Special Geometric Structures in String Theory
Bonn, 8th–11th September, 2001

MAXIMAL SUPERSYMMETRY IN TEN AND ELEVEN DIMENSIONS

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ABSTRACT. This is the written version of a talk given in Bonn on September 11th, 2001 during a workshop on *Special structures in string theory*. We report on joint work in progress with George Papadopoulos aimed at classifying the maximally supersymmetric solutions of the ten- and eleven-dimensional supergravity theories with 32 supercharges.

1. Eleven-dimensional supergravity

Eleven-dimensional supergravity was predicted by Nahm [Nah78] and constructed soon thereafter by Cremmer, Julia and Scherk [CJS78]. We will only be concerned with the bosonic equations of motion. The geometrical data consists of (M^{11}, g, F) where (M, g) is an eleven-dimensional lorentzian manifold with a spin structure and $F \in \Omega^4(M)$ is a closed 4-form. The equations of motion generalise the Einstein–Maxwell equations in four dimensions. The Einstein equation relates the Ricci curvature to the energy momentum tensor of F. More precisely, the equation is

$$\mathrm{Ric}(g) = T(g, F) \tag{1}$$

where the symmetric tensor

$$T(X, Y) = \tfrac{1}{2}\left\langle \imath_X F, \imath_Y F\right\rangle - \tfrac{1}{6} g(X, Y)|F|^2 \ ,$$

is related to the energy-momentum tensor of the (generalised) Maxwell field F. In the above formula, $\langle -, - \rangle$ denotes the scalar product on forms, which depends on g, and $|F|^2 = \langle F, F\rangle$ is the associated (indefinite) norm. The generalised Maxwell equations are now nonlinear:

$$d \star F = \tfrac{1}{2} F \wedge F \ . \tag{2}$$

Definition 1. *A triple (M, g, F) satisfying the equations* (1) *and* (2) *is called a* ***(classical) solution*** *of eleven-dimensional supergravity.*

Let $\$$ denote the bundle of spinors[1] on M. It is a real vector bundle of rank 32 with a spin-invariant symplectic form $(-,-)$. A differential form on M gives rise to an endomorphism of the spinor bundle via the composition

$$c : \Lambda T^*M \xrightarrow{\cong} C\ell(T^*M) \to \operatorname{End}\$ \ ,$$

where the first map is the bundle isomorphism induced by the vector space isomorphism between the exterior and Clifford algebras, and the second map is induced from the action of the Clifford algebra $C\ell(1,10)$ on the spinor representation S of $\mathrm{Spin}(1,10)$. In signature $(1,10)$ one has the algebra isomorphism

$$C\ell(1,10) \cong \mathrm{Mat}(32,\mathbb{R}) \oplus \mathrm{Mat}(32,\mathbb{R}) \ ,$$

hence the map $C\ell(1,10) \to \operatorname{End} S$ has kernel. In other words, the map c defined above involves a choice. This comes down to choosing whether the (normalised) volume element in $C\ell(1,10)$ acts as $\pm$ the identity. We will assume that a choice has been made once and for all.

Definition 2. *We say that a classical solution (M,g,F) is **supersymmetric** if there exists a nonzero spinor $\varepsilon \in \Gamma(\$)$ which is parallel with respect to the **supercovariant connection***

$$D : \Gamma(\$) \to \Gamma(T^*M \otimes \$)$$

defined, for all vector fields X, by

$$D_X\varepsilon = \nabla_X\varepsilon - \Omega_X(F)\varepsilon \ ,$$

where ∇ is the spin connection and $\Omega(F) : TM \to End\$$ is defined by

$$\Omega_X(F) = \tfrac{1}{12}c(X^\flat \wedge F) - \tfrac{1}{6}c(\imath_X F) \ ,$$

with $X^\flat$ the one-form dual to X.

