SOME RESULTS ON KNOTS AND LINKS IN ALL DIMENSIONS

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An *(oriented) (ordered) m-component n-(dimensional) link* is a smooth, oriented submanifold $L = \{K_1, \ldots, K_m\}$ of S^{n+2}, which is the ordered disjoint union of m manifolds, each PL homeomorphic to the standard n-sphere. (If $m = 1$, then L is called a *knot*.) We say that m-component n-dimensional links, L_0 and L_1, are *(link-)concordant* or *(link-)cobordant* if there is a smooth oriented submanifold $\widetilde{C}=\{C_1, \ldots, C_m\}$ of $S^{n+2} \times [0,1]$, which meets the boundary transversely in $\partial\widetilde{C}$, is PL homeomorphic to $L_0 \times [0,1]$ and meets $S^{n+2} \times \{l\}$ in L_l $(l = 0, 1)$.

We work in the smooth category.

§1

Let S^3_1 and S^3_2 be 3-spheres embedded in the 5-sphere S^5 and intersect transversely. Then the intersection C is a disjoint collection of circles. Thus we obtain a pair of 1-links C in S^3_i, and a pair of 3-knots S^3_i in S^5.

Conversely let (L_1, L_2) be a pair of 1-links and (X_1, X_2) be a pair of 3-knots. It is natural to ask whether (L_1, L_2) is obtained as the intersection of X_1 and X_2.

In this paper we give a complete answer to the above question.

Definition. (L_1, L_2, X_1, X_2) is called a *quadruple of links* if the following conditions (1), (2) and (3) hold.
(1) $L_i = (K_{i1}, \ldots, K_{im_i})$ is an oriented ordered m_i-component 1-dimensional link $(i = 1, 2)$. (2) $m_1 = m_2$. (3) X_i is an oriented 3-knot.

Definition. A quadruple of links (L_1, L_2, X_1, X_2) is said to be *realizable* if there exists a smooth transverse immersion $f : S^3_1 \coprod S^3_2 \looparrowright S^5$ satisfying the following conditions.
(1) $f|S^3_i$ is a smooth embedding and defines the 3-knot $X_i (i = 1, 2)$ in S^5.
(2) For $C = f(S^3_1) \cap f(S^3_2)$, the inverse image $f^{-1}(C)$ in S^3_i defines the 1-link $L_i (i =$

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$1,2$). Here, the orientation of C is induced naturally from the preferred orientations of S_1^3, S_2^3, and S^5, and an arbitrary order is given to the components of C.

The following theorem characterizes the realizable quadruples of links.

Theorem 1.1. *A quadruple of links (L_1, L_2, X_1, X_2) is realizable if and only if (L_1, L_2, X_1, X_2) satisfies one of the following conditions (i) and (ii).*

(i) Both L_1 and L_2 are proper links, and

$$\mathrm{Arf}(L_1) = \mathrm{Arf}(L_2).$$

(ii) Neither L_1 nor L_2 is proper, and

$$lk(K_{1j}, L_1 - K_{1j}) \equiv lk(K_{2j}, L_2 - K_{2j}) \mod 2 \quad \text{for all } j.$$

Let $f : S^3 \looparrowright S^5$ be a smooth transverse immersion with a connected self-intersection C in S^5. Then the inverse image $f^{-1}(C)$ in S^3 is a knot or a 2-component link. For a similar realization problem, we have:

Theorem 1.2.
(1) All 2-component links are realizable as above.
(2) All knots are realizable as above.

Remark. By Theorem 1.1 a quadruple of links (L_1, L_2, X_1, X_2) with K_1 being the trivial knot and K_2 being the trefoil knot is not realizable. But by Theorem 1.2, the two component split link of the trivial knot and the trefoil knot is realizable as the self-intersection of an immersed 3-sphere.

§2

We discuss the high dimensional analogue of §1.

