Integration Theory in Financial Mathematics

P. Muldowney, W. Wojdowski

1 Introduction

For a majority of people, the use of mathematics is limited to counting and simple arithmetic, mostly in the handling of money. But since the work of Black, Scholes [2] and Merton [7] in 1973, sophisticated mathematics has been required in order to conduct transactions in financial instruments such as stock options and other kinds of derivative assets.

A financial asset or security is a claim to some payment. It may take the physical form of a piece of paper on which a legal contract specifying the claim is written. Assets are frequently traded — bought and sold. We are concerned with establishing the monetary value, here and now, of such entities. Because if their correct monetary value is unknown, they cannot be traded fairly.

A *forward contract* is a simple kind of derivative asset. It is an agreement to trade a security at a future time T for a price K called the *delivery* price. The party who is going to buy is said to hold the *long position* in the forward contract, and the party who is going to sell holds the *short position*.

At the time of writing the forward contract, no money changes hands between the two parties, so the initial value of the forward is zero. But at some time t between the time of writing the contract and its time of expiry, the underlying security value may increase or decrease. So the value of the long position will become positive or negative, and the value of the short position will, respectively, become negative or positive.

Suppose that you hold the long position in the contract. At any time between the initial writing of the contract and its expiry date, you can go to a third party and negotiate a transaction — either selling off your long position if its value is now positive, or buying your way out of it if its value is negative. Likewise if you are the holder of the short position. Our problem here is to determine the correct monetary values involved in such a transaction.

So we have an underlying security, concerning which the forward contract is written, but now both the long and short positions in the forward have values (the one is the negative of the other) and are themselves negotiable instruments. In other words, the forward contract is a *derivative* security.

If the forward contract is written at time 0, and its expiry date is time T, let t denote any intermediate time. Let $x(t)$ denote the price at time t of the underlying security, and let $f(t)$ denote the corresponding value of the forward contract. It is not difficult to express $f(t)$ in terms of other known quantities, such as the risk-free interest rate r. It is shown in Hull [3] that

$$f(t) = x(t) - Ke^{-r(T-t)}. \quad (1)$$

But it is not so easy to establish the values of other kinds of derivative assets. However, Black, Scholes [2], and Merton [7] opened up the subject of *option pricing*.

The holder of the long position in a forward contract is obliged to purchase the underlying asset at time T for the contracted price K. In contrast, the holder of the long position in a European call option on the asset has the *right, but not the obligation* to purchase the underlying asset at time T for the price K.

The Black-Scholes formula for the value at time t of a European call option is

$$x(t)\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \quad (2)$$

where $0 \leq \Phi(d_j) \leq 1$, $j = 1, 2$, and $\Phi(d)$ is a value of a cumulative normal probability distribution which depends, through d, on the behaviour of the underlying asset price.

Our purpose now is to examine in more detail the Black-Scholes-Merton analysis, and how it may be improved by using the non-absolute, generalised Riemann integration of Henstock.

2 The Black-Scholes Model

The basic model is *Brownian motion.* If a process $y(t)$ is a Brownian motion (or, in alternative terminology, a Wiener process) it satisfies:

1. For any $t_1 < t_2$ the increments $y(t_2) - y(t_1)$ are normally distributed;
2. The sample paths $\{y(t) : 0 \leq t \leq T\}$ are continuous functions of t.

A *normally distributed* random variable with mean value μ and variance σ^2 is said to be

$$N(\mu, \sigma^2).$$

The increments $y(t_2)-y(t_1)$ of a *generalised Brownian motion* will have mean value and variance each proportional to $t_2 - t_1$, so the increments are

$$N(\mu(t_2 - t_1), \sigma^2(t_2 - t_1)).$$

We then say that the process y has a *drift rate* μ and *variance rate* σ^2. The Itô calculus [5] can be used to represent y. This approach uses *stochastic differential equations* (in reality, equations involving *stochastic integrals*). The SDE used to represent a generalised Brownian motion y is

$$dy(t) = \sigma dw(t) + \mu dt$$

where $w(t)$ is a standard Brownian motion with zero drift and constant unit variance rate; so $w(t_2) - w(t_1)$ is $N(0, t_2 - t_1)$ for all $t_1 < t_2$.