A nonzero spinor ε which is parallel with respect to D is called a **Killing spinor**. This is a generalisation of the usual geometrical notion of Killing spinor (see, for example, [BFGK90]). The name is apt because Killing spinors are "square roots" of Killing vectors. Indeed, one has the following

Proposition 1. *Let ε_i, $i = 1,2$ be Killing spinors: $D\varepsilon_i = 0$. Then the vector field V defined, for all vector fields X, by*

$$g(V,X) = (\varepsilon_1, X \cdot \varepsilon_2)$$

is a Killing vector.

There is a vast literature on supersymmetric solutions of eleven-dimensional supergravity, but so far very few results of a general nature. This problem comes down to studying the supercovariant connection D. Alas, D is not induced from

[1] As David Calderbank likes to remind me, there is big money in spin geometry.

a connection on the tangent bundle and in fact, it does not even preserve the symplectic structure. In fact, one has the following

Proposition 2. *The holonomy of D is generically* $\mathrm{GL}(32, \mathbb{R})$.

2. Kaluza–Klein reduction and type IIA supergravity

Suppose that (M, g, F) is a classical solution of eleven-dimensional supergravity admitting a free circle (or $\mathbb{R}$) action leaving g and F invariant. Let ξ denote the Killing vector generating this action. We will assume that ξ is spacelike, so that its norm is everywhere positive. Let N denote the space of orbits. For definiteness we can consider the case of a free circle action. Let $\pi : M \to N$ be the canonical projection sending a point in M to the (unique) orbit it belongs to. For every $m \in M$, the tangent space to M at m splits into vertical and horizontal subspaces:

$$T_m M = \mathcal{V}_m \oplus \mathcal{H}_m \ ,$$

where $\mathcal{V}_m$ is the one-dimensional subspace spanned by $\xi(m)$ and $\mathcal{H}_m = \mathcal{V}_m^\perp$ is its perpendicular complement. The projection π_* defines an isomorphism $\mathcal{H}_m \cong T_{\pi m} N$ and there is a unique metric h on N for which this is also an isometry.

The horizontal distribution $\mathcal{H}$ defines a one-form ω such that $\ker \omega = \mathcal{H}$ and normalised so that $\omega(\xi) = 1$. Introducing a coordinate θ adapted to the circle action, we have $\xi = \partial_\theta$ and $\omega = d\theta + A$, where A is a horizontal one-form called the **RR one-form potential**. Its field-strength pulls back to the curvature $d\omega$ of the principal connection, which is both horizontal and invariant, hence basic.

Finally, the metric on the fibres is described by a function Φ on N, called the **dilaton**. In terms of these data, the eleven-dimensional metric can be written as

$$g = \pi^* h + e^{\pi^* \Phi} \omega \otimes \omega \ . \tag{3}$$

Similarly we can decompose the four-form F as follows

$$F = \omega \wedge \imath_\xi F + K \ ,$$

where $\imath_\xi K = 0$. It follows from the fact that F is closed and invariant, that $\imath_\xi F$ and K are basic. Therefore there are forms $H \in \Omega^3(N)$ and $G \in \Omega^4(N)$ on N such that

$$\imath_\xi F = \pi^* H \qquad \text{and} \qquad K = \pi^* G \ ,$$

whence

$$F = \omega \wedge \pi^* H + \pi^* G \ . \tag{4}$$

The closed 3-form H is called the **NSNS 3-form** and G is called the **RR 4-form field-strength**.

The data (N, h, Φ, H, A, G) is then a solution of the equations of motion of ten-dimensional type IIA supergravity theory. These equations are obtained from equations (1) and (2) by simply inserting the expressions (3) for the metric and (4) for the four-form.

The data (N, h, Φ, H) defines the **common sector** of type II supergravity in ten dimensions. In this context, the NSNS 3-form H can be interpreted as the torsion three-form of a metric connection on (N, h). This gives rise to a variety of torsioned geometries discussed at this conference by Friedrich and Papadopoulos.