Definition. (K_1, K_2) is called a *pair of n-knots* if K_1 and K_2 are n-knots. (K_1, K_2, X_1, X_2) is called a *quadruple of n-knots and $(n+2)$-knots* or a *quadruple of $(n, n+2)$-knots* if K_1 and K_2 constitute a pair of n-knots (K_1, K_2) and X_1 and X_2 are diffeomorphic to the standard $(n+2)$-sphere.

Definition. A quadruple of $(n, n+2)$-knots (K_1, K_2, X_1, X_2) is said to be *realizable* if there exists a smooth transverse immersion $f : S_1^{n+2} \coprod S_2^{n+2} \looparrowright S^{n+4}$ satisfying the following conditions.

1. $f|S_i^{n+2}$ defines X_i (i=1,2).
2. The intersection $\Sigma = f(S_1^{n+2}) \cap f(S_2^{n+2})$ is PL homeomorphic to the standard sphere.
3. $f^{-1}(\Sigma)$ in S_i^{n+2} defines an n-knot K_i $(i = 1, 2)$.

A pair of n-knots (K_1, K_2) is said to be *realizable* if there is a quadruple of $(n, n+2)$-knots (K_1, K_2, X_1, X_2) which is realizable.

The following theorem characterizes the realizable pairs of n-knots.

Theorem 2.1. *A pair of n-knots (K_1, K_2) is realizable if and only if (K_1, K_2) satisfies the condition that*

$$\begin{cases} (K_1,K_2) \text{ is arbitrary} & \text{if } n \text{ is even,} \\ \mathrm{Arf}(K_1) = \mathrm{Arf}(K_2) & \text{if } n = 4m+1, \quad (m \geqq 0, m \in \mathbb{Z}). \\ \sigma(K_1)=\sigma(K_2) & \text{if } n = 4m+3, \end{cases}$$

There exists a mod 4 periodicity in dimension similar to the periodicity of high-dimensional knot cobordism and surgery theory. ([CS1,2] and [L1,2].)

We have the following results on the realization of a quadruple of $(n, n+2)$-knots.

Theorem 2.2. *A quadruple of $(n, n+2)$-knots $T = (K_1, K_2, X_1, X_2)$ is realizable if K_1 and K_2 are slice.*

Kervaire proved that all even dimensional knots are slice ([Ke]). Hence we have:

Corollary 2.3. *If n is even, an arbitrary quadruple of $(n, n+2)$-knots $T = (K_1, K_2, X_1, X_2)$ is realizable.*

In order to prove Theorem 2.1, we introduce a new knotting operation for high dimensional knots, *high dimensional pass-moves.* The 1-dimensional case of Definition 2.1 is discussed on p.146 of [K1].

Definition. Let $(2k+1)$-knot K be defined by a smooth embedding $g : \Sigma^{2k+1} \hookrightarrow S^{2k+3}$, where Σ^{2k+1} is PL homeomorphic to the standard $(2k+1)$-sphere. ($k \geqq 0$.) Let $D_x^{k+1}=\{(x_1, \dots, x_{k+1}) | \ \Sigma x_i^2 < 1\}$ and $D_y^{k+1}=\{(y_1, \dots, y_{k+1}) | \ \Sigma y_i^2 < 1\}$. Let $D_x^{k+1}(r)=\{(x_1, \dots, x_{k+1}) | \ \Sigma x_i^2 \leqq r^2\}$ and $D_y^{k+1}(r)=\{(y_1, \dots, y_{k+1}) | \ \Sigma y_i^2 \leqq r^2\}$. A local chart (U, ϕ) of S^{2k+3} is called a *pass-move-chart of K* if it satisfies the following conditions.