Our model for a security price $x(t)$ is *geometric Brownian motion* (or *exponential Wiener process*): $x(t)$ is a geometric Brownian motion if $\ln x(t) = y(t)$ is a Brownian motion.

If the Brownian motion $y(t)$ has drift rate μ and variance rate σ^2, then, using the Itô calculus [1], we find the following SDE for $x(t)$:

$$dx(t) = x(t)(\sigma dw(t) + \mu dt), \tag{3}$$

which can be solved using the Itô calculus to give

$$x(t) = \exp\left[\sigma w(t) + (\mu - \frac{1}{2}\sigma^2)t\right].$$

Writing $\mu - \frac{1}{2}\sigma^2 = \nu$, we say that the security price process $x(t)$ has *growth rate* ν and *volatility* σ.

Implicit in this model of the security price process as a geometric Brownian motion is a probability measure on the process sample space which governs the actual changes in time of the security price $x(t)$. But we must impose a new and different set of probabilities (in place of the "real-world" or actual probabilities which allow for a risk-premium) on the sample space in order to accomplish the so-called *risk-neutral valuation* which is used to establish the present value of the derivative price.

Let P denote the natural or real-world probability measure, as implied in the SDE (3) above. Our next step is to impose a different probability measure Q on the sample space of paths $\{x(t) : 0 \leq t \leq T\}$. This Q will be the risk-neutral probability measure that we require.

To see the connection between two probability measures P and Q, we use the Radon-Nikodym Theorem. First, we suppose that P and Q are *equivalent* measures; that is, for any event A of the sample space, $P(A)$ is positive if and only if Q is positive.

Let $\frac{dQ}{dP}$ be the Radon-Nikodym derivative. The Radon-Nikodym Theorem tells us that

$$E^Q\left(x(T)|x(0)\right) = E^P\left(\frac{dQ}{dP}x(T)|x(0)\right).$$

This specifies $\frac{dQ}{dP}$ at time T. Let $\zeta(t) := E^P(\frac{dQ}{dP}|x(t))$. For $t > s$ it can be shown that

$$E^Q\left(x(t)|x(s)\right) = \frac{1}{\zeta(s)}E^P\left(\zeta(t)x(t)|x(s)\right).$$

(The notation $\cdots|x(s))$ indicates that $x(t)$ is determined (or known) for all times less than or equal to s, but is unknown or random for all times t greater than s.) This establishes the relationship between the "natural" probability measure P of the security price process $x(t)$ and any equivalent probability measure Q on the spaces determined by selecting values for s and t:

$$\{x(\tau) : s \leq \tau \leq t\}, \;\; 0 \leq s < t \leq T.$$

We will require to determine such a measure Q which will change a geometric Brownian motion with a given, non-zero drift, to a geometric Brownian motion with zero drift.

This is the fundamental step in the valuation of derivatives (and it applies equally to the classical theory which we are discussing at present, and to the theory based on Henstock integration which we will discuss later).

In the classical Itô calculus version of derivative pricing the Girsanov Theorem is the key to determining Q. (But when we use the Henstock integral, a much simpler approach does the trick.) Loosely, the Girsanov Theorem says: *If $w(t)$ is a P-Brownian motion and $\gamma(t)$ is a pre-visible process satisfying the boundedness condition*

$$E^P\left(\exp\left[\frac{1}{2}\int_0^T \gamma(t)^2 dt\right]\right) < \infty,$$

then there exists a probability measure Q such that Q is equivalent to P,

$$\begin{aligned} \frac{dQ}{dP} &= \exp\left(-\int_0^T \gamma(t)dw(t) - \frac{1}{2}\int_0^T \gamma(t)^2 dt\right), \ \textit{and} \\ \bar{w}(t) &= w(t) + \int_0^t \gamma(s)ds \ \textit{is a } Q\textit{- Brownian motion.} \end{aligned}$$

In other words, given the sample space $\{y\}$ we can impose a probability measure on it so that the resulting Brownian motion $y(t)$ has whatever drift rate we want.

As a consequence of this, if $x(t)$ is a geometric Brownian motion with SDE

$$dx(t) = x(t)(\sigma dw(t) + \mu dt)$$

where $w(t)$ is a P-Brownian motion, then we can use the Girsanov Theorem with $\gamma(t) = (\mu - \nu)/\sigma$ to obtain a measure Q so that

$$dx(t) = x(t)(\sigma d\bar{w}(t) + \nu dt)$$

where $\bar{w}(t)$ is a Q-Brownian motion.

We can now formulate the classical Black-Scholes model. For $0 \le t \le T$, let $f(t)$ denote the price process of a derivative security, the underlying security price process being $x(t)$. We need an amount of cash which grows at the risk-free rate – a cash bond – so we represent this by the bond price

$$b(t) = \exp[rt].$$

Let σ be the volatility of the security and let μ be its growth rate. Define the *discounted claim process* D as follows:

$$D(t) := E^Q\left(b(T)^{-1}f(T)|x(t)\right) = E^Q\left(\exp\left[-rT\right]f(T)|x(t)\right),$$

where Q has been chosen so that the *discounted stock process* S,

$$S(t) := b(t)^{-1}x(t) = \exp\left(-rt\right)x(t),$$

is a Q-martingale. The construction of D implies $D(t)$ is also a martingale under Q. *(It should be noted that, where we have used x, in many books S is used to represent the stock price process.)*

In effect, we use the Radon-Nikodym and Girsanov Theorems to select Q so that $S(t)$ has zero growth rate. The existence of such a Q is guaranteed, in the absence of arbitrage, by the Fundamental Theorem of Asset Pricing [4].

The next step is to use the Martingale Representation Theorem. As a process whose increments are governed by the probability measure Q, $S(t)$ is a martingale because of the way Q is selected. $D(t)$ is also a Q-martingale. The Martingale Representation Theorem then implies that, if the values of $S(t)$ and $D(t)$ are known at time t, there exists a function $\Delta(t)$ so that the increments in $S(t)$ and $D(t)$ satisfy the stochastic differential equation

$$dD(t) = \Delta(t)dS(t). \tag{4}$$

When the value $x(t)$ is known, and hence $S(t)$ and $D(t)$) are known, the function $\Delta(t)$ is deterministic in terms of the SDE just given. But since x is stochastic, $\Delta(t)$ is globally stochastic. The term *pre-visible* is sometimes used to describe this property of Δ. We can think of Δ as being continuous from the left.

The implication of (4) is that, under the probability measure Q, the value of the derivative is growing ("on average", in terms of expectations relative to Q) at the risk-free rate, just as the value of the stock is.

The basic Black-Scholes result is the following. Suppose r, μ and σ are constant. Suppose the derivative claim f is determined at time T. At time t, $x(s)$ is taken to be known for $0 \leq s \leq t$, so $x(s)$ is not a random variable for any such s. Since $f(t)$ is determined by $x(s)$ $(0 \leq s \leq t)$, neither $f(t)$ nor $D(t)$ are random variables. (Randomness only occurs for times greater than t. This is the conditionality implied by the notation $\cdots|x(t)$.) Then the arbitrage price $f(t)$ of such a claim is given by

$$\begin{aligned} b(t)^{-1}f(t) &= E^Q\left(b(T)^{-1}f(T)|x(t)\right) \qquad \text{or} \\ f(t) &= b(t)E^Q\left(b(T)^{-1}f(T)|x(t)\right) \end{aligned}$$

This gives us the valuation we need. The derivative value at time t is

$$f(t) = e^{-r(T-t)}E^Q\left(f(T)|x(t)\right). \tag{5}$$

¿From this we can deduce the values of various kinds of derivative instruments, whose values at maturity are given by appropriate functions $f(T)$.

This is the outline of the classical Black-Scholes model. The above is, for the most part, a summary of an account of the classical continuous-time-continuous-values model given in the book [1] of Baxter and Rennie, which is a good place to start reading this theory, before attempting the deeper mathematical treatments of the subject.

3 Pricing a European Call Option

A European call option gives the holder the right, but not the obligation, to buy a unit of stock for a pre-determined amount K on a particular date in the future, say T. If the stock price $x(T)$ at time T is less than K, the option is not exercised and its value is zero. If $x(T)$ is greater than K, the option is exercised, and a profit of $x(T) - K$ can be realised if the stock is immediately sold.

The value of the claim at expiry time T is therefore

$$f(T) = \max(x(T) - K, 0). \tag{6}$$

The Black-Scholes theory then gives the present value of the derivative as $f(t) = e^{-r(T-t)} E^Q \left(f(T) | x(t) \right)$. Letting $t = 0$ represent the present, for a European call option this reduces to

$$f(0) = e^{-rT} E^Q \left(\max(x(T) - K, 0) \right),$$

where Q is the martingale measure for the discounted stock price process $S(t)$. To evaluate $f(0)$ we need an explicit expression for the probability distribution of $x(T)$ under Q. If we look at the process for $x(t)$ in terms of the standard (drift rate zero, variance rate 1) Brownian motion $\bar{w}(t)$, we have

$$\begin{aligned} d(\ln x(t)) &= \sigma d\bar{w}(t) + (r - \tfrac{1}{2}\sigma^2)dt, \\ \ln x(t) &= \ln x(0) + \sigma \bar{w}(t) + (r - \tfrac{1}{2}\sigma^2)t, \\ x(t) &= x(0) \exp[\sigma \bar{w}(t) + (r - \tfrac{1}{2}\sigma^2)t] \end{aligned}$$

Thus, if we let z denote a normally distributed random variable with parameters

$$N\left(-\frac{1}{2}\sigma^2 T, \sigma^2 T\right),$$

we can write

$$x(T) = x(0) e^{z + rT}$$

so, letting $b = \ln\left(\frac{K}{x(0)}\right) - rT$,

$$\begin{aligned} f(0) &= e^{-rT} E\left(\max(x(0)e^{z+rT} - K, 0) \right) \\ &= \tfrac{1}{\sqrt{2\pi\sigma^2 T}} \int_b^\infty \left(x(0)e^s - Ke^{-rT} \right) \exp\left(-\tfrac{(s + \frac{1}{2}\sigma^2 T)^2}{2\sigma^2 T} \right) ds. \end{aligned}$$

Now let

$$v = -\frac{s + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}},$$

so

$$f(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} \left(x(0) e^{-\sigma\sqrt{T}v - \frac{1}{2}\sigma^2 T} - K e^{-rT} \right) e^{-\frac{1}{2}v^2} dv$$

where

$$a = \frac{\ln \frac{x(0)}{K} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

Since

$$e^{-\sigma\sqrt{T}v - \frac{1}{2}\sigma^2 T - \frac{1}{2}v^2} = e^{-\frac{1}{2}(v + \sigma\sqrt{T})^2},$$

we get

$$f(0) = x(0) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a+\sigma\sqrt{T}} e^{-\frac{1}{2}v^2} dv \right) - K e^{-rT} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{1}{2}v^2} dv \right), \quad (7)$$

and this is the celebrated Black-Scholes formula for the price of a European call option. Compare its form to the form of the expression for the value of a forward contract (1) above. If the present value $x(0)$ of the underlying asset is substantially larger than the exercise price K in the option, then the option is very likely to be exercised, and, to some extent, performs financially like a forward contract. In that case a will be large, and the bracketed factors in (7) above will approach 1. In another notation,

$$f(0) = x(0)\Phi(a + \sigma\sqrt{T}) - K e^{-rT}\Phi(a) \quad (8)$$

where $\Phi(a)$ is the probability that a standard normal variable v (with mean zero and variance 1) is less than or equal to a.

4 Derivative Pricing and Henstock Integrals

The classical Black-Scholes-Merton method for pricing European call options uses the Itô calculus to model the processes involved. But it is possible to model stochastic process using Henstock integrands instead of Itô differentials (or stochastic integrals), and to derive the Black-Scholes partial differential equation and pricing formulae using elementary methods. This is done in [8]. We outline some of the basic ideas here.

The Black-Scholes model [2] assumes that the price of an economic asset, as a random function of time, is a geometric Brownian motion. This implies that if the value x_{j-1} occurs at time t_{j-1}, the probability of the outcome that, at time t_j the process takes a value x_j between u_j and v_j, is related to

$$\int_{u_j}^{v_j} \frac{1}{A_j} \frac{1}{x_j} \exp\left[-\frac{(\ln x_j - \ln x_{j-1})^2}{2\sigma^2(t_j - t_{j-1})}\right] dx_j$$

where A_j is a normalising factor $(2\pi\sigma^2(t_j - t_{j-1}))^{\frac{1}{2}}$.

When pricing a derivative asset, such as a European call option whose value depends on the movements in the value of an underlying asset, the probabilities involved turn out to have the form

$$\int_{u_j}^{v_j} g_j(\mu) dx_j \tag{9}$$

where $g_j(\mu)$ is

$$\frac{1}{A_j} \frac{1}{x_j} \exp\left[-\frac{1}{2\sigma^2} \left(\frac{\ln x_j - \ln x_{j-1} - (\mu - \frac{1}{2}\sigma^2)(t_j - t_{j-1})}{t_j - t_{j-1}}\right)^2 (t_j - t_{j-1})\right] \tag{10}$$

with μ being the actual growth rate of the underlying asset values.

¿From this, the probability of the outcome that, at times t_j, the underlying asset price process x takes values x_j in the range $[u_j, v_j[$ for $1 \le j \le n$ will be given by integrating from u_j to v_j, $j = 1, 2, \ldots, n$, giving an integral of the form

$$\int_{u_1}^{v_1} \cdots \int_{u_n}^{v_n} \left\{\prod_{j=1}^{n} g_j(\mu)\right\} dx_1 \cdots dx_n. \tag{11}$$

In order to accomplish the risk-neutral valuation described in Section 2 above, we construct probability functions which give the underlying asset values $x(t)$ a growth rate equal to the risk-free interest rate r. This is done simply by replacing the parameter μ by r in (10) above, giving

$$\frac{1}{A_j} \frac{1}{x_j} \exp\left[-\frac{1}{2\sigma^2} \left(\frac{\ln x_j - \ln x_{j-1} - (r - \frac{1}{2}\sigma^2)(t_j - t_{j-1})}{t_j - t_{j-1}}\right)^2 (t_j - t_{j-1})\right] \tag{12}$$

which is $g_j(r)$, and then forming the density functions

$$\int_{u_1}^{v_1} \cdots \int_{u_n}^{v_n} \left\{\prod_{j=1}^{n} g_j(r)\right\} dx_1 \cdots dx_n. \tag{13}$$

These are the risk-neutral probabilities for the asset price process x. That the risk-neutral probabilities are, firstly, so easily established, and, secondly, specified sufficiently in (13), distinguishes the Henstock integral approach sharply from the pricing theory that has developed over the past twenty five years or so, as described in Section 2 above. The latter approach requires that the simple sets of outcomes described above be extended, using the Kolmogorov Theorem, to a sigma-algebra of measurable sets in an infinite-dimensional sample space whose representative elements are continuous paths; that the processes involved be represented by appropriate stochastic differential equations; that a suitable probability measure for the sample space be found by means of the Girsanov and Radon-Nikodym Theorems; and that the derivative asset valuation be then determined by means of an expectation using Lebesgue integration.

The binomial model of derivative valuation (see [3]), in which only a finite number of times t_j and a finite number of asset and derivative values are considered, is much simpler than the continuous time model in which every possible time t is allowed. The difficulty arises because of the complicated structures of measurable sets which the continuous-time-continuous-values model requires in the classical Itô model described in Section 2 above.

However, to calculate expectation using Henstock integration, the machinery of measurable sets is not required. It is sufficient that the probabilities be defined for $u_j \leq x(t_j) < v_j$, $1 < j \leq n$. And in this situation, it is remarkably simple to determine the change of measure needed for risk-neutral valuation. The simple sets we have just described are, of course, measurable sets in the classical Itô theory of Section 2. But since that theory requires us to deal with probability measure on general measurable sets, which are harder to visualise than the simple sets we have just described, the classical theory uses the very abstract notation and methods of stochastic differential equations, and tends to lose sight of the basic transition probabilities of (11) which are fundamental to both the Itô and Henstock approaches.

Let us take, for instance, a European call option, whose claim value depends on the underlying asset value at time T in a very simple way, as we have seen in (6). If we seek to obtain the claim value by trying to compute a statistical expectation by integrating the discounted claim value, in n dimensions only, with respect to the probabilities defined by the n-dimensional integrals of (11) above, we get a result similar to that which is obtained by the Lebesgue integral-based continuous-time model. To see this, take

$$0 = t_0 < t_1 < \cdots < t_n = T; \quad x_j := x(t_j) \text{ for } 0 \le j \le n,$$

and, following the reasoning involved in (5), as discussed in Section 2, estimate the expectation of

$$\exp[-rT] \max(x(T) - K, 0)$$

with respect to the probabilities in (11) above. We may suspect that this involves evaluation of an integral of the form

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi dx_1 \cdots dx_n.$$

(Remember that x_0 is the present, known value of the underlying asset, and is not, therefore, a random variable.) The integrand ϕ is

$$e^{-rT} \max(x_n - K, 0) \times \prod_{j=1}^{n} g_j(r). \tag{14}$$

When the integration with respect to x_j $(1 \le j \le n)$ is performed, we get the very same expression as that found in Section 3 for the value of a European call option.

Another pointer to a new approach to derivative valuation is obtained from the Black-Scholes partial differential equation. The problem of option valuation was originally solved (with r and σ assumed constant) by Black and Scholes [2], not by the the risk-neutral probability method described in Sections 2 and 3, but by solving the following partial differential equation, where x_0, t_0 are written as ξ, τ, respectively:

$$\frac{\partial f}{\partial \tau} + r\xi \frac{\partial f}{\partial \xi} + \frac{1}{2}\sigma^2 \xi^2 \frac{\partial^2 f}{\partial \xi^2} = rf, \tag{15}$$

subject to the boundary condition

$$f(T) = \max(x(T) - K, 0).$$

Note that the integrand ϕ of (14) depends on the parameters $\xi = x_0$, $\tau = t_0$ If we perform on ϕ the various partial differentiations that appear in (15), we obtain a partial differential equation which is very similar to the Black-Scholes equation. We might then hope to take expectations (using some system of integration) involving risk-neutral probabilities, and, switching the order of integration and differentiation, obtain the Black-Scholes partial differential equation (15).

These simplistic observations indicate that there may be a way to avoid using the Itô calculus and other advanced mathematical theories in formulating a continuous time model for pricing derivatives, and this is further motivation for examining the problem in terms of Henstock integration in the manner of [8].

References

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