How about supersymmetry? The circle action lifts to an action on the spinor bundle, which is infinitesimally generated by the **spinorial Lie derivative** introduced by Lichnerowicz. If ε is any spinor, then

$$\mathcal{L}_\xi \varepsilon = \nabla_\xi \varepsilon + \tfrac{1}{4} c(d\xi^\flat)\varepsilon \ .$$

An invariant Killing spinor

$$D_X \varepsilon = 0 \qquad \text{and} \qquad \mathcal{L}_\xi \varepsilon = 0$$

gives rise to a IIA Killing spinor, and viceversa (at least locally). Notice that the IIA Killing spinor equation has a purely algebraic component, namely

$$(\mathcal{L}_\xi - \nabla_\xi)\, \varepsilon = 0 \ ,$$

called the **dilatino equation**.

A useful principle in this game is the fact that supersymmetric solutions to IIA supergravity can be lifted to invariant supersymmetric solutions of eleven-dimensional supergravity. This procedure does not involve any loss of supersymmetry; although it may sometimes result in accidental supersymmetry in eleven dimensions. This means that it is often more convenient to work with eleven-dimensional supergravity than with IIA supergravity.

3. Maximal supersymmetry

Definition 3. *A classical solution of eleven-dimensional or type IIA supergravity is called **maximally supersymmetric** if the space of Killing spinors is of maximal dimension, namely* 32.

If (M, g, F) is a maximally supersymmetric classical solution of eleven-dimensional supergravity, the supercovariant connection D is flat. Solving the flatness equations of the supercovariant connection one arrives at the following theorem.

Theorem 1 ([KG84, FOP]). *Let (M, g, F) be a maximally supersymmetric solution of eleven-dimensional supergravity. Then (M, g) has constant scalar curvature s, and depending on the value of s one has the following classification:*

- *If* $s > 0$, *then* (M, g) *is locally isometric to* $\mathrm{AdS}_7 \times S^4$, *where* AdS_7 *is the lorentzian space-form of constant negative curvature* $-7s$ *and* S^4 *is the round sphere with constant positive curvature* $8s$; *and* $F = \sqrt{6s}\,\mathrm{dvol}(S^4)$.
- *If* $s < 0$, *then* (M, g) *is locally isometric to* $\mathrm{AdS}_4 \times S^7$, *where* AdS_4 *has constant negative curvature* $8s$ *and* S^7 *is the round sphere with constant positive curvature* $-7s$; *and* $F = \sqrt{-6s}\,\mathrm{dvol}(\mathrm{AdS}_4)$.
- *If* $s = 0$ *there are two possibilities:*
  - (M, g) *is flat and* $F = 0$; *or*
  - (M, g) *is locally isometric an indecomposable lorentzian symmetric space with solvable transvection group, and* $F \neq 0$.

The classification of symmetric spaces in indefinite signature is hindered by the fact that there is no splitting theorem saying that if the holonomy representation is reducible, the space is locally isometric to a product. In fact, local splitting implies both reducibility *and* a nondegeneracy condition on the factors [Wu64]. This means that one has to take into account reducible yet indecomposable holonomy representations. The general semi-riemannian case is still open, but indecomposable lorentzian symmetric spaces were classified by Cahen and Wallach [CW70] more than thirty years ago. At least for dimension $n \geq 3$, there are three types of indecomposable lorentzian symmetric spaces:

- dS_n (de Sitter space), the space form with constant positive curvature,
- AdS_n (anti de Sitter space), the space form with constant negative curvature, and
- an $(n - 3)$-dimensional family of "pp-waves" with solvable transvection group.

It is precisely this last class of symmetric spaces which, for $n = 11$, describes the gravitational part of a maximally supersymmetric solution of eleven-dimensional supergravity.

