarchy have Bi-Hamiltonian form

$$A_t = Q_A(\pi(F_{n+1})) = \bar{P}_A(\pi(F_n)). \quad (33)$$

It must be emphasized, that the recursion operator  $\Lambda_{\pm}^L$  appears here for the first time. The recursion operator that has been considered up to now in relation to the LL equation is the one found in [1]. Both operators (ours and that one in [1]) relate two compatible Poisson structures, but in the present paper the second structure appears in clear algebraic settings which is not the case for the recursion operator used until now. Next, the operator in [1] reduces to  $\Lambda_{\pm}^2$  ( $\Lambda_{\pm}$  is the recursion operator for HF) when the parameters  $r_i$  in the Landau–Lifshitz equation tend to zero (and hence the LL equation reduces to HF) while the operator  $\Lambda_{\pm}^L$  reduces to  $\Lambda_{\pm}$  in the same limiting case. Since the operator found in [1] is not the square of  $\Lambda_{\pm}^L$ , we have a new recursion operator. The flaw of using  $\Lambda_{\pm}^L$  seems could be ascribed to the need to invert the operator  $\mathbf{A}$ . This actually amounts to the problem of solving for  $\mathbf{c}_n$  the chain system at each step. However, the above flaw is apparent, as we have shown in [2] that there exists a nice formula, expressing  $\mathbf{c}_n$  through  $\mathbf{b}_0, \dots, \mathbf{b}_{n-1}$ . Thus  $\Lambda_{\pm}^L$  are recursion operators for the LL equation at least on the same reasoning as the operators found by Barouch *et al* [1].

## 6. Conclusion

We have outlined some of the applications of the theory of the bundles of Lie algebras, related to infinite-dimensional integrable systems. As can be seen, these applications are still rather limited, despite the success in the finite-dimensional case. We strongly believe however, that the recent developments in the theory of the Landau–Lifshitz equation will attract the interest of the specialists. These developments make even more interesting than before the question about the equivalence

between different bundles of the Lax pairs – the elliptic one (usually considered) and the polynomial one, based on the new notion of the alternative Lie algebra structures.

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