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KT AND HKT GEOMETRIES IN STRINGS AND IN BLACK HOLE MODULI SPACES

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ABSTRACT. Some selected applications of KT and HKT geometries in string theory, supergravity, black hole moduli spaces and hermitian geometry are reviewed. It is shown that the moduli spaces of a large class of five-dimensional supersymmetric black holes are HKT spaces. In hermitian geometry, it is shown that a compact, conformally balanced, strong KT manifold whose associated KT connection has holonomy contained in $SU(n)$ is Calabi-Yau. The implication of this result in the context of some string compactifications is explained.

1. INTRODUCTION

In physics, geometries with torsion a three form have found applications in string theory, in supersymmetric quantum mechanics [1, 2] and in the investigation of geometry of black hole moduli spaces [2, 3, 4, 5, 6]. (For other applications see [7, 8].) The typical geometric structure that appears is a triplet (M, g, H), where M is a n-dimensional manifold with a metric g which in addition is equipped with a three form H. Such manifolds apart from the usual Levi-Civita connection ∇ associated with the metric g also admit two more metric connections $\nabla^{\pm}$ which have torsion $\pm H$. We shall refer to $(M, g, H, \nabla^{\pm})$ as *T-manifolds.* The emphasis is on the connections $\nabla^{\pm}$ because many applications in physics involve the reduction of the holonomy of these connections to an appropriate subgroup of $SO(n)$.

In string theory, M is the spacetime, g is the Lorentzian spacetime metric and H is the (closed) three-form associated with the NS$\otimes$NS field strength. In supersymmetric quantum mechanics, M is the manifold that a supersymmetric particle propagates, g is a Riemannian metric, and H is a three-form which appears in some fermion couplings. In the case of black holes, M is the black hole moduli space, g is the moduli metric and H is a three-form on the moduli space.

In mathematics, geometries with torsion a three-form arise in the context of hermitian manifolds which are *not* Kähler. A hermitian manifold is a triplet (M, g, J)

of a manifold M with Riemannian metric g and a complex structure J such that $g(JX, JY) = g(X, Y)$ for any vector fields X and Y. For such manifolds, the complex structure is not parallel with respect to the Levi-Civita connection. However it has been known for sometime (see for example [9]) that there is *unique* connection $\hat{\nabla}$ with torsion a three-form H such that $\hat{\nabla}g = \hat{\nabla}J = 0$, ie the metric and complex structure are parallel with respect to $\hat{\nabla}$. We shall refer to $(M, g, J, \hat{\nabla})$ as a Kähler manifold with torsion or KT manifold for short [2]. On every complex manifold there is always a KT-structure. This is because given a complex structure it is always possible to find a metric which satisfies the hermiticity property. Given a hermitian metric and a complex structure, one can construct a unique connection $\hat{\nabla}$. We shall refer to $\hat{\nabla}$ as KT-connection. If the torsion H is closed, $dH = 0$, the KT-structure on M is called strong otherwise it is called weak.

A hyper-Kähler manifold with torsion or HKT manifold $(M, g, J_r, \hat{\nabla})$ is a manifold with hypercomplex structure[1] $\{J_r; r = 1, 2, 3\}$, a tri-hermitian metric g, $g(J_rX, J_rY) = g(X, Y)$, and a metric connection $\hat{\nabla}$ with torsion a three-form H such that all three complex structures are parallel with respect to $\hat{\nabla}$, $\hat{\nabla}J_r = 0$. Clearly the HKT structure on a hypercomplex manifold is the analogue of KT structure on a complex manifold. However it is not known whether every hypercomplex manifold can always admit a HKT structure unlike the case of a complex manifold which always admits a KT structure. If the torsion H is closed, $dH = 0$, the HKT-structure on M is called strong otherwise it is called weak in analogy with the KT case. The definition of the HKT structure was given in [10] and various properties have been investigated in [10, 2, 11, 12]. Many examples of HKT manifolds have been found which include group manifolds [13] and homogeneous spaces [14, 15]. For generalizations see [16, 14, 17, 18].

In this paper, we shall begin with a summary of the main properties of complex and hypercomplex geometry with emphasis on the KT and HKT structures. Then we shall describe how the T-geometries that appear in supersymmetric quantum mechanics and in string theory are related to KT and HKT geometries. We shall present two main results the following:

- The moduli space of supersymmetric five-dimensional black holes which preserve four supersymmetries is a weak HKT manifold.
- Compact, strong, conformally balanced, KT-manifolds for which the holonomy of the KT-connection is contained in $SU(n)$, $\mathrm{hol}(\hat{\nabla}) \subseteq SU(n)$, are necessarily Calabi-Yau.

[1]A manifold is hypercomplex if it admits three complex structures J_r that obey the algebra of imaginary unit quaternions $J_1^2 = J_2^2 = -1$ and $J_3 = J_1J_2 = -J_2J_1$.

The definition of a conformally balanced hermitian manifold will be given in the next section. An application of the latter result in string theory is that there are no supersymmetric warped compactifications of the common sector of type II string theory with non-vanishing NS⊗NS three-form and $\mathrm{hol}(\hat{\nabla}) \subseteq SU(n)$.

The material that I present on the geometry of black hole moduli spaces is a selection of the work done in collaboration with Jan Gutowski in [4, 6, 20]. Most of the material that I describe on KT-manifolds with holonomy $\mathrm{hol}(\hat{\nabla}) \subseteq SU(n)$ has been done in collaboration with Stefan Ivanov in [25, 26].

This paper has been organized as follows: In section two, a summary of the main properties of hermitian, KT and HKT manifolds is presented. In section three, the relation between supersymmetric mechanics and geometries with torsion is explained. In section four, it is shown that the moduli spaces of five-dimensional black holes which preserve four supersymmetries are HKT manifolds. In section five, the relation between type II supergravity and geometries with torsion is explained. In section six, it is shown that a class of compact, conformally balanced, KT manifolds with $\mathrm{hol}(\hat{\nabla}) \subseteq SU(n)$ are in fact Calabi-Yau and an application to string compactifications is presented. In addition, a non-compact example of a KT manifold with $\mathrm{hol}(\hat{\nabla}) \subseteq SU(n)$ is given.

2. Hermitian, KT and HKT manifolds

Let (M, g, J) be a hermitian manifold. Using the hermiticity condition of the metric g, $g_{ij} = g_{k\ell} J^k{}_i J^\ell{}_j$, we can define a Kähler two-form Ω on M as $\Omega_{ij} = g_{ik} J^k{}_j$. For hermitian manifolds which are *not* Kähler, Ω is *not* closed, $d\Omega \neq 0$. Observe that $dvol(M) = \frac{1}{2^n}\Omega^n$ and so M is oriented; $(\mathrm{dim}M = 2n)$.

There are several connections on M which preserve the hermitian structure, ie they have the property that both the metric and complex structure are parallel. Here we shall focus on two such connections. One is the Chern connection defined as

$$\tilde{\nabla}_i Y^j = \nabla_i Y^j + \frac{1}{2} J^m{}_i d\Omega_{mkn} g^{nj} Y^k \ ,$$

where Y is a vector field, ∇ is the Levi-Civita connection associated with the metric g and $i, j, k = 1, \ldots, \mathrm{dim}M$. The torsion of this connection is

$$C_{ijk} = \frac{1}{2}\big(J^m{}_i d\Omega_{mjk} - J^m{}_j d\Omega_{mik}\big) \ .$$

(We have lowered the upper index of the torsion using the metric g.) Observe that $\tilde{\nabla} g = \tilde{\nabla} J = 0$. The main property of the Chern connection is that the curvature two-form is (1,1) with respect to J and therefore $\tilde{\nabla}$ is compatible with the complex structure of the tangent bundle of M.

The KT-connection $\hat{\nabla}$, which appears in physics applications, is the unique hermitian connection which has as torsion a three-form. The connection $\hat{\nabla}$ is

$$\hat{\nabla}_i Y^j = \nabla_i Y^j + \frac{1}{2} g^{jm} H_{mik} Y^k \ , \tag{2.1}$$

where Y is a vector field and the torsion H is

$$H_{ijk} = -3 J^m{}_{[i} d\Omega_{jk]m} \ . \tag{2.2}$$

(Again we have lowered the upper index of the torsion using the metric g.) Observe that $\hat{\nabla} g = \hat{\nabla} J = 0$. Therefore $(M, g, J, \hat{\nabla})$ is a KT-manifold. For generic hermitian structures, $dH \neq 0$, and so the associated KT structures are weak.

In what follows, we shall use a relation between the curvature[2] $\tilde{R}$ of Chern connection $\tilde{\nabla}$ and curvature $\hat{R}$ of KT connection $\hat{\nabla}$. For this define

$$u = -\frac{1}{4} \tilde{R}_{ijkl} \Omega^{ij} \Omega^{kl} \tag{2.3}$$

$$b = -\frac{1}{2} \hat{R}_{ijkl} \Omega^{ij} \Omega^{kl} \ . \tag{2.4}$$

It has been shown in [25, 26] that

$$2u = b + C_{ijk} C^{ijk} + \frac{1}{4} (dH)_{ijkl} \Omega^{ij} \Omega^{kl} \ . \tag{2.5}$$

This formula is valid for any hermitian manifold.

The *Lee* form of a hermitian manifold (M, g, J) is defined as follows:

$$\theta = -J^m{}_i \nabla^k \Omega_{km} dx^i \ . \tag{2.6}$$

We say that (M, g, J) is *conformally balanced*, if the Lee form θ is *exact*. In the case that the Lee form *vanishes* $\theta = 0$, (M, g, J) is called *balanced*. It can be shown that if (M, g, J) is conformally balanced with $\theta = df$, then $(M, e^{\frac{f}{1-n}} g, J)$ is balanced $(n > 1)$.

For HKT manifolds $(M, g, J_r, \hat{\nabla})$, the connection $\hat{\nabla}$ is defined for each complex structure as in (2.1). In particular, this implies that the three KT torsions (2.2) associated with the three complex structures J_r are equal.

Some HKT geometries arise from a HKT potential [11, 3]. Let (M, J_r) be a hypercomplex manifold and a function μ on M. Then a HKT structure can be

[2]Our conventions for the curvature R of a connection ∇ are as follows: $[\nabla_i, \nabla_j] Y^k = R_{ij}{}^k{}_\ell Y^\ell$ and $R_{ijk\ell} = -g_{km} R_{ij}{}^m{}_\ell$.

defined on (M, J_r) as

$$
\begin{aligned}
ds^2(M) &= \big(\partial_i\partial_j + \sum_{r=1}^{3} (J_r)^k{}_i (J_r)^\ell{}_j \partial_k\partial_\ell\big)\mu\, dx^i dx^j \\
H &= d_1 d_2 d_3 \mu
\end{aligned}
\qquad (2.7)
$$

provided that μ can be chosen such that the HKT metric above is well defined, where $d_r = i(\partial_r - \bar{\partial}_r)$; ∂_r is the holomorphic exterior derivative with respect to J_r complex structure.

Some of the applications of KT and HKT geometries in physics are as follows:

- Strong KT and HKT geometries have applications in type II string theory and in two-dimensional supersymmetric sigma models [29, 7, 8].
- Weak KT and HKT geometries have applications in supersymmetric quantum mechanics [1, 2].
- Strong and weak HKT geometries have applications in the moduli spaces of gravitational solitons and black holes [2, 3, 4].

3. Supersymmetric mechanics

The supersymmetry algebra in one-dimension is spanned by the generators $\{Q_I, T;\ I = 1, \ldots, \mathcal{N}\}$, where Q_I are the supersymmetry generators and T is the translation generator, subject to the anti-commutator relation

$$
Q_I Q_J + Q_J Q_I = 2\delta_{IJ} T\ . \qquad (3.8)
$$

Supersymmeric mechanical systems are those that are invariant under (infinitesimal) symmetries which realize the above algebra. There are different realizations of the above supersymmetry algebra and have been investigated in [1].

3.1. $\mathcal{N} = 1$ supersymmetry.

To find a system invariant under one supersymmetry, let $(M, g, H, \nabla^{\pm})$ be a T-manifold M equipped with metric g and a three-form H. Next consider a map $X:\ \mathbb{R} \to M$. A class of $\mathcal{N}$=1 supersymmetric mechanics models[3] can be described by the action [1]

$$
I = \frac{1}{2}\int_{\mathbb{R}} dt\, \big(g_{ij}\partial_t X^i \partial_t X^j + i g_{ij}\lambda^i \nabla_t^+ \lambda^j - \frac{1}{24}(dH)_{ijkl}\lambda^i\lambda^j\lambda^k\lambda^l\big)\ , \qquad (3.9)
$$

where λ is a (worldline) fermion on $\mathbb{R}$ which geometrically can be described as section of the bundle $S \otimes X^*TM$; S is the spinor bundle over $\mathbb{R}$ and X^*TM is the

[3] The torsion three-form H in supersymmetric mechanics is usually denoted with c.

pull-back of the tangent bundle of M with respect to the map X. In addition, ∇^+_t is the pull-back of the connection ∇^+ on $\mathbb{R}$ with respect to X. So

$$\nabla^+_t \lambda^i = \partial_t \lambda^i + (\Gamma^+)^i_{jk} \partial_t X^j \lambda^k \ ,$$

where $(\Gamma^+)^i_{jk} = \Gamma^i_{jk} + \frac{1}{2} H^i{}_{jk}$ and Γ^i_{jk} is the Levi-Civita connection of g.

The action (3.9) is supersymmetric because it can be written in terms of superfields X as

$$I = -\frac{1}{2} \int_\Xi dt\, d\theta \left(i g_{ij} DX^i \partial_t X^j + \frac{1}{12} H_{ijk} DX^i DX^j DX^k \right) \ , \tag{3.10}$$

where $D^2 = i\partial_t$, $D = \partial_\theta + i\theta\partial_t$, and $X : \Xi \to M$; Ξ is a supermanifold with an even coordinate t and an odd coordinate θ. The infinitesimal supersymmetry transformation is $\delta X^i = \eta Q X^i$, where $Q = \partial_\theta - i\theta\partial_t$ and η is the parameter; $DQ + QD = 0$. The action (3.10) is invariant under this supersymmetry transformation because it is a full superspace integral.

To derive the action (3.9) from (3.10), we integrate over the odd coordinate θ which is equivalent to differentiating with respect to D and evaluating the resulting expression at $\theta = 0$. The maps X and fermions λ in (3.9) are given in terms of the superfields[4] X as $X^i = X^i|_{\theta=0}$ and $\lambda^i = DX^i|_{\theta=0}$.

The $\mathcal{N}$=1 supersymmetric mechanics system described by the action (3.9) or equivalently by (3.10) is not the most general one with $\mathcal{N}$=1 supersymmetry. More general models have been constructed in [1]; see also [2].

3.2. $\mathcal{N} = 2B$ **and** $\mathcal{N} = 4B$ **supersymmetry.** It is expected that the dynamics of black holes at small velocities, ie in the moduli approximation, is described by a action similar to (3.9) which however is invariant under at four supersymmetries instead of one. For this, we investigate the conditions for (3.9) to be invariant under one and three additional supersymmetries. The infinitesimal transformations of the additional supersymmetries are most easily written in terms of $\mathcal{N} = 1$ superfields X as

$$\delta X^i = \eta^r (J_r)^i{}_j DX^j \ ,$$

where η^r are the anti-commuting infinitesimal parameters and J_r are endomorphisms of TM; $r = 1$ or $r = 1, 2, 3$.

Requiring that the above transformations satisfy the supersymmetry algebra (3.8) and leave the action (3.10) invariant, one finds the following:

[4] It is customary to denote the superfield and its first component with the same symbol.

- For models with two supersymmetries ($\mathcal{N}$=$2B$)[5], (M, g, H) is a KT manifold $(M, g, J, \hat{\nabla})$, where $J = J_1$ and $\hat{\nabla} = \nabla^+$
- For models with four supersymmetries ($\mathcal{N}$=$4B$), (M, g, H) is a HKT manifold $(M, g, J_r, \hat{\nabla})$, where J_r is the hypercomplex structure and $\hat{\nabla} = \nabla^+$.

In fact the above described geometric conditions for a model to admit two or four supersymmetries are sufficient but no necessary. The derivation of the above conditions as well as some more general results can be found in [1, 2].

4. Black hole moduli spaces

Supersymmetric black holes are black hole solutions of supergravity theories which in addition admit a number of Killing spinors. Killing spinors are solutions of Killing spinor equations and an example of such equations will be described in detail in section five. Supersymmetric black hole solutions in supergravity theories apart from the spacetime metric also involve non-vanishing Maxwell and scalar fields. The mass of the black holes is related to their charges which is a consequence of a BPS type of condition. Several supersymmetric black holes can be superposed together to form a static configuration because there is a balance of forces acting on them. Although the presence of Maxwell and scalar fields in the solution are essential for the existence for such superpositions, in what follows we shall focus on the spacetime metric of a supersymmetric system with N black holes. The spacetime metric of a typical solution which describes N supersymmetric black holes in superposition of a supergravity theory can be expressed as

$$ds^2 = -A^2(\mathbf{x}, \mathbf{y}_A)dt^2 + B^2(\mathbf{x}, \mathbf{y}_A)|d\mathbf{x}|^2 \ , \tag{4.11}$$

where $(\mathbf{x}, t)$ are the spacetime coordinates, $|\cdot|$ is the Euclidean inner product in $\mathbb{R}^k$ and $\{\mathbf{y}_A \in \mathbb{R}^k; A = 1, \cdots, N\}$ are the positions of the black holes. Note that the components A^2, B^2 of the metric depend of the space coordinates $\mathbf{x}$ and the positions $\mathbf{y}_A$ of the black holes. As we shall see for the description of many aspects of the geometry of black hole moduli spaces, the details of the supergravity action that (4.11) is a solution are not essential.

The moduli space $\mathcal{M}^k_N$ of a (supersymmetric) N-black hole solution is the space of positions of black holes. This can be identified with the space of N-particles in $\mathbb{R}^k$, ie

$$\mathcal{M}^k_N = \times^N \mathbb{R}^k - \Delta \ ,$$

[5] The letter 'B' has been added to denote a particular realization of the supersymmetry algebra with two supercharges according to the terminology used in [2].

where $\Delta = \{(\mathbf{y}_1, \dots, \mathbf{y}_N) \in \times^N \mathbb{R}^k;\ \mathbf{y}_i = \mathbf{y}_j, i \neq j\}$ is the diagonal. The dimension of the moduli space is kN. If the black holes have the same masses and carry the same charges, then the metric (4.11) is invariant under the action of the permutation group Σ_N acting on the positions of the black holes. For such black holes, the moduli space is

$$\tilde{\mathcal{M}}^k_N = \mathcal{M}^k_N / \Sigma_N$$

which is the configuration space of N-indistinguishable particles in $\mathbb{R}^k$.

The geometry on the moduli space of black holes that preserve four supersymmetries[6] is expected to be that of the 'target' manifold of supersymmetric mechanical systems which are invariant under the same number of supersymmetries. This is because the symmetries of a supergravity solution are expected to be realized as symmetries of the associated effective theory. Thus for black hole systems which have as an effective theory the supersymmetric mechanics models presented in section three, it is expected that their moduli space is a HKT manifold.

The cases of interest are those of black holes in four and five spacetime dimensions. The computation of the metric on the moduli space is done as follows:

- The positions of the black holes $\mathbf{y}_A$ are allowed to depend on time t.
- The metric and the other fields, like Maxwell fields, are perturbed by first order terms in the black hole velocities.
- These perturbations of the fields are determined by using the field equations.
- The moduli metric is read by substituting the perturbed solution into the appropriate supergravity action and by collecting the quadratic in the velocity terms.

The actual computation of the metric on the black hole moduli space is long and complicated. However Gutowski and I found that for most multi-black hole solutions, those that preserve at least four supersymmetries, the black hole moduli metric can be determined from the components of the spacetime metric by a simple relation that will be described below [6]. We have shown this by an explicit computation for the electrically charged black holes of five- and four-dimensional supergravities which preserve four superymmetries and are coupled to any number of Maxwell fields [4, 6]. (Our results include the moduli metrics of the Reissner-Nordström and the graviphoton black holes which had been previously found in [19] and [3], respectively.) We then conjectured that the same relation between the spacetime metric and the moduli metric holds for all systems of N-black holes that preserve at least four supersymmetries. Our conjectured is based on duality.

[6]This means that these solutions admit four non-vanishing Killing spinors, see section five.

The metric on the moduli space of black holes can be determined from the associated N-black hole spacetime metric as follows:

First define a function[7] μ, *the moduli potential*, on the moduli space $\mathcal{M}^k_N$ $(k = 3, 4)$ as

$$\mu(\mathbf{y}_1, \cdots, \mathbf{y}_N) = \int_{\mathbb{R}^k} d^kx\ A^{-2}B^2(\mathbf{x}, \mathbf{y}_1, \cdots, \mathbf{y}_N)\ . \tag{4.12}$$

The metric on the moduli space of four- and five-dimensional black holes can be determined from μ. In particular for five-dimensional black holes, the metric on $\mathcal{M}^4_N$ is

$$ds^2(\mathcal{M}^4_N) = \big[\partial_{mA}\partial_{nB} + \sum_{r=1}^{3}(I_r)^k{}_m(I_r)^\ell{}_n\partial_{kA}\partial_{\ell B}\big]\mu\ dy^{mA}dy^{nB}\ , \tag{4.13}$$

where $\{I_r; r = 1, 2, 3\}$ is a constant hypercomlex structure on $\mathbb{R}^4$ associated say with a basis of self-dual two forms on $\mathbb{R}^4$.

The moduli space $\mathcal{M}^4_N$ is a HKT manifold. To show this, one has to find a hypercomplex structure on $\mathcal{M}^4_N$ and identify the moduli potential μ given above with the HKT potential in section two. A hypercomplex structure on the moduli space is

$$(\mathbf{I}_r)^{mA}{}_{nB} = \delta^A{}_B(I_r)^m{}_n\ . \tag{4.14}$$

It is clear that this hypercomplex structure is induced from that on $\mathbb{R}^4$. Comparing (4.13) and (2.7) using (4.14), we can conclude that the moduli metric (4.13) is a HKT metric with potential μ. The torsion on the moduli space is then given as in (2.7). Therefore we have shown the following:

- The moduli space, $\mathcal{M}^4_N$, of five-dimensional black holes which preserve four supersymmetries is a HKT manifold whose geometry is determined by the HKT potential given in (4.12).

The metric on the moduli space of four-dimensional black holes can be determined in a similar way. In this case the geometry on the moduli space and the associated supersymmetric classical mechanics system are more involved and they will not be presented here. For more details see [5, 6].

4.1. **STU black holes.** A large class of black hole solutions which preserve four supersymmetries are those of the STU supergravity in five-dimensions with eight supersymmetries. The bosonic fields of this supergravity are a graviton (a metric), three Maxwell fields and two scalars. In what follows, the details of the action

[7]This function may not be well-defined on the moduli space because the integral may not converge. However, it can be shown that the moduli metric is well-defined.

of the STU supergravity theory are not important. The spacetime metric of the multi-black hole solution is

$$ds^2 = -(f_1 f_2 f_3)^{-\frac{2}{3}} dt^2 + (f_1 f_2 f_3)^{\frac{1}{3}} |d\mathbf{x}|^2 \ ,$$

where

$$f_i = h_i + \sum_{A=1}^{N} \frac{\lambda_{iA}}{|\mathbf{x} - \mathbf{y}_A|^2}$$

for $i = 1, 2, 3$ which are harmonic functions on $\mathbb{R}^4$. The constants $\{h_i; i = 1, 2, 3\}$ are related to the asymptotic values of the two scalars of the theory and the constants $\{\lambda_{iA}; i = 1, 2, 3; A = 1, \dots, N\}$ are interpreted as the charge of the A-th black-hole with respect to the i-th Maxwell gauge potential; $\{\mathbf{y}_A; A = 1, \dots, N\}$ are the positions of the black holes. It is clear the solution is not invariant under the action of the permutation group Σ_N acting on the positions of black holes unless the charges λ_{iA} are equal.

The moduli potential for the black holes of the STU model is

$$\mu = \int_{\mathbb{R}^4} d^4x \, f_1 f_2 f_3 \ . \tag{4.15}$$

For the masses of the black holes to be positive and for the black hole spacetime metric not to have naked singularities, we take $h_i, \lambda_{iA} > 0$. The moduli potential (4.15) gives rise to the moduli metric [6]

$$\begin{aligned} ds^2 &= V_3 \sum_A [h_2 h_3 \lambda_{1A} + h_1 h_3 \lambda_{2A} + h_1 h_2 \lambda_{3A}] |d\mathbf{y}_A|^2 \\ &+ V_3 \sum_{A \neq B} [h_2 \lambda_{1A} \lambda_{3B} + h_1 \lambda_{2A} \lambda_{3B} + h_3 \lambda_{1A} \lambda_{2B}] \frac{|d\mathbf{y}_{AB}|^2}{|\mathbf{y}_{AB}|^2} \\ &+ \frac{V_3}{2} \sum_{\{A \neq B\}, C} \tau_{ABC} |d\mathbf{y}_{AB}|^2 \Big[\frac{1}{|\mathbf{y}_{AC}|^2 |\mathbf{y}_{AB}|^2} + \frac{1}{|\mathbf{y}_{BC}|^2 |\mathbf{y}_{AB}|^2} - \frac{1}{|\mathbf{y}_{AC}|^2 |\mathbf{y}_{BC}|^2} \Big] \\ &- 2 \sum_{A \neq B \neq C} \int d^4x \, \tau_{ABC} \frac{[(dy_A^{[m} dy_B^{n]})^-}{|\mathbf{x} - \mathbf{y}_C|^2} \partial_m \Big(\frac{1}{|\mathbf{x} - \mathbf{y}_A|^2} \Big) \partial_n \Big(\frac{1}{|\mathbf{x} - \mathbf{y}_B|^2} \Big) \ , \end{aligned} \tag{4.16}$$

where $\mathbf{y}_{AB} = \mathbf{y}_A - \mathbf{y}_B$, V_3 is the volume of the unit three-sphere, $(dy_A^{[m} dy_B^{n]})^-$ is the anti-self-dual part of $dy_A^{[m} dy_B^{n]}$ and

$$\tau_{ABC} = [\lambda_{1A} \lambda_{2B} \lambda_{3C} + \lambda_{1C} \lambda_{2A} \lambda_{3B} + \lambda_{1B} \lambda_{2C} \lambda_{3A}] \ .$$

The moduli metric has a free term for N particles, and two-and three-body velocity dependent interactions. Observe that part of the moduli metric that contains three-body interactions is not given explicitly since the last term in (4.16) involves an integration over the spatial coordinates $\mathbf{x}$. This term has been investigated in [20] and it was found that it can be determined by the one-loop three-point amplitude of a ϕ^3 theory. We remark that if the masses of the black holes

$$m_A = h_2 h_3 \lambda_{1A} + h_1 h_3 \lambda_{2A} + h_1 h_2 \lambda_{3A}$$

are not equal, then the centre of mass motion does not decouple from the dynamics of the system. For a discussion of the properties of the two-body system see [20].

The two-black hole moduli space $\mathcal{M}_2^4$ is geodesically complete. In the case that the centre of mass decouples, the relative moduli space is the connected sum of two $\mathbb{R}^4$, $\mathbb{R}^4 \# \mathbb{R}^4$. One asymptotic region is associated with black holes at small separations while the other is associated with black holes at large separations. Thus $\mathcal{M}_2^4 = \mathbb{R}^4 \times (\mathbb{R}^4 \# \mathbb{R}^4)$, where $\mathbb{R}^4$ is associated with the free motion of the centre of mass. For $N > 2$, it is not known whether $\mathcal{M}_N^4$ is geodesically complete.

It has been demonstrated in [6] that the STU black holes at small separations exhibit a superconformal structure following a similar work for the graviphoton black hole in [3]. For applications of the superconformal symmetry in HKT geometry see [21].

5. Type II supergravity

In physics, there has been much activity in understanding the soliton-like solutions of supergravity theories because of their applications in the investigation of non-perturbative properties of string theory. One property of such solutions is that they are *supersymmetric*. This means that these supergravity solutions satisfy in addition a set of at most first-order in the spacetime derivatives equations acting linearly on a spinor ϵ for some $\epsilon \neq 0$. These are called *Killing spinor* equations. The *non-vanishing* solutions ϵ of Killing spinor equations are called *Killing spinors.* In supergravity theories, the Killing spinor equations arise as the vanishing conditions of the supersymmetry transformations of the fermions of the supergravity theories and ϵ is the supersymmetry infinitesimal parameter. Some of the Killing spinor equations are *parallel transport* equations for the spinor ϵ with respect to a connection of a spin bundle of spacetime. The integrability conditions of the Killing spinor equations imply some of the field equations of the supergravity theory. The existence of Killing spinors is closely related to the stability of a supergravity solution against small fluctuations. The *number of supersymmetries* preserved by a solution is the *number of linearly independent* Killing spinors.

The simplest supergravity system for which some of the Killing spinor equations have a direct geometric interpretation as parallel transport equations is that of the common sector or NS$\otimes$NS sector of type II ten-dimensional supergravities

[22, 23, 24]. We shall focus on the type IIA theory. The discussion for the common sector of type IIB supergravity is similar. It can be easily extended to the case of heterotic string as well.

The bosonic fields of common sector of type IIA supergravity theory are the spacetime metric g, a closed three-form field strength H ($dH = 0$) and a scalar field ϕ called dilaton. The spacetime (M, g, H) is therefore a T-manifold. The field equations of the common sector of type IIA supergravity are

$$\begin{aligned} R_{MN} - \frac{1}{4} H^R{}_{ML} H^L{}_{NR} + 2\nabla_M \partial_N \phi &= 0 \\ \nabla_M \big(e^{-2\phi} H^M{}_{RL} \big) &= 0 \,. \end{aligned} \tag{5.17}$$

In fact there is an additional field equation that of the dilaton ϕ but it is implied from the above two equations. Let $\{\Gamma^M; M = 0, \dots, 9\}$ be a basis of the Clifford algebra Cliff$(\mathbb{R}^{1,9})$, ie $\Gamma^M\Gamma^N + \Gamma^N\Gamma^M = 2g^{MN}$. Then the Killing spinor equations[8] are

$$\begin{aligned} \nabla^\pm \epsilon_\pm &= 0 \\ \big(\Gamma^M \partial_M \phi \mp \frac{1}{12} H_{MNR} \Gamma^{MNR}\big) \epsilon_\pm &= 0 \,, \end{aligned} \tag{5.18}$$

where

$$\nabla^\pm_M Y^N = D_M Y^N \pm \frac{1}{2} H^N{}_{MR} Y^R$$

and $\epsilon_\pm$ are sections of the spin bundles $S_\pm$, respectively[9]. It is clear that the first two of the Killing spinor equations are parallel transport equations for the connections $\nabla^\pm$. Since these connections of the spin bundles $S_\pm$ are induced from the tangent bundle of the spacetime, the investigation of the Killing spinor equations is greatly simplified. This is not the case in general, some of Killing spinor equations of supergravity theories are parallel transport equations of a spin bundle but typically the associated connections are not induced from the tangent bundle of spacetime[10].

According to the definition given in the beginning of this section, a solution of the field equations (5.17) is supersymmetric if it satisfies the Killing spinor equations (5.18) for some (non-vanishing) Killing spinors $\epsilon_\pm$.

[8] We have used the notation $\Gamma^{M_1 \dots M_k} = \Gamma^{[M_1} \dots \Gamma^{M_k]}$.

[9] The spin group $Spin(1,9)$ has two inequivalent irreducible sixteen-dimensional spinor representations and $S_\pm$ are the associated bundles.

[10] The Killing spinor equations of $D = 11$ supergravity are parallel transport equations with respect to a connection of the spin bundle which is not induced from the tangent bundle of spacetime.

5.1. **The NS5-brane.** We shall illustrate the relation between the holonomy of the connections $\nabla^{\pm}$ and the number of supersymmetries preserved with a simple example. Consider the NS5-brane solution [28] of IIA supergravity

$$\begin{aligned} ds^2 &= ds^2(\mathbb{R}^{1,5}) + f ds^2(\mathbb{R}^4) \\ H &= \star df \\ e^{2\phi} &= f \ , \end{aligned} \tag{5.19}$$

where $\star$ is the Hodge star operation in $\mathbb{R}^4$ and $f = 1 + \frac{Q_5}{|\mathbf{x}|^2}$, $\mathbf{x} \in \mathbb{R}^4$; Q_5 is the charge (per unit volume) of the NS5-brane.

Since the spacetime metric, three-form field strength and dilaton do not depend of the coordinates of $\mathbb{R}^{1,5}$, the non-trivial part of the solution is

$$\begin{aligned} ds^2 &= f ds^2(\mathbb{R}^4) \\ H &= \star df \\ e^{2\phi} &= f \ . \end{aligned} \tag{5.20}$$

This metric describes the geometry of a smooth four-dimensional manifold M equipped with a closed three-form, ie (M, g, H) is a T-manifold. As $|\mathbf{x}| \to \infty$, the metric (5.20) becomes that of $\mathbb{R}^4$ while as $|\mathbf{x}| \to \mathbf{0}$ the metric becomes that of $\mathbb{R} \times S^3$. In fact M admits two constant hypercomplex structures $\{J_r; r = 1, 2, 3\}$ and $\{I_r; r = 1, 2, 3\}$. One is associated with a basis of self-dual two forms on $\mathbb{R}^4$ and the other with a basis of anti-self-dual forms on $\mathbb{R}^4$, respectively. It turns out that $\nabla^+ J_r = 0$ and $\nabla^- I_r = 0$. This implies that the holonomy of both $\nabla^{\pm}$ connections is contained in $Sp(1)$. It turns out that their holonomy is precisely $Sp(1)$ and there are sixteen parallel spinors. The remaining two Killing spinor equations in (5.18) are satisfied without additional conditions. So one concludes the following:

- The NS5-brane admits two HKT structures and preserves sixteen supersymmetries.

5.2. **Supersymmetric compactifications on KT-manifolds.** String theory is formulated in ten-dimensions. To relate it to physics in (d+1) dimensions ($d < 9$), one uses solutions of the type

$$\begin{aligned} ds^2 &= ds^2(\mathbb{R}^{1,d}) + ds^2(X) \\ H &= H(y) \\ \phi &= \phi(y) \ , \end{aligned} \tag{5.21}$$

where H and ϕ are a three-form and a function on X, respectively; X is a compact manifold. At low energies the (d+1)-dimensional theory emerges as the fluctuations of the (5.21) geometry in ten dimensions. Again the non-trivial part of the geometry (5.21) is described by the T-manifold $(X, g, H, \nabla^{\pm})$. Such compactifications have been considered in [29]; Such compactifications for which the dilaton is *not* constant are also called *warped* compactifications.

A special class of solutions of this type are those for which $H = 0$ and ϕ is a constant. In such a case, the Einstein equations imply that the Ricci tensor vanishes. Moreover the Killing spinor equations imply that the Killing spinors are parallel with respect to the Levi-Civita connection. For both field equations and Killing spinor equations to have solutions, X must be the product of suitable irreducible Riemannian manifolds with holonomy in $SU(n)$ $(n = 2, 3, 4)$, G_2, $Sp(2)$ or $Spin(7)$.

We shall focus on the investigation of compactifications for which $H \neq 0$ and the holonomy of one of the connections $\nabla^{\pm}$, say ∇^{+}, is a subgroup of $SU(n)$ $(\mathrm{hol}(\nabla^{+}) \subseteq SU(n))$. The investigation of the geometry of X that follows is due to [29] but we use the terminology of [25] to describe it. Since $\mathrm{hol}(\nabla^{+}) \subseteq SU(n)$, there is a ∇^{+}-parallel spinor η such that

$$J^i{}_j = -i\eta^{\dagger}\Gamma^i{}_j\eta$$

is an almost complex structure. Since $\eta^{\dagger}\eta$ is constant, we have normalized η such that $\eta^{\dagger}\eta = 1$. In fact it can be shown that J is an *integrable* complex structure, *parallel* with respect to ∇^{+}, $\nabla^{+}J = 0$, and the metric g is *hermitian* with respect to J. Therefore, we conclude the following:

- If $\mathrm{hol}(\nabla^{+}) \subseteq SU(n)$, then $(X, g, J, \hat{\nabla})$ is a KT manifold with $\hat{\nabla} = \nabla^{+}$.

Next we consider the second Killing spinor equation given by

$$\big(\Gamma^i\partial_i\phi - \frac{1}{12}\Gamma^{ijk}H_{ijk}\big)\eta = 0$$

on the parallel spinor η. This equation implies an additional condition. In particular multiplying the above Killing spinor equation and its conjugate with Γ^m, and using the definition of J, we find that

$$2\partial_i\phi - \frac{1}{2}J^m{}_iH_{mjk}\Omega^{jk} = 0\ ,$$

where Ω is the Kähler form of X. In [25, 26], it was observed using $\hat{\nabla}J = 0$ that the one-form

$$\theta = J^m{}_iH_{mjk}\Omega^{jk}dy^i$$

is the *Lee form*, θ, of the Hermitian manifold X as defined in (2.6). Thus we have that

$$\theta_i = 2\partial_i \phi \ .$$

Since the Lee form is *exact*, ϕ is a real function on X, X is a *conformally balanced* hermitian manifold. Therefore we find that supersymmetric compactifications of type II strings for which $H \neq 0$ and $\mathrm{hol}(\nabla^+) \subseteq SU(n)$ are associated with manifolds X which have the following properties:

- $(X, g, J, \hat{\nabla})$ is a compact, conformally balanced, strong KT-manifold whose KT connection $\hat{\nabla}$ has holonomy which is a subgroup $SU(n)$.

An important property of the above manifolds is that they admit a non-vanishing holomorphic (n,0)-form [29]. To be precise, we have the following:

- Let $(X, g, J, \hat{\nabla})$ be a conformally balanced KT-manifold with $\mathrm{hol}(\hat{\nabla}) \subseteq SU(n)$, then X admits a holomorphic (n,0) form.

To prove this, since $\mathrm{hol}(\hat{\nabla}) \subseteq SU(n)$, there is a parallel (n,0)-form α, $\hat{\nabla}\alpha = 0$. Since X is conformally balanced, then the Lee form can be written as $\theta = 2d\phi$ for some function ϕ on X. Then the (n,0)-form $\tilde{\alpha} = e^{-2\phi}\alpha$ is holomorphic. We can easily demonstrate this by computing $\bar{\partial}\tilde{\alpha}$ using the fact that α is parallel and $\theta = 2d\phi$. $\diamond$

This concludes the description of the geometry of the manifolds X that arise in the compactifications which have been investigated in this section. In the next section we shall address the question whether such manifolds can exist.

6. Conformally Balanced KT-manifolds with $\mathrm{hol}(\hat{\nabla}) \subseteq SU(n)$

To find whether compact, conformally balanced, strong KT-manifold with $\mathrm{hol}(\hat{\nabla}) \subseteq SU(n)$ can exist, we define the holomorphic Laplace operator on a function f as

$$L(f) := -2g^{\alpha\bar{\beta}}\partial_\alpha \partial_{\bar{\beta}} f$$

and observe that it can be rewritten[11] as

$$L(f) = \Delta f + g^{ij}\theta_i \partial_j f \ ,$$

where $\Delta = -\nabla^i \partial_i$ is the standard Laplace operator. The main result shown in [25, 26] is the following:

- Let $(X, g, J, \hat{\nabla})$ be a compact, strong, conformally balanced, KT-manifold with $\mathrm{hol}(\hat{\nabla}) \subseteq SU(n)$, then X is a Calabi-Yau manifold.

[11] This corrects a misprint in [25].

To show this, assume that (X, g, J) in non-Kähler. From the assumptions of the theorem and the results of the previous section, X admits a holomorphic (n,0)-form $\tilde{\alpha}$. Set $f = -\frac{1}{2}|\tilde{\alpha}|^2$, where $|\cdot|$ is the norm with respect to the metric g. Then

$$L(f) = -\frac{1}{2}\Delta|\tilde{\alpha}|^2 - \frac{1}{2}g^{ij}\theta_i\partial_j|\tilde{\alpha}|^2 \ .$$

On the other hand using the holomorphicity of $\tilde{\alpha}$, we find that

$$L(f) = 2u|\tilde{\alpha}|^2 + |\tilde{\nabla}\tilde{\alpha}|^2 \ ,$$

where u is defined in (2.3), section two. Next observe that

$$2u = C_{ijk}C^{ijk} > 0 \tag{6.22}$$

and so $u > 0$ for a KT but non-Kähler manifold. This follows from the (2.5) and the assumptions of the theorem which imply that $b = 0$ because $\mathrm{hol}(\hat{\nabla}) \subseteq SU(n)$ and $dH = 0$ because X is strong KT manifold.

Since $u > 0$ and $\tilde{\alpha} \neq 0$, $L(f) > 0$. From the Hopf maximum principle follows that either $\tilde{\alpha} = 0$ or $C = 0$. Since $\tilde{\alpha} \neq 0$, it follows that $C = 0$, ie the torsion of the Chern connection vanishes and X is Kähler, so X is a Calabi-Yau manifold. ◇

One could reach the same conclusion from (6.22) using a Kodaira vanishing theorem. The above result can also be derived under somewhat weaker assumptions [25, 26]. However the proof is more involved. Observe that if the manifold X is *weak* KT, then it is not necessarily the case that $u>0$ because $dH\neq 0$ in (2.5). This implies that compact, weak, conformally balanced, KT-manifold with $\mathrm{hol}(\hat{\nabla}) \subseteq SU(n)$ can exist. For some more vanishing theorems on hermitian manifolds see [27].

An application of the above result [25] to type II string theory is the following:

- The only smooth supersymmetric compactifications of common sector of type II strings with $\mathrm{hol}(\nabla^+) \subseteq \mathrm{SU(n)}$ are the Calabi-Yau compactifications of [30] for which $H = 0$ and the dilaton ϕ is constant.

In other words that are no such warped compactifications of the common sector of type II string theory. In the case of heterotic string though, it may be possible to find compactifications with $H \neq 0$ because $dH \neq 0$ due to the anomaly cancellation. Therefore the relevant compact manifolds are weak KT and as we have mentioned such smooth manifolds can exist.

6.1. **An example of non-compact, KT-manifold with** $\mathrm{hol}(\hat{\nabla}) \subseteq SU(3)$**.** The assumption that X is compact in the theorem of the previous section is necessary. This is because there are *non-compact*, strong, conformally balanced, KT-manifolds

$(X, g, J, \hat{\nabla})$ for which $\mathrm{hol}(\hat{\nabla}) \subseteq SU(n)$. Such an example was found in [31] and interpreted in [32] as a gravitational dual of pure $N = 1$ supersymmetric Yang-Mills theory in four dimensions. The geometric interpretation of the IIA supergravity solution below in terms of KT-geometry with $\mathrm{hol}(\hat{\nabla}) \subseteq SU(3)$ was given in [33]. Let $d\sigma^i = -\frac{1}{2}\epsilon^i{}_{jk}\sigma^j \wedge \sigma^k$ be a basis of left-invariant one-forms in S^3. The KT geometry is

$$\begin{aligned} ds^2 &= dr^2 + e^{2g(r))}\big(d\theta^2 + \sin^2\theta d\varphi^2\big) + \frac{1}{4}\sum_{i=1}^{3}(\sigma^i - A^i)^2 \\ H &= -\frac{1}{4}(\sigma^1 - A^1) \wedge (\sigma^2 - A^2) \wedge (\sigma^2 - A^2) + \frac{1}{4}\sum_{i=1}^{3} F^i \wedge (\sigma^i - A^i) \\ e^{2\phi} &= e^{2\phi_o}\frac{2e^g}{\sinh r} \ , \end{aligned} \tag{6.23}$$

where (θ, φ) are the usual angular coordinates on S^2, r is a radial coordinate and ϕ_0 is an integration constant. In addition

$$A^1 = a(r)d\theta \qquad A^2 = a(r)\sin\theta d\varphi \qquad A^3 = \cos\theta d\varphi$$

and

$$\begin{aligned} a &= \frac{2r}{\sinh r} \\ e^{2g} &= r\coth 2r - \frac{r^2}{\sinh^2 2r} - \frac{1}{4} \ . \end{aligned} \tag{6.24}$$

The manifold associated with (6.23) is complete and admits a strong, conformally balanced, KT-structure with $\mathrm{hol}(\hat{\nabla}) = SU(3)$. The Kähler form [33] is

$$\begin{aligned} \Omega &= \frac{1}{2}dr \wedge (\sigma^3 - A^3) + X(r)e^g\big(\sin\theta(\sigma^1 - A^1) \wedge d\varphi - (\sigma^2 - A^2) \wedge d\theta\big) \\ &+ P(r)\big(-\frac{1}{4}(\sigma^1 - A^1) \wedge (\sigma^2 - A^2) + e^{2g}\sin\theta d\theta \wedge d\varphi\big) \ , \end{aligned} \tag{6.25}$$

where

$$P = \frac{\sinh 4r - 4r}{2\sinh^2 2r} \qquad X = \sqrt{1 - P^2} \ .$$

It would be of interest to construct other examples of non-compact, conformally balanced, strong (smooth) KT manifolds with $\mathrm{hol}(\hat{\nabla}) \subseteq SU(3)$ because of the applications that they might have in supersymmetric gauge theories.

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