Special Geometric Structures in String Theory
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SPECIAL GEOMETRIES, FROM REAL TO QUATERNIONIC

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ABSTRACT. Special geometry is most known from 4-dimensional $N = 2$ supergravity, though it contains also quaternionic and real geometries. In this review, we first repeat the connections between the various special geometries. Then the constructions are given starting from the superconformal approach. Without going in detail, we give the main underlying principles. We devote special attention to the quaternionic manifolds, introducing the notion of hypercomplex geometry, being manifolds close to hyperkähler manifolds but without a metric. These are related to supersymmetry models without an action.

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1. Introduction

Special geometry is related to the supersymmetric theories with 8 real supercharges. In 4 dimensions this corresponds to $N = 2$, and special geometry was first found in this form [1, 2]. The first motivation to study $N = 2$ supergravity was given in [3] as 'Extended supergravity theories offer prospects for unified theories of gravitation with other fundamental interactions. Although much is know for $N = 1$ supergravity, not many complete results exist for higher-N theories'. Thus the first motivation was just to find more general theories than those that already existed at the time. The corresponding geometry in rigid supersymmetry was investigated in [4, 5]. A new boost came with the advent of Calabi–Yau compactifications of string theory. It was found that special geometry occurs in the moduli space of these manifolds [6, 7, 8, 9, 10, 11]. E.g. in this period, a coordinate-independent formulation of special geometry was developed [12, 13]. In the research on black holes a few years later, the $N = 2$ theories were a popular tool, and the special geometry was particularly useful [14, 15]. The research on dualities started with the Seiberg–Witten papers, which were based on the use of (rigid) special geometry [16, 17]. The AdS/CFT correspondence [18] gave new applications of special geometries, and with brane-world scenarios as in [19, 20], also the 5-dimensional variants ('very special', see below) of special geometry got a lot of attention. The relevant actions were first found in [21] and connected to special geometry in [22]. A full account of the present knowledge was given in [23]. The mathematicians got interested in special geometry due to its relation with quaternionic geometry [7], which lead to new results on the classification of homogeneous quaternionic spaces [24, 25].

Why are the theories with 8 supersymmetries so interesting? The maximal supergravities[1] contain 32 supersymmetries. These are the $N = 8$ theories in 4 dimensions, and exist in spaces of Lorentzian signature with at most 11 dimensions, i.e. (10,1) space–time dimensions. If one allows more time directions, 32 supersymmetries is possible in 12 dimensions with (10,2) or (6,6) signature. However, these theories allow no matter multiplets (multiplets with fields up to spin 1). For the geometry, determined by the kinetic terms of the scalars, this means that the manifold is fixed once the dimension is given. For all theories with 32 supersymmetries this is a symmetric space.

Matter multiplets are possible if one limits the number of supersymmetries to 16 (thus $N = 4$ in 4 dimensions). These theories exist up to 10 dimensions with

[1] The restriction is due to interacting field theory descriptions, which e.g. in 4 dimensions does not allow fields with spin larger than 2.

Lorentzian signature. In this case, the geometry is fixed to a particular coset geometry once one gives the number of matter multiplets that are coupled to supergravity.

The situation becomes more interesting if the number of supersymmetries is 8. Now there are functions, which can be varied continuously, that determine the geometry. This makes the geometries much more interesting. Of course, if one further restricts to 4 supersymmetries, more geometries would be possible. In 4 dimensions, e.g., general Kähler manifolds appear. For 8 supersymmetries, these are restricted to 'special Kähler manifolds', determined by a holomorphic prepotential. However, this restriction makes the class of manifolds very interesting and manageable. The holomorphicity is a useful ingredient, and was e.g. essential to allow the solution of the theory in the Seiberg–Witten model [16, 17]. The theories with 8 supersymmetries are thus the maximally supersymmetric that are not completely determined by the number of fields in the model, but allow arbitrary functions in their definition, i.e. continuous deformations of the metric of the manifolds.

In the following section, we will give an overview of the matter multiplets with 8 supersymmetries. Then, in section 3 we will present the family of special geometries. Section 4 will show why superconformal methods are useful when one wants to understand the extra symmetries (related to duality) that appear in special geometry. This sets the stage for the main section (section 5) where we will explain the construction of the models using 3 steps: introduction of the multiplets in the superconformal framework, construction of the action, and gauge fixing. We will devote special emphasis to hypermultiplets and quaternionic geometry. The presentation here is new, and introduces models without an action, related to hypercomplex geometry. We give some conclusions in section 6.

2. Matter multiplets with 8 supersymmetries

The maximal spacetime for 8 supersymmetries is 8-dimensional, with signature (8,0) or (4,4). We restrict ourselves to Lorentzian signature, and as such the maximal dimension is 6. An overview of the matter multiplets is given in table 1. We now discuss the geometries defined by the kinetic action of the scalar fields in these multiplets.

In 6 dimensions there are tensor multiplets, vector multiplets and hypermultiplets. The former have a 2-form gauge field with self-dual field strength. They have been studied first in [26]. The scalars of n tensor multiplets coupled to supergravity describe a coset space $\mathrm{SO}(1,n)/\mathrm{SO}(n)$. They can be coupled to vector multiplets. These couplings are governed by constants c_{IMN}, where I labels the

Table 1. *Matter multiplets with 8 supersymmetries*

$D=6$	$D=5$	$D=4$	$D=3$
tensor multiplet $\mathbb{R}$	vector/tensor $\mathbb{R}$	vector multiplet $\mathbb{C}$	hypermultiplet $\mathbb{Q}$
vector multiplet *no scalar*			
hypermultiplet $\mathbb{Q}$	hypermultiplet $\mathbb{Q}$	hypermultiplet $\mathbb{Q}$	

tensor multiplets, and M and N the vector multiplets, but these couplings do not influence the geometry of the scalar manifolds, as vector multiplets in 6 dimensions do not contain scalars. A full account of these couplings is given in [27]. The hypermultiplets determine a separate sector of the manifold.

Dimensional reduction deforms a multiplet in table 1 to the one to its right. Vector and tensor multiplets appear in one box in 5 dimensions, as Abelian vectors are dual to antisymmetric tensors in $D = 5$. As long as we only discuss the geometry, we do not have to make a distinction. When one considers gauged symmetry groups then the vector and tensor multiplets play a different role [28], but for the geometry they can be considered as equivalent. They reduce to vector multiplets in 4 dimensions, as illustrated in table 2. These have complex scalars,

Table 2. *Reduction of matter multiplets between 5,4 and 3 dimensions.*

	$D=5$		$D=4$		$D=3$		after duality
spin 1/2	2	$\rightarrow$	2	$\rightarrow$	4	$\rightarrow$	4
spin 1	1	$\rightarrow$ $\searrow$	1	$\rightarrow$ $\searrow$	1	$\searrow$	
spin 0	1	$\rightarrow$	2	$\rightarrow$	3	$\rightarrow$	4
	$\mathbb{R}$		$\mathbb{C}$				$\mathbb{Q}$

which can be seen as the 5th and 6th components of the vectors of 6 dimensions. In 3 dimensions, Abelian vectors are dual to scalars. This leads to the last column of table 2. Thus, in 3 dimensions all the multiplets are equivalent to hypermultiplets, i.e. the multiplets with only spin-0 and spin-1/2 fields.

The hypermultiplets in all the dimensions look alike. Indeed, the scalar sector does not depend on the spacetime dimension. There are in any case 4 scalars for each multiplet. As we will show below, they appear in a quaternionic structure. The spin-1/2 fields are differently organised in 3,4 and 5 dimensions, but that is not visible in the geometry.

Therefore, in the following we will concentrate on the upper line of table 1, starting from the vector/tensor multiplets in 5 dimensions with real scalars, the vector multiplets in 4 dimensions with complex scalars, and the hypermultiplets in 3 dimensions (and which describe thus the geometry as well for 4,5 and 6 dimensions) with quaternionic scalars. We will leave the rather obvious geometry of the tensor multiplets in 6 dimensions aside.

When one considers the supergravity theory, there is an extra contribution from the reduction of the supergravity multiplet. The supergravity multiplets in 5 dimensions (as in 4) contains the graviton, gravitini and a graviphoton (spin 1 field of the gravity multiplet). When reducing to 4 dimensions, the graviton leads to a graviton plus a vector and a scalar in 4 dimensions. The gravitino gives an extra spin 1/2, and the graviphoton gives an extra scalar. Thus, we end up with an extra vector multiplet. This shows that there is a mapping from a $D = 5$ supergravity theory with n vector multiplets to a $D = 4$ supergravity theory with $n + 1$ vector multiplets. The same procedure leads further (after the duality transformations of the vectors) to a $D = 3$ supergravity theory with $n + 2$ hypermultiplets. For the geometry, this means a mapping from a real n-dimensional manifold, to a complex $n + 1$ dimensional manifold, which is called the **r**-map [24], and from there to a quaternionic $n+2$-dimensional manifold, the so-called **c**-map [7]. Note that e.g. in the 5-dimensional theory one has $n+1$ vectors when there are n vector multiplets. These $n+1$ vectors can better be treated together, rather than separating them in a graviphoton and n vectors of vector multiplets. In fact, for duality transformations they should be considered together. The same applies in 4 dimensions. This remark will be the starting point in section 4.

3. The family of special geometries

We can distinguish between theories that appear in rigid supersymmetry, and those in supergravity. This leads to the overview in the upper part of table 3. The geometries that are related to rigid supersymmetry have been called 'affine' in the mathematics literature [29, 30], while those for supergravity are called 'projective' (and these are the default, in the sense that e.g. special Kähler refers to the geometry that is found in supergravity). The analogous manifolds with 3 complex structures got already a name in the literature: the ones that occur in rigid supersymmetry are the hyperkähler manifolds, while those in supergravity are the

	$D=5$ vector multiplets	$D=4$ vector multiplets	hypermultiplets
rigid (affine)	affine very special real	affine special Kähler	hyperkähler
local (projective)	(projective) very special real	(projective) special Kähler	quaternionic-Kähler

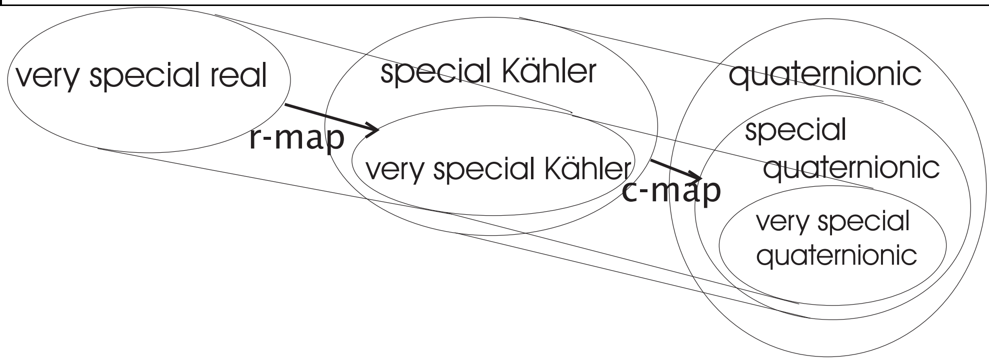

TABLE 3. *Geometries from supersymmetric theories with 8 real supercharges, and the connections provided by the* **r***-map and the* **c***-map.*

quaternionic-Kähler manifolds[2]. The manifolds that are defined by the scalar sector of $N=2$, $D=4$ vector multiplets are called special Kähler. When one considers $N=1$ supergravity in 4 dimensions, all Kähler manifolds can occur. The presence of 8 real supercharges restricts the possibilities, and the restricted class is denoted by 'special Kähler' [11]. Definitions of such manifolds independent of supergravity were given in [31], and a review appeared in [32]. Another definition has been given in [29, 30] and has been discussed in this workshop in the talk of V. Cortés. The real manifolds that appear in five dimensions got the name of very special real manifolds. In [33] a definition in more mathematical terms is given.

The **r**-map and **c**-map, discussed in the previous section, induce a terminology for subclasses of special Kähler and quaternionic-Kähler manifolds. The image of the very special real manifolds under the **r**-map define the very special Kähler manifolds. The fact that some of the scalar fields have their origin in components of vector fields of 5 dimensions leads to isometries for all these manifolds. Thus, not all special Kähler manifolds are in this image (as we will illustrate below for the symmetric spaces), and very special Kähler manifolds are a non-trivial

[2]Mathematicians include hyperkähler as a special case of what they call 'quaternionic-Kähler', while physicists reserve the name quaternionic to the manifolds that have non-vanishing SU(2) curvature, which excludes the hyperkähler ones. Furthermore, we will restrict ourselves to the quaternionic-Kähler manifolds of negative scalar curvature, as those are the only ones that appear in supergravity. For manifolds with continuous isometries, this implies that they are non-compact.

subclass of special Kähler manifolds. The **c**-map defines in the same way the 'special quaternionic-Kähler' manifolds as a non-trivial subclass of quaternionic-Kähler manifolds, and in an obvious way, also very special quaternionic-Kähler manifolds are defined, see the lower part of table 3.

Homogeneous and symmetric spaces are the most known manifolds. These are spaces of the form G/H, where G are the isometries and H is its isotropy subgroup. The group G is not necessarily a semi-simple group, and thus not all the homogeneous spaces have a clear name. The symmetric spaces are those for which the algebra splits as $g = h + k$ and all commutators $[k, k] \subset h$. The homogeneous special manifolds are classified in [24].

It turns out that homogeneous special manifolds are in one-to-one correspondence to realizations of real Clifford algebras with signature $(q+1,1)$ for real, $(q+2,2)$ for Kähler, and $(q+3,3)$ for quaternionic manifolds. Thus, the spaces are identified by giving the number q, which specifies the Clifford algebra, and by specifying its representation. If q is not a multiple of 4, then there is only one irreducible representations, and we thus just have to mention the multiplicity P of this representation. The spaces are denoted as $L(q,P)$. If $q = 4m$ then there are two inequivalent representations, chiral and antichiral, and the spaces are denoted as $L(q,P,\dot{P})$. If we use n as the complex dimension of the special Kähler space, the dimension of these manifolds is ($\dot{P} = 0$ if $q \neq 4m$)

$$n = 3+q+(P+\dot{P})D_{q+1}\,, \qquad \left\{ \begin{array}{l} \dim_{\mathbb{R}}[\text{very special real } L(q,P,\dot{P})] = n-1 \\ \dim_{\mathbb{R}}[\text{special Kähler } L(q,P,\dot{P})] = 2n \\ \dim_{\mathbb{R}}[\text{quaternionic-Kähler } L(q,P,\dot{P})] = 4(n+1). \end{array} \right. \tag{3.1}$$

where D_{q+1} is the dimension of the irreducible representation of the Clifford algebra in $q+1$ dimensions with positive signature, i.e.

$$\begin{array}{lll} D_{q+1} = 1 \ \text{ for } q = -1, 0\,, & D_{q+1} = 2 \ \text{ for } q = 1\,, & D_{q+1} = 4 \ \text{ for } q = 2\,, \\ D_{q+1} = 8 \text{ for } q = 3, 4\,, & D_{q+1} = 16 \text{ for } q = 5, 6, 7, 8\,, & D_{q+8} = 16\, D_q \end{array} \tag{3.2}$$

The very special manifolds are defined by coefficients C_{IJK} as we will see below. For the homogeneous ones, we can write them as

$$C_{IJK}\, h^I h^J h^K = 3 \left\{ h^1 \left(h^2\right)^2 - h^1 \left(h^\mu\right)^2 - h^2 \left(h^i\right)^2 + \gamma_{\mu ij}\, h^\mu\, h^i\, h^j \right\}. \tag{3.3}$$

We decomposed the indices $I = 1, \ldots, n$ into $I = 1, 2, \mu, i$, with $\mu = 1, \ldots, q+1$ and $i = 1, \ldots, (P+\dot{P})D_{q+1}$. Here, $\gamma_{\mu ij}$ is the Clifford algebra representation that we mentioned. Note that these models have predecessors in 6 dimensions, with $q+1$ tensor multiplets and $(P+\dot{P})D_{q+1}$ vector multiplets. The gamma matrices are then the corresponding coupling constants between the vector and tensor multiplets.

TABLE 4. *Homogeneous manifolds. In this table, q, P, $\dot{P}$ and m denote positive integers or zero, and $q \neq 4m$. SG denotes an empty space, which corresponds to supergravity models without scalars. Furthermore, $L(4m, P, \dot{P}) = L(4m, \dot{P}, P)$. The horizontal lines separate spaces of different rank. The first non-empty space in each column has rank 1. Going to the right or down a line increases the rank by 1. The manifolds indicated by a $\star$ did not get a name.*

	real	Kähler	quaternionic
$L(-3,P)$			$\frac{\mathrm{USp}(2P+2,2)}{\mathrm{USp}(2P+2)\otimes\mathrm{SU}(2)}$
SG_4		SG	$\frac{\mathrm{U}(1,2)}{\mathrm{U}(1)\otimes\mathrm{U}(2)}$
$L(-2,P)$		$\frac{\mathrm{U}(P+1,1)}{\mathrm{U}(P+1)\otimes\mathrm{U}(1)}$	$\frac{\mathrm{SU}(P+2,2)}{\mathrm{SU}(P+2)\otimes\mathrm{SU}(2)\otimes\mathrm{U}(1)}$
SG_5	SG	$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$	$\frac{G_2}{\mathrm{SU}(2)\otimes\mathrm{SU}(2)}$
$L(-1,P)$	$\frac{\mathrm{SO}(P+1,1)}{\mathrm{SO}(P+1)}$	$\star$	$\star$
$L(4m,P,\dot{P})$	$\star$	$\star$	$\star$
$L(q,P)$	$X(P,q)$	$H(P,q)$	$V(P,q)$

Considering further the table 4, we find in the quaternionic spaces the homogeneous ones that were found in [34], together with those that were discovered in [24] (the ones with a $\star$ except for the series $L(0,P,\dot{P})$, which were already in [34], and denoted there as $W(P,\dot{P})$).

Observe that the classification of homogeneous spaces exhibits that the quaternionic projective spaces have no predecessor in special geometry, and that the complex projective spaces have no predecessor in very special real manifolds. Similarly, only those with $q \geq -1$ can be obtained from 6 dimensions [with the scalars of tensor multiplets describing $\mathrm{SO}(1,q+1)/\mathrm{SO}(q)$] and $L(-1,0)$ corresponds to pure supergravity in 6 dimensions.

There are still symmetric spaces in the range $q \geq -1$. These are shown in table 5. For the symmetric special Kähler spaces, this reproduces the classification obtained in [35], while the quaternionic symmetric spaces contain the 'Wolf'-spaces. The full set of isometries for all the homogeneous manifolds has been obtained in [36], and all these results have been summarised in [37].

TABLE 5. *Symmetric very special manifolds. Note that the very special real manifolds $L(-1,P)$ are symmetric, but not their images under the* $\mathbf{r}$ *map.*

	real	Kähler	quaternionic
$L(-1,0)$	$\mathrm{SO}(1,1)$	$\left[\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right]^2$	$\frac{\mathrm{SO}(3,4)}{(\mathrm{SU}(2))^3}$
$L(-1,P)$	$\frac{\mathrm{SO}(P+1,1)}{\mathrm{SO}(P+1)}$		
$L(0,P)$	$\mathrm{SO}(1,1)\otimes\frac{\mathrm{SO}(P+1,1)}{\mathrm{SO}(P+1)}$	$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\otimes\frac{\mathrm{SO}(P+2,2)}{\mathrm{SO}(P+2)\otimes\mathrm{SO}(2)}$	$\frac{\mathrm{SO}(P+4,4)}{\mathrm{SO}(P+4)\otimes\mathrm{SO}(4)}$
$L(1,1)$	$\frac{\mathrm{S}\ell(3,\mathbb{R})}{\mathrm{SO}(3)}$	$\frac{\mathrm{Sp}(6)}{\mathrm{U}(3)}$	$\frac{F_4}{\mathrm{USp}(6)\otimes\mathrm{SU}(2)}$
$L(2,1)$	$\frac{\mathrm{S}\ell(3,C)}{\mathrm{SU}(3)}$	$\frac{\mathrm{SU}(3,3)}{\mathrm{SU}(3)\otimes\mathrm{SU}(3)\otimes\mathrm{U}(1)}$	$\frac{E_6}{\mathrm{SU}(6)\otimes\mathrm{SU}(2)}$
$L(4,1)$	$\frac{\mathrm{SU}^*(6)}{\mathrm{Sp}(3)}$	$\frac{\mathrm{SO}^*(12)}{\mathrm{SU}(6)\otimes\mathrm{U}(1)}$	$\frac{E_7}{\mathrm{SO}(12)\otimes\mathrm{SU}(2)}$
$L(8,1)$	$\frac{E_6}{F_4}$	$\frac{E_7}{E_6\otimes\mathrm{U}(1)}$	$\frac{E_8}{E_7\otimes\mathrm{SU}(2)}$

4. Superconformal methods for extra symmetry

We already mentioned that it is advantageous to treat the graviphoton on equal footing with the other vectors. Indeed, duality transformations can mix the field strengths of all these vectors. You can thus expect that a formalism where these are treated democratically will show more of the symmetry structure. Therefore, we want to use a formalism with $n+1$ vector multiplets (for n denoting the number of physical vector multiplets). But that would lead to too many spin-1/2 and spin-0 fields. Let us concentrate first on $D=5$. A formalism with $n+1$ vector multiplets would thus have also $n+1$ scalars, of which only n can be physical. One scalar should therefore be a gauge degree of freedom. The same applies to one of the fermions. The extra symmetries that are used in this context can naturally be combined in the supergravity group to enlarge it to a superconformal group.

What we have in mind can be illustrated first for pure gravity. We show how Poincaré supergravity is obtained after gauge fixing a conformal invariant action. The conformal invariant action for a scalar ϕ (in 4 dimensions) is

$$\begin{aligned} L &= \sqrt{g}\left[\tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)-\tfrac{1}{12}R\phi^2\right], \\ \delta\phi &= \Lambda_D\phi\,, \qquad \delta g_{\mu\nu}=-2\Lambda_D g_{\mu\nu}\,, \end{aligned} \tag{4.1}$$

where the second line gives the local dilatation symmetry that leaves this action invariant. Now, we can gauge fix this dilatation symmetry by choosing[3] the gauge

[3]A gauge fixing can be interpreted as choosing better coordinates such that only one field still transforms under the corresponding transformations. Then, the invariance is expressed as the absence of this field from the action. In this case we would use $g'_{\mu\nu}=g_{\mu\nu}\phi^2$ as D-invariant metric. One can check that this redefinition also leads to (4.2) in terms of the new field.

$\phi = \sqrt{6}$. This leads to the pure Poincaré action

$$L = -\tfrac{1}{2}\sqrt{g}R\,. \tag{4.2}$$

Pure Poincaré is in this way obtained from a conformal action of a scalar after gauge fixing. This scalar, which we will denote further as 'compensating scalar', thus has no physical modes. Observe that the action (4.1) describes a scalar with negative kinetic energy in order that (4.2) describes a graviton with positive kinetic energy. The negative kinetic energy of the compensating scalar will be important below.

In the full models, this scalar is unified with all the other scalars in the theory, and this is useful to clarify some isometries of these theories.

After this motivation for using the superconformal methods, let us repeat the structure of the superconformal groups that we use. They have 3 parts:

conformal group: consisting of translations and rotations (the Poincaré group), dilatations, and special conformal transformations, i.e. the group $\mathrm{SO}(D,2)$.

supersymmetries: ordinary (that appear also in the super-Poincaré group) and special supersymmetries that are the counterparts of the special conformal transformations. For the theories that we consider here, there are 8 real components in the ordinary supersymmetries and 8 in the special ones.

R-symmetry: the bosonic group is in general the product of the conformal group and an extra bosonic group that rotates the supersymmetries. This group appears in the anticommutator of ordinary with special supersymmetries. In 5 dimensions this R-symmetry group is $\mathrm{SU}(2)$, in 4 dimensions it is $\mathrm{SU}(2)\times\mathrm{U}(1)$ (always for theories with 8 ordinary supersymmetries).

The whole supergroup is $\mathrm{F}^2(4)$ (the '2' indicating the real form) for $D = 5$, and $\mathrm{SU}(2,2|2)$ for $D = 4$. See other reviews [38, 39] for more details and references on the supergroups.

5. Construction in 3 steps

We will now review the essential steps in the construction of these theories. There are 3 steps involved. First, we define multiplets as representations of the superconformal algebra. Second, we define an invariant action. This should thus be a superconformal invariant, the analogue of (4.1). Finally we gauge-fix the superfluous symmetries, i.e. all those that belong to the superconformal group and not to the super-Poincaré group. These methods are already known since [40, 41, 42].

Often these steps are mixed. In many (most) papers one treats at the same time the action and the transformation laws. We will, however, keep them strictly separated. This will give insight in what parts of the structure follows from the algebra, and what follows from the action. We will even see that in some cases the construction of an action is not possible, while still we have dynamical equations. In view of this concept, we will put special emphasis on the hypermultiplets and quaternionic geometry.

5.1. **Multiplets in the superconformal algebra.** We have to warn immediately that the terminology 'superconformal algebra' has to be interpreted with care in field theory. It is not really the group concept for 2 reasons.

First, the algebra is 'soft'. Consider the commutator of two supersymmetry transformations with parameters ϵ_1 and ϵ_2 in a gauge theory. The result should be a translation, which on the scalars is a derivative of the field, but in a gauge theory, derivatives all appear as covariant derivative, i.e. we will get

$$[\delta(\epsilon_1),\delta(\epsilon_2)]\,\phi=\bar\epsilon_1\gamma^\mu\epsilon_2\,D_\mu\phi\,,\qquad\text{with}\qquad D_\mu\phi=\partial_\mu\phi-A_\mu^\alpha T_\alpha\phi\,,\tag{5.1}$$

where the covariant derivative involves all gauge transformations of the scalar, $T_\alpha\phi$ multiplied by the corresponding gauge fields A_μ^α. The last term thus means that the commutator of two supersymmetries contains a gauge transformation with parameter $\bar\epsilon_1\gamma^\mu\epsilon_2\,A_\mu^\alpha$. The latter expression should be the structure 'constant'. But it clearly is no constant, rather it is a 'structure function'. Jacobi identities (or their generalizations in field theory) imply that also other commutators involve fields. Especially the fields that belong to the multiplet involving the gauge fields can be expected in structure functions.

A second special feature of the algebra is that it can be 'open'. This terminology is used to indicate that the closure of the algebra involves equations of motion. In fact, a commutator of two symmetries of an action should always be a linear combination of symmetries. But there are trivial symmetries. Any action $S(\phi)$ is invariant under the 'trivial' transformation (here it is presented for bosonic fields, but the extension to fermionic fields is obvious)

$$\delta_{\rm triv}\phi^i=\eta^{ij}\frac{\delta S}{\delta\phi^j}\,,\tag{5.2}$$

where η^{ij} is any antisymmetric tensor (constant or field-dependent). Indeed, the transformation of the action is

$$\delta_{\rm triv}S=\frac{\delta S}{\delta\phi^i}\,\eta^{ij}\frac{\delta S}{\delta\phi^j}=0\,.\tag{5.3}$$

So, the trivial symmetries are part of the full set of symmetries, and in general a commutator of symmetries only closes modulo trivial transformations:

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\,\phi^i = \text{superconformal} + \eta^{ij}(\epsilon_1, \epsilon_2)\,\frac{\delta S}{\delta \phi^j}\,. \tag{5.4}$$

However, these trivial transformations are clearly model-dependent: they are determined only when the full action is known. In many cases, we do not yet want to decide on the full action when we give the transformation laws. To avoid this situation, one can often enlarge the set of fields by so-called 'auxiliary fields'. With a suitable choice of auxiliary fields the last term of (5.4) can be absent, and the transformations are independent of the action. The algebra closes using only the symmetries that are the basis of the theory, not using the 'trivial' ones.

Such a set of auxiliary fields can not always be found, and is also not always necessary. When there is no suitable set of auxiliary fields, then the closure of the algebra already leads to field equations. So, the transformation laws fix the equations of motion. The transformation laws can thus only be used with a fixed physical content. When we consider multiplets that are used in different actions, we do not want to change their multiplet structure each time. Thus in this case, we need auxiliary fields, in order to use the multiplet transformations in these different situations. Fortunately, in the superconformal multiplets that are used for different actions, we always find a suitable set of auxiliary fields.

After these preliminary remarks on the meaning of 'multiplets of the superconformal algebra' we can discuss the main multiplets. The first one that we should discuss is the *'Weyl multiplet'* [3]. This is the name that is given to the multiplet which contains the gauge fields of all the symmetries in the superconformal algebra. There are 3 types of fields in this multiplet:

independent gauge fields: these are the vielbein e_μ^a as gauge field of the translations, the gravitini ψ_μ^i as gauge fields of the ordinary supersymmetries, the field b_μ, gauge field of dilatations, and the gauge fields of the R-symmetry $V_\mu^{(ij)}$.

dependent gauge fields: As is well-known already in Poincaré supergravity, the spin-connection, which is the gauge field of Lorentz transformations, is not an independent field, but is constrained to be some function of the vielbein. In supergravity it is also a function of the gravitino, and in the superconformal setup it is a function $\omega_\mu{}^{ab}(e_\mu^a, \psi_\mu^i, b_\mu)$. Also the gauge fields of special supersymmetry and of special conformal transformations are not independent fields, but functions of the other fields of the multiplet.

auxiliary fields: to close the algebra. There are different possibilities for the fields that can be used here [43, 44].

This multiplet satisfies a soft algebra with all the symmetries of the superconformal algebra. The structure functions between generators of the algebra thus depend on the fields of this multiplet. For 5 dimensions, this has been constructed recently in [44, 45]. The first reference explains our methods in more detail and has references to the constructions in other dimensions.

Then we turn to the *vector multiplets*. As the name says, these involve vectors. These vectors are gauge fields of symmetries that do not belong to the superconformal group. They commute with the superconformal symmetries. The vector multiplets involve also other fields as we saw before: the 'gauginos' and the scalars (and auxiliary fields). The multiplet is defined 'in the background of the Weyl multiplet'. This means that the transformation laws satisfy the soft algebra that the Weyl multiplet has defined, but also the gauge transformations of the vectors appear in the algebra. E.g. (without going into notational details), the ordinary supersymmetries commute to

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \text{superconformal algebra with modified } D_\mu + \delta_G(\bar{\epsilon}_2\epsilon_1 X)\,, \qquad (5.5)$$

where the first term represents the algebra as it is satisfied by the Weyl multiplet, except that covariant derivatives are replaced by an expression as in (5.1) including the new gauge transformations. These gauge transformations also appear in the last term, where the parameter contains X, the scalar of the multiplet. Observe that, in a solution of the theory with non-vanishing fields of the vector multiplet, these extra terms give rise to central charges.

Auxiliary fields can be introduced to close the algebra. Note the hierarchy in the multiplets. We first had to define the Weyl multiplet before we could introduce the vector multiplets. This is so because the Weyl multiplet does not involve the gauge transformations gauged by the vectors of the vector multiplets. On the other hand, the fields in the vector multiplets transform of course under the superconformal symmetries, and we thus need the Weyl multiplet to be able to introduce the vector multiplets.

Then we can introduce *hypermultiplets*. These are defined in the background of the Weyl multiplet and possibly also in the background of the vector multiplet [this is illustrated at the end of this section, see (5.17) and (5.18]. The latter is the case if one considers hypermultiplets that transform non-trivially under the gauge transformations of the vector multiplets. Auxiliary fields to close the algebra (in the sense explained before that 'open' means closed including the trivial symmetries) exist for the simplest quaternionic manifolds, or can be introduced if one uses the methods of harmonic superspace. However, we can avoid this. We do not need auxiliary fields any more at this point. This is because the hypermultiplets are at the end of the hierarchy line. We are not going to introduce any further multiplet in the background of the hypermultiplets, as these do not introduce new gauge symmetries. When we considered the vector multiplets, the construction had to take into account that the multiplets can be used for various possible actions (including hypermultiplets or not). When we come to the hypermultiplets, however, we know what we are looking for. The hypermultiplet transformations define one particular physical system.

The algebra thus closes only modulo equations of motion. This means that we have already the physical theory although we have no action yet. The multiplet and the transformation rules are defined, and as we will illustrate below, the complex structures are defined on this manifold. But there is no metric. This corresponds to the notion of a hypercomplex manifold in the Mathematics literature [46]. Homogeneous examples of hypercomplex manifolds that are not hyperkähler were constructed by [47, 48, 49] and some constructions of non-homogeneous hypercomplex manifolds were proposed by D. Joyce [48], A. Swann and H. Pedersen [50]. Various aspects have been treated in two workshops with mathematicians and physicists [51, 50].

As this aspect is new, let us make it more explicit. We start with $4r$ scalar fields and $2r$ spinors. The scalars are denoted by q^X with $X = 1, \ldots, 4r$ and the spinors by ζ^A with $A = 1, \ldots, 2r$. We use the formulation in 5 dimensions, and first for rigid supersymmetry. For details on spinor properties and our conventions, we refer to [44]. We consider general transformations for the scalars under the two supersymmetries with parameters ϵ^i, $i = 1, 2$:

$$\delta_Q(\epsilon) q^X = -\mathrm{i}\bar{\epsilon}^i \zeta^A f^X_{iA}(q) \,, \tag{5.6}$$

where the 'vielbeins' $f^X_{iA}(q)$ satisfy a reality condition

$$\rho_A{}^B E_i{}^j (f^X_{jB})^* = f^X_{iA} \,, \tag{5.7}$$

defined by matrices $E_i{}^j$ and $\rho_A{}^B$ that satisfy

$$E\, E^* = -\mathbb{1}_2 \,, \qquad \rho\, \rho^* = -\mathbb{1}_{2r} \,. \tag{5.8}$$

One may choose a standard antisymmetric form for ρ and identify E with ε by a choice of basis. The transformations on variables with an A index are by the reality condition restricted to $\mathrm{G}\ell(r, \mathbb{Q}) = \mathrm{SU}^*(2r) \times \mathrm{U}(1)$.

The supersymmetry algebra can be closed on the scalars by choosing the supersymmetry transformations of the fermions as

$$\delta_Q(\epsilon)\zeta^A = \tfrac{1}{2}\mathrm{i}\partial\!\!\!/ q^X f^{iA}_X(q)\epsilon_i - \zeta^B \omega_{XB}{}^A(q) \left[\delta(\epsilon) q^X\right] , \tag{5.9}$$

where the f that appear here have indices in opposite position as in (5.6), indicating that they are the inverse matrices as $4r \times 4r$ matrices

$$f^{iA}_Y f^X_{iA} = \delta_Y{}^X , \qquad f^{iA}_X f^X_{jB} = \delta_j{}^i \delta_B{}^A . \tag{5.10}$$

The functions $\omega_{XB}{}^A(q)$ in (5.9) appear in an integrability condition:

$$\mathfrak{D}_Y f^X_{iB} \equiv \partial_Y f^X_{iB} - \omega_{YB}{}^A(q) f^X_{iA} + \Gamma^X_{ZY}(q) f^Z_{iB} = 0 , \tag{5.11}$$

where $\Gamma^X_{ZY}(q) = \Gamma^X_{YZ}(q)$ is any symmetric function.

This can then be interpreted geometrically. $\omega_{XB}{}^A(q)$ are seen as gauge fields for the $\mathrm{G}\ell(r,\mathbb{Q})$. Obviously, we can interpret $\Gamma^X_{YZ}(q)$ as an affine connection. Also complex structures can be defined as ($\alpha = 1,2,3$ and using the three sigma matrices)

$$J_X{}^{Y\alpha} \equiv -\mathrm{i} f^{iA}_X (\sigma^\alpha)_i{}^j f^Y_{jA} . \tag{5.12}$$

The complex structures satisfy, due to (5.10), the quaternion algebra

$$J^\alpha J^\beta = -\mathbb{1}_{4r} \delta^{\alpha\beta} + \varepsilon^{\alpha\beta\gamma} J^\gamma . \tag{5.13}$$

This defines the space of the scalars to be a hypercomplex manifold.

One can then check that with the given transformations and identities (5.10) and (5.11) the supersymmetry algebra on the fermions closes modulo terms proportional to

$$\Gamma^A \equiv \not{\mathfrak{D}} \zeta^A + \tfrac{1}{2} \varepsilon^{ij} f^X_{jC} f^Y_{iD} R_{XYB}{}^A \zeta^B \bar{\zeta}^D \zeta^C . \tag{5.14}$$

There are more terms if the fields transform under the gauge group of a vector multiplet, see [52]. Putting this equal to zero (demanding an on-shell algebra) gives an equation of motion for the fermions. The supersymmetry transformation of this equation gives then also an equation of motion for the bosons. As announced before, we thus have already physical equations despite the absence of an action.

All this can be generalized to a superconformal setup. The essential new ingredient is then a 'homothetic Killing vector' k^X [53], which in 5 dimensions satisfies

$$\mathfrak{D}_Y k^X \equiv \partial_Y k^X + \Gamma^X_{YZ} k^Z = \tfrac{3}{2} \delta_Y{}^X . \tag{5.15}$$

The presence of this vector allows one to extend the transformations of rigid supersymmetry to the superconformal group [53, 54, 52], with e.g. transformations under the dilatations and SU(2) R-symmetry group:

$$\delta_{D,\mathrm{SU}(2)}(\Lambda_D, \Lambda^\alpha) q^X = \Lambda_D k^X + \tfrac{2}{3} \Lambda^\alpha k^Y J^\alpha{}_Y{}^X . \tag{5.16}$$

To illustrate the concept of the background of vector multiplets and Weyl multiplet for the hypermultiplet, we can give the full forms of (5.9). The given formula is applicable for rigid supersymmetry. If the hypermultiplet transforms moreover under a gauge transformation, $\delta q^X = \Lambda^I k^I_X(q)$ where Λ^I are the parameters, and this gauge transformation is associated to the vector multiplet whose scalars are h^I, then the transformation is

$$\delta_Q(\epsilon)\zeta^A = \tfrac{1}{2}\mathrm{i}\not{D}q^X f_X^{iA}(q)\epsilon_i - \zeta^B\omega_{XB}{}^A(q)\left[\delta(\epsilon)q^X\right] + \tfrac{1}{2}h^I k_I^X f_X^{iA}\epsilon_i\,. \tag{5.17}$$

The covariant derivative includes now a term $-A_\mu^I k_I^X$. If we consider the multiplet in the local superconformal background, then the transformation is

$$\delta_Q(\epsilon)\zeta^A = \tfrac{1}{2}\mathrm{i}\not{D}q^X f_X^{iA}(q)\epsilon_i - \zeta^B\omega_{XB}{}^A(q)\left[\delta(\epsilon)q^X\right] + \tfrac{1}{3}\gamma^{ab}T_{ab}k^X f_X^{iA}\epsilon_i + \tfrac{1}{2}h^I k_I^X f_X^{iA}\epsilon_i\,, \tag{5.18}$$

where T_{ab} is one of the auxiliary fields of the Weyl multiplet. Also here, the latter term can be present or not depending on whether the gauge symmetry is gauged or not.

We thus stress that all this is obtained independent of an action. A similar result can be obtained for tensor multiplets in 5 dimensions. As mentioned, in the Abelian case they are dual to vector multiplets. However, in the non-Abelian theory, the inclusion of these multiplets leads to more general possibilities [55]. These multiplets are also at the end of the hierarchy as that they are not used for the definition of further multiplets. As for hypermultiplets, their algebra is not closed. We thus again obtain equations of motion without the presence of an action [52]. This mechanism of having physical theories without an action is a concept that is known in other cases too. E.g. with self-dual tensors, an action is difficult to construct, and as such, type IIB supergravity [56] is a first example.[4] Theories without action are also often used in the group manifold approach, see e.g. [59].

5.2. **Define an invariant action.** To define an invariant action, we need in each case some extra ingredient apart from the transformation laws. E.g. in 4 dimensions one can start from a holomorphic function $F(X)$ of the complex scalars of these multiplets. This is the object that in superspace would be integrated over the chiral superspace. Thus given the $F(X)$, the action is determined. For conformal symmetry F should be homogeneous of degree 2 [2].

For vector multiplets in 5 dimensions, we still need a constant tensor C_{IJK} where I, J, K label the vector multiplets. If we want to use their duals, the tensor multiplets, then we need also an antisymmetric metric Ω_{MN}, where M, N label the tensor multiplets. As this has to be non-degenerate, it implies that the tensor multiplets appear with an even number (and they are in symplectic representations of the gauge group). Observe that this requirement only comes from the requirement

[4]There are methods to have an action for these theories, either by breaking explicit Lorentz invariance, or by including extra auxiliary fields and gauge transformation [57, 58], but these theories can be, and usually are, considered without a proper action.

of an action, and can thus be avoided if one does not insist on the existence of an action.

For the hypermultiplets one also needs an antisymmetric metric in the fermion sector C_{AB}. The supersymmetry algebra on the hypermultiplets closes only modulo functions Γ^A (5.14), and we now want to interpret them as field equations. The relation (5.2) says that these functions should be proportional to the Euler-Lagrange equations with antisymmetric coefficients[5]

$$\frac{\delta S}{\delta \bar{\zeta}^A} = 2C_{AB}\Gamma^B\,. \tag{5.19}$$

The consistency equations of field theory (which one can get e.g. from the field-antifield formalism) imply that C_{AB} are covariantly constant, and can be chosen to be constant. The requirement that this tensor is invariant reduces the symmetry group from $\mathrm{SU}(2)\times \mathrm{G}\ell(r,\mathbb{Q})$ to $\mathrm{SU}(2)\times \mathrm{USp}(2p,2q)$. Note that the SU(2) is not gauged in rigid supersymmetry. In the (local) superconformal approach, there is the SU(2) gauge field in the Weyl multiplet, but that is so far independent of the scalars that constitute the manifold. We thus have here a hyperkähler manifold.

The metric is given by

$$g_{XY} = f_X^{iA} f_Y^{jB}\varepsilon_{ij}C_{AB}\,. \tag{5.20}$$

Its signature is given by the product of C_{AB} with the matrix $\rho_A{}^B$ that determined the reality properties, see e.g. (5.7).

5.3. **Gauge fixing.** Finally we have to gauge fix the symmetries of the superconformal algebra that do not belong to the super-Poincaré algebra. We are most interested in the bosonic part, and thus have to consider the gauge fixing of dilatations and of the R-symmetry group.[6] But one extra ingredient is the field equation of an auxiliary field of the Weyl multiplet, which is called D.

In 5 dimensions for a general theory with vector multiplets and hypermultiplets, we can fix the dilatations by fixing the value of one of the scalars of the vector multiplets, as it was already foreseen in section 4. The scalars of the hypermultiplets transform under the SU(2) R-symmetry group, as we saw in (5.16). Thus the SU(2) gauge fixes 3 scalars of a hypermultiplet. The field equation of the auxiliary field D eliminates the fourth. In 4 dimensions, the R-symmetry group has also a U(1) factor, and the scalars of the vector multiplet are complex, such that also there one

[5]It was antisymmetric for bosons and should be symmetric for fermions, but a charge conjugation matrix that is implicit in taking the derivative w.r.t. $\bar{\zeta}$, which is antisymmetric, implies that the C_{AB} should be antisymmetric.

[6]Gauge fixing of the special conformal transformations is done by fixing the value of the gauge field of dilatations b_μ.

vector multiplet loses his scalar by the gauge fixing. Thus the scalars of one vector multiplet and one hypermultiplet get fixed. The corresponding spin-1/2 fields are fixed by a similar procedure: the gauge fixing of the S-supersymmetry and the field equation of a fermion auxiliary field of the Weyl multiplet.

The conclusion is thus that one full hypermultiplet and one vector multiplet (apart from its vector being the graviphoton) is 'compensating'. We thus have to start from the Weyl multiplet, $n+1$ vector multiplets and $r+1$ hypermultiplets to end up with the super-Poincaré gravity coupled to n vector multiplets and r hypermultiplets. Similar as we mentioned in the remark after (4.2), we have to start with negative kinetic terms for the compensating vector multiplets to end up with a super-Poincaré theory with positive kinetic terms.

In practice, the field equation of the auxiliary field D looks similar as a gauge condition of dilatations. In fact, one has a linear combination of this field equation and the dilatational gauge condition for the vector multiplet and another linear combination for the hypermultiplet. We may consider these as two independent scaling conditions.

Let us now see what this means for the geometry. Consider first the $D=5$ vector multiplets. Before the gauge fixing one has a $n+1$ vector multiplets with scalars h^I and a geometry that is determined by the C_{IJK} coefficients, i.e. $\mathrm{d}s^2 = C_{IJK}h^I\mathrm{d}h^J\mathrm{d}h^k$. This is an affine geometry. Then the gauge fixing is imposed to fix the scale transformations of the h^I. It is most appropriate to choose as gauge condition [60] a fixed value for the length $\rho \equiv C_{IJK}h^Ih^Jh^K$. Notice that this is a dimensionful quantity, and this gauge fixing thus introduces a length scale that is the gravitational constant.[7] The 'projective very special geometry' thus lives on the slice $\rho = M_P^3$, as illustrated in figure 1.

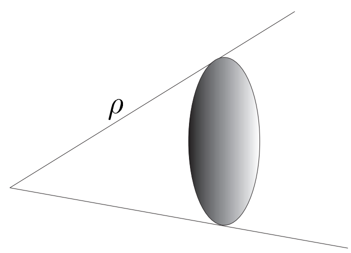

FIGURE 1. *Projective very special geometry in the cone of the* $(n{+}1)$-*dimensional space.*

A similar gauge fixing was performed for vector multiplets in 4 dimensions [2]. In this case, the slice as in figure 1 leads to a manifold on which there is a U(1)

[7]More details on this way of presenting the procedure are given in [61].

structure. This is a Sasakian manifold. Once also the field equation of the U(1) gauge field of the Weyl multiplet is used, the geometry is modified, and the Kähler curvature is non-zero. The manifold in fact becomes a Kähler-Hodge manifold. This is the projective special Kähler geometry.

In quaternionic geometry, we start similarly from a hyperkähler manifold with a homothetic Killing vector (a 'hyperkähler cone') [62]. The slice that is taken here is determined by a condition as

$$k^X g_{XY} k^Y = M_P^3 \,. \tag{5.21}$$

This slice has still an SU(2) symmetry and defines a tri-Sasakian manifold [63, 54]. Now the SU(2) gauge field of the Weyl multiplet gets by its field equation a value $V_\mu^\alpha = \omega_X^\alpha(q)\partial_\mu q^X + \ldots$, where $\omega_X^\alpha(q)$ becomes the gauge field of an SU(2) in the scalar manifold. As such, the manifold gets non-trivial SU(2) curvature and is promoted to a quaternionic-Kähler manifold.

With this procedure we can associate the picture in table 6, similar to table 3, illustrating the mapping from the affine geometries [with signature $(-+\ldots+)$] to projective geometries.

	$D=5$ vector multiplets	$D=4$ vector multiplets	hypermultiplets
conformal (affine)	affine very special real	affine special Kähler	hyperkähler
	$\downarrow$ gauge fix D	$\downarrow$ gauge fix D, U(1)	$\downarrow$ gauge fix D, SU(2)
Poincaré (projective)	(projective) very special real	(projective) special Kähler	quaternionic-Kähler

TABLE 6. *Construction of projective special geometries from conformal affine geometries, with signature* $(-+\ldots+)$.

6. Conclusions

Special geometry is thus a framework of related theories whose structure is restricted by 8 real supersymmetry transformations. The restrictions that are inherent to these models make them particularly useful for investigations in issues such as dualities. The superconformal construction gives insight in hidden symmetries that these theories possess. This construction leads at the end to general actions for matter-coupled supergravity theories [64]. We have schematically indicated here how these can be obtained from the picture in table 6. Similar results have been obtained using other methods. In particular the classic papers for general couplings in 4 and 5 dimensions are [65, 23] used the group manifold approach. In this paper we did not spend much attention to the way in which the gauging acts on the hypermultiplets, although this is automatically obtained by having the hypermultiplets defined 'in the background' of the gauge symmetries of the vector multiplets, as shortly illustrated by (5.17). More details on this issue have been given in [66].

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