(1) $\phi(U) \cong \mathbb{R}^{2k+3} = (0,1) \times D_x^{k+1} \times D_y^{k+1}$
(2) $\phi(g(S^{2k+1}) \cap U) = [\{\frac{1}{2}\} \times D_x^{k+1} \times \partial D_y^{k+1}(\frac{1}{3})] \amalg [\{\frac{2}{3}\} \times \partial D_x^{k+1}(\frac{1}{3}) \times D_y^{k+1}]$

Let $g_U : \Sigma^{2k+1} \hookrightarrow S^{2k+3}$ be an embedding such that:

(1) $g|\{\Sigma^{2k+1} - g^{-1}(U)\} = g_U|\{\Sigma^{2k+1} - g^{-1}(U)\}$, and
(2) $\phi(g_U(\Sigma^{2k+1}) \cap U) = [\{\frac{1}{2}\} \times D_x^{k+1} \times \partial D_y^{k+1}(\frac{1}{3})] \amalg$
$[\{\frac{2}{3}\} \times \partial D_x^{k+1}(\frac{1}{3}) \times (D_y^{k+1} - D_y^{k+1}(\frac{1}{2}))]$
$\cup [[\frac{1}{3}, \frac{2}{3}] \times \partial D_x^{k+1}(\frac{1}{3}) \times \partial D_y^{k+1}(\frac{1}{2})]$
$\cup [\{\frac{1}{3}\} \times \partial D_x^{k+1}(\frac{1}{3}) \times D_y^{k+1}(\frac{1}{2})\}]$

Let K_U be the $(2k+1)$-knot defined by g_U. Then we say that K_U is obtained from K by the *(high dimensional) pass-move in U.* We say that $(2k+1)$-knot K and K' are *(high dimensional) pass-move equivalent* if there exist $(2k+1)$-knots $K_1, \dots, K_{q+1}$ and pass-move charts U_i $(i = 1, \dots, q)$ of K_1 such that (1) $K_1 = K$, $K_{q+1} = K'$, and (2) K_{i+1} is obtained from K_i by the high dimensional pass-move in U_i.

High dimensional pass-moves have the following relation with the Arf invariant and the signature of knots.

Theorem 2.4. *For simple $(2k+1)$-knots K_1 and K_2, the following two conditions are equivalent. $(k \geqq 1.)$*

(1) *K_1 is pass-move equivalent to K_2.*

(2) *K_1 and K_2 satisfy the condition that* $\begin{cases} \mathrm{Arf}(K_1) = \mathrm{Arf}(K_2) & \textit{when } k \textit{ is even} \\ \sigma(K_1) = \sigma(K_2) & \textit{when } k \textit{ is odd.} \end{cases}$

See [L2] for simple knots.

Theorem 2.5. *For $(2k+1)$-knots K_1 and K_2, the following two conditions are equivalent. $(k \geqq 0.)$*

(1) *There exists a $(2k+1)$-knot K_3 which is pass-move equivalent to K_1 and cobordant to K_2.*

(2) *K_1 and K_2 satisfy the condition that* $\begin{cases} \mathrm{Arf}(K_1) = \mathrm{Arf}(K_2) & \textit{when } k \textit{ is even} \\ \sigma(K_1) = \sigma(K_2) & \textit{when } k \textit{ is odd.} \end{cases}$

The case $k = 0$ of Theorem 2.5 follows from [K1],[K2].

§**3**

We discuss the case when three spheres intersect in a sphere.

Let F_i be closed surfaces $(i = 1, 2, \dots, \mu)$. A *surface-$(F_1, F_2, \dots, F_\mu)$-link* is a smooth submanifold $L = (K_1, K_2, \dots, K_\mu)$ of S^4, where K_i is diffeomorphic to F_i. If F_i is orientable we assume that F_i is oriented and K_i is an oriented submanifold which is orientation preserving diffeomorphic to F_i. If $\mu = 1$, we call L a *surface-F_1-knot.*

An (F_1, F_2)-link $L = (K_1, K_2)$ is called a *semi-boundary link* if

$$[K_i] = 0 \in H_2(S^4 - K_j; \mathbb{Z}) \ (i \neq j)$$

following [S].

An (F_1, F_2)-link $L = (K_1, K_2)$ is called a *boundary link* if there exist Seifert hypersurfaces V_i for K_i $(i = 1, 2)$ such that $V_1 \cap V_2 = \phi$.

An (F_1, F_2)-link (K_1, K_2) is called a *split link* if there exist B^4_1 and B^4_2 in S^4 such that $B^4_1 \cap B^4_2 = \phi$ and $K_i \subset B^4_i$.

Definition. Let $L_1 = (K_{12}, K_{13})$, $L_2 = (K_{23}, K_{21})$, and $L_3 = (K_{31}, K_{32})$ be surface-links. (L_1, L_2, L_3) is called a *triple of surface-links* if K_{ij} is diffeomorphic to K_{ji}. ((i,j)=(1,2),(2,3),(3,1).) (Note that the knot type of K_{ij} is different from that of K_{ji}.)

Definition. Let $L_1 = (K_{12}, K_{13})$, $L_2 = (K_{23}, K_{21})$, and $L_3 = (K_{31}, K_{32})$ be surface-links. A triple of surface-links (L_1, L_2, L_3) is said to be *realizable* if there exists a transverse immersion $f : S^4_1 \amalg S^4_2 \amalg S^4_3 \looparrowright S^6$ such that (1)$f|S^4_i$ is an embedding(i=1,2,3), and (2) $(f^{-1}(f(S^4_i) \cap f(S^4_j)),\ f^{-1}(f(S^4_i) \cap f(S^4_k))\ )$ in S^4_i is L_i. ((i,j,k)=(1,2,3), (2,3,1), (3,1,2).)

Note. If (L_1, L_2, L_3) is realizable, then K_{ij} are orientable and are given an orientation naturally. From now on we assume that, when we say a triple of surface-links, the triple of surface-links consists of oriented surface-links.

We state the main theorem.

Theorem 3.1. *Let L_i $(i = 1, 2, 3)$ be semi-boundary surface-links. Suppose the triple of surface-links (L_1, L_2, L_3) is realizable. Then we have the equality*

$$\beta(L_1) + \beta(L_2) + \beta(L_3) = 0,$$

where $\beta(L_i)$ is the Sato-Levine invariant of L_i.

Refer to [S] for the Sato-Levine invariant. Since there exists a triple of surface-links (L_1, L_2, L_3) such that $\beta(L_1)$=0, $\beta(L_2)$=0 and $\beta(L_3)$=1 ([R] and [S]), we have:

Corollary 3.2. *Not all triples of oriented surface-links are realizable.*

We have sufficient conditions for the realization.

Theorem 3.3. *Let L_i $(i = 1, 2, 3)$ be split surface-links. Then the triple of surface-links (L_1, L_2, L_3) is realizable.*

Theorem 3.4. *Suppose L_i are (S^2, S^2)-links. If L_i are slice links$(i = 1, 2, 3)$, then the triple of surface-links (L_1, L_2, L_3) is realizable.*

It is well known that there exists a slice-link which is neither a boundary link nor a ribbon link. Hence we have:

Corollary 3.5. *There exists a realizable triple of surface-links (L_1, L_2, L_3) such that neither L_i are boundary links and all L_i are semi-boundary links.*

Besides the above results, we prove the following triple are realizable.

Theorem 3.6. *There exists a realizable triple of surface-links (L_1, L_2, L_3) such that neither L_i are semi-boundary links.*

Here we state:

Problem (1). *Suppose $\beta(L_1)$+$\beta(L_2)$+$\beta(L_3)$=0. Then is the triple of surface-links (L_1, L_2, L_3) realizable?*

Using a result of [O], we can make another problem from Problem (1).

Problem (2). *Is every triple of (S^2, S^2)-links realizable?*

Note. By Theorem 3.4, if the answer to Problem (2) is negative, then the answer to an outstanding problem: "Is every (S^2, S^2)-link slice?" is "no." (Refer [CO] to the slice problem.)

§4

An *(oriented) n-(dimensional) knot* K is a smooth oriented submanifold of $\mathbb{R}^{n+1} \times \mathbb{R}$ which is PL homeomorphic to the standard n-sphere. We say that n-knots K_1 and K_2 are *equivalent* if there exists an orientation preserving diffeomorphism $f : \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}^{n+1} \times \mathbb{R}$ such that $f(K_1)$=K_2 and $f|_{K_1} : K_1 \to K_2$ is an orientation preserving diffeomorphism. Let π: $\mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}^{n+1}$ be the natural projection map. A subset P of $\mathbb{R}^{n+1}$ is called the *projection* of an n-knot K if there exists an orientation preserving diffeomorphism $\beta : X \to K$ and a smooth transverse immersion $\gamma : X \to \mathbb{R}^{n+1}$ such that $\pi|_K \circ \beta = \gamma$ and $\pi|_K \circ \beta(K) = \gamma(X) = P$. The *singular point set* of the projection of an n-knot is the set $\{ x \in \pi|_K (K) \mid \sharp (\pi|_K)^{-1}(x) \geqq 2. \}$.

It is well-known that the projection of any 1-dimensional knot is the projection of a 1-knot equivalent to the trivial knot. This fact is used in a way to define the Jones polynomial and in another way to define the Conway-Alexander polynomial.

We consider the following problem.

Problem. (1) Let K be an n-knot diffeomorphic to the standard sphere. Let P be the projection of K. Is P the projection of an n-knot equivalent to the trivial knot?

(2) Furthermore, suppose that the singular point set of P consists of double points. Is P the projection of an n-knot equivalent to the trivial knot?

The author proved that the answer to Problem (2) in the case of $n \geqq 3$ is negative and hence that the answer to Problem (1) in the case of $n \geqq 3$ is also negative. We prove:

Theorem 4.1. *Let n be any integer greater than two. There exists an n-knot K such that the projection P has the following properties.*

(1) *P is not the projection of any knot equivalent to the trivial knot.*
(2) *The singular point set of P consists of double points.*
(3) *K is diffeomorphic to the standard sphere.*

Note. Problem (1) in the case of $n = 2$ remains open. Dr. Taniyama has informed that the answer to Problem (2) in the case of $n = 2$ is positive, that the proof is easy, but that he has not published the proof.

§5

We have the following outstanding open problems.

Problem (1). *Classify n-links up to link concordance for $n \geqq 1$.*

Problem (2). *Is every even dimensional link slice?*

Problem (3). *Is every odd dimensional link concordant to (a sublink of) a homology boundary link?*

The author has modified Problem (2) to formulate the following Problem (4). We consider the case of 2-component links. Let $L = (K_1, K_2) \subset S^{n+2} \subset B^{n+3}$ be a $2m$-link ($2m \geq 2$). By Kervaire's theorem in [Ke] there exist D_i^{2m+1} $(i = 1, 2)$ embedded in B^{2m+3} such that $D_i^{2m+1} \cap S^{2m+2} = \partial D_i^{2m+1} = K_i$. Then D_1^{2m+1} and D_2^{2m+1} intersect mutually in general. Furthermore $D_1^{2m+1} \cap D_2^{2m+1}$ in D_i^{2m+1} defines $(2m-1)$-link.

Problem (4). *Can we remove the above intersection $D_1^{2m+1} \cap D_2^{2m+1}$ by modifying embedding of D_1^{2m+1} and D_2^{2m+1}?*

If the answer to Problem (4) is positive, the answer to Problem (2) is positive.

Here we make another problem from Problem (4).

Problem (5). *What is obtained as a pair of $(2m-1)$-links ($D_1^{2m+1} \cap D_2^{2m+1}$ in D_1^{2m+1}, $D_1^{2m+1} \cap D_2^{2m+1}$ in D_2^{2m+1}) by modifying embedding of D_1^{2m+1} and D_2^{2m+1}?*

We have the following theorem, which is an answer to Problem (5).

Theorem 5.1. *For all 2-component $2m$-link $L = (K_1, K_2)$ ($m \geq 0$), there exist D_1^{2m+1} and D_2^{2m+1} as above such that each of $(2m-1)$-links $D_1^{2m+1} \cap D_2^{2m+1}$ in D_1^{2m+1} and $D_1^{2m+1} \cap D_2^{2m+1}$ in D_2^{2m+1} is the trivial knot.*

We have the following Theorem 5.2. We say that n-dimensional knots, K and K', are *(link-)concordant* or *(link-)cobordant* if there is a smooth oriented submanifold C of $S^{n+2} \times [0,1]$, which meets the boundary transversely in ∂C, is PL homeomorphic to $L_0 \times [0,1]$, and meets $S^{n+2} \times \{l\}$ in L_l ($l = 0, 1$). Then we call C a *concordance-cylinder of K and K'*.

Theorem 5.2. *For all 2-component n-link $L = (K_1, K_2)$ ($n > 1$), there exist a boundary link $L' = (K_1', K_2')$ satisfying that K_i' is concordant to K_i and a concordance-cylinder $\begin{cases} C_1 \\ C_2 \end{cases}$ of $\begin{cases} K_1 \\ K_1' \end{cases}$ and $\begin{cases} K_2 \\ K_2' \end{cases}$ such that each of $(n-1)$-links, $C_1 \cap C_2$ in C_1 and $C_1 \cap C_2$ in C_2, is the trivial knot.*

When n is even, Theorem 5.2 is Theorem 5.1. Because all even dimensional boundary links are slice.

By the following exciting theorem of Cochran and Orr, when n is odd, Theorem 5.2 is best possible from a viewpoint.

Theorem. *[CO] Not all 2-component odd dimensional links are concordant to boundary links.*

§6

Let D_1^n, D_2^n, D_3^n be submanifolds of S^{n+2} diffeomorphic to the n-disc such that Int $(D_i^n) \cap$ Int $(D_j^n)=\phi$ (for $i \neq j$) and $\partial D_1^n = \partial D_2^n = \partial D_3^n$. Then $D_1^n \cup D_2^n \cup D_3^n$ is called an *n-dimensional θ-curve in $\mathbb{R}^{n+2}$. The set of the constituent knots of an n-dimensional θ-curve θ in $\mathbb{R}^{n+2}$* is a set of three n-knots in S^{n+2}, which are made from $D_1^n \cup D_2^n$, $D_2^n \cup D_3^n$, and $D_1^n \cup D_3^n$.

The definitions in the case of the PL category are written in [Y].

Problem. Take any set of three n-knots. Is it the set of the constituent knots of an n-dimensional θ-curve?

In [Y] it is proved that if K_1, K_2, and K_3 are ribbon n-knots, then the set (K_1, K_2, K_3) is the set of the constituent knots of an n-dimensional θ-curve.

We discuss the case of non-ribbon knots. The following theorems hold both in the smooth category and in the PL category.

We have the following theorems.

Theorem 6.1. *Let n be any positive integer. Let K_1 and K_2 be trivial knots. There exist an n-dimensional θ-curve θ in $\mathbb{R}^{n+2}$ and a non-ribbon knot K_3 such that (K_1, K_2, K_3) is the set of the constituent knots of the n-dimensional θ-curve θ in $\mathbb{R}^{n+2}$.*

Furthermore we have the following.

Theorem 6.2. *Let m be any odd positive integer. Let K_1 and K_2 be trivial knots.*

(1) There exist m-dimensional θ-curves θ in $\mathbb{R}^{m+2}$ and a non-ribbon and non-slice knot K_3 such that (K_1, K_2, K_3) is the set of the constituent knots of the m-dimensional θ-curve θ in $\mathbb{R}^{m+2}$.

(2) There exist m-dimensional θ-curves θ in $\mathbb{R}^{m+2}$ and a non-ribbon and slice knot K_3 such that (K_1, K_2, K_3) is the set of the constituent knots of the m-dimensional θ-curve θ in $\mathbb{R}^{m+2}$.

We have the following.

Theorem 6.3. *When $n = 2m + 1 (m \geq 1)$, there exists a set of three n-knots which is never the set of the constituent knots of any θ-curve.*

The above problem remains open.

§7

We use Theorem 1.1 in §1 to give an answer to a problem of Fox.

In [F] Fox submitted the following problem about 1-links. Here, note that "slice link" in the following problem is now called "ordinary sense slice link," and "slice link in the strong sense" in the following problem is now called "slice link" by knot theorists.

Problem 26 of [F]. Find a necessary condition for L to be a slice link; a slice link in the strong sense.

Our purpose is to give some answers to the former part of this problem. The latter half is not discussed here. The latter half seems discussed much more often than the former half. See e.g. [CO], [L3], etc.

We review the definition of ordinary sense slice links and that of slice links, which we now use.

We suppose m-component 1-links are oriented and ordered.

Let $L = (K_1, \dots, K_m)$ be an m-component 1-link in $S^3 = \partial B^4$. L is called a *slice 1-link*, which is "a slice link in the strong sense" in the sense of Fox, if there exist 2-discs $D_i^2 (i = 1, \dots, m)$ in B^4 such that $D_i^2 \cap \partial B^4 = \partial D_i^2$, $D_i^2 \cap D_j^2 = \phi (i \neq j)$, and $(\partial D_1^2, \dots, \partial D_m^2)$ in ∂B^4 defines L.

Take a 1-link L in S^3. Take S^4 and regard S^4 as $(\mathbb{R}^3 \times \mathbb{R}) \cup \{\infty\}$. Regard the 3-sphere S^3 as $\mathbb{R}^3 \cup \{\infty\}$ in S^4. L is called an *ordinary sense slice 1-link*, which is "a slice link" in the sense of Fox, if there exists an embedding $f : S^2 \hookrightarrow \mathbb{R}^3 \times \mathbb{R}$ such that f is transverse to $\mathbb{R}^3 \times \{0\}$ and $f(S^2) \cap (\mathbb{R}^3 \times \{0\})$ in $\mathbb{R}^3 \times \{0\}$ defines L. Suppose f defines a 2-knot X. Then L is called a *cross-section* of the 2-knot X.

From now on we use the terms in the ordinary sense now current.

Ordinary sense slice 1-links have the following properties.

Theorem 7.1. *Let L be a 1-dimensional ordinary sense slice link. Then the followings hold.*

(1) *L is a proper link.*
(2) $\mathrm{Arf}(L) = 0$.

§8

Let K_1 and K_2 be smooth submanifolds of S^{n+2} diffeomorphic to an n-dimensional closed smooth manifold M. The notion of cobordism between K_1 and K_2 is defined naturally. A Seifert surface of K_i and a Seifert matrix of K_i are defined naturally. The notion of matrix cobordism between two Seifert matrices is defined naturally.

It is also natural to ask the following problem.

Problem. Are K_1 and K_2 as above cobordant?

The author thinks that there exists a kind of surgery exact sequence.

The author obtained the following results.

Theorem 8.1. *There exist a $(2n+1)$-dimensional closed oriented smooth manifold M and smooth submanifolds K_1 and K_2 of S^{2n+3} diffeomorphic to M such that (1) K_1 and K_2 are not cobordant, and (2) the Seifert matrices of K_1 and K_2 are matrix cobordant.*

Theorem 8.2. *There exist a $2n$-dimensional closed oriented smooth manifold M and smooth submanifolds K_1 and K_2 of S^{2n+2} diffeomorphic to M such that K_1 and K_2 are not cobordant*

In the both cases the obstructions live in certain homotopy groups.

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