4. The Cahen–Wallach pp-waves

The Cahen–Wallach n-dimensional pp-waves are constructed as follows. Let V be a real vector space of dimension $n-2$ endowed with a euclidean structure $\langle -, - \rangle$. Let V^* denote its dual. Let Z be a real one-dimensional vector space and Z^* its dual. We will identify Z and Z^* with $\mathbb{R}$ via canonical dual bases $\{e_+\}$ and $\{e_-\}$, respectively. Let $A \in S^2V^*$ be a symmetric bilinear form on V. Using the euclidean structure on V we can associate with A an endomorphism of V also denoted A:

$$\langle A(v), w \rangle = A(v, w) \qquad \text{for all } v, w \in V.$$

We will also let $\flat : V \to V^*$ and $\sharp : V^* \to V$ denote the musical isomorphisms associated to the euclidean structure on V.

Let $\mathfrak{g}_A$ be the Lie algebra with underlying vector space $V \oplus V^* \oplus Z \oplus Z^*$ and with Lie brackets

$$\begin{aligned} [e_-, v] &= v^\flat \\ [e_-, \alpha] &= A(\alpha^\sharp) \\ [\alpha, v] &= A(v, \alpha^\sharp) e_+ \ , \end{aligned} \tag{5}$$

for all $v \in V$ and $\alpha \in V^*$. All other brackets not following from these are zero. The Jacobi identity is satisfied by virtue of A being symmetric. Notice that since its second derived ideal is central, $\mathfrak{g}_A$ is (three-step) solvable.

Notice that $\mathfrak{k}_A = V^*$ is an abelian Lie subalgebra, and its complementary subspace $\mathfrak{p}_A = V \oplus Z \oplus Z^*$ is acted on by $\mathfrak{k}_A$. Indeed, it follows easily from (5) that

$$[\mathfrak{k}_A, \mathfrak{p}_A] \subset \mathfrak{p}_A \qquad \text{and} \qquad [\mathfrak{p}_A, \mathfrak{p}_A] \subset \mathfrak{k}_A \ ,$$

whence $\mathfrak{g}_A = \mathfrak{k}_A \oplus \mathfrak{p}_A$ is a symmetric split. Lastly, let $B \in \left(S^2\mathfrak{p}_A^*\right)^{\mathfrak{k}_A}$ denote the invariant symmetric bilinear form on $\mathfrak{p}_A$ defined by

$$B(v, w) = \langle v, w \rangle \qquad \text{and} \qquad B(e_+, e_-) = 1 \ ,$$

for all $v, w \in V$. This defines on $\mathfrak{p}_A$ a $\mathfrak{k}_A$-invariant lorentzian inner product of signature $(1, n-1)$.

We now have the required ingredients to construct a (lorentzian) symmetric space. Let G_A denote the connected, simply-connected Lie group with Lie algebra $\mathfrak{g}_A$ and let K_A denote the Lie subgroup corresponding to the subalgebra $\mathfrak{k}_A$. The lorentzian inner product B on $\mathfrak{p}_A$ induces a lorentzian metric g on the space of cosets

$$M_A = G_A / K_A \ ,$$

turning it into a symmetric space.

Proposition 3 ([CW70])**.** *The metric on M_A defined above is indecomposable if and only if A is nondegenerate. Moreover, M_A and $M_{A'}$ are isometric if and only if A and A' are related in the following way:*

$$A'(v, w) = cA(Ov, Ow) \qquad \textit{for all } v, w \in V,$$

for some orthogonal transformation $O : V \to V$ and a positive scale $c > 0$.

From this result one sees that the moduli space $\mathcal{M}_n$ of indecomposable such metrics in n dimensions is given by

$$\mathcal{M}_n = \left(S^{n-3} - \Delta\right) / \mathfrak{S}_{n-2} \ ,$$

where

$$\Delta = \left\{(\lambda_1, \dots, \lambda_{n-2}) \in S^{n-3} \subset \mathbb{R}^{n-2} \mid \lambda_1 \cdots \lambda_{n-2} = 0\right\}$$

is the singular locus consisting of eigenvalues of degenerate A's, and $\mathfrak{S}_{n-2}$ is the symmetric group in $n-2$ symbols, acting by permutations on $S^{n-3} \subset \mathbb{R}^{n-2}$.

A remarkable fact which is still not properly understood is the following

Minor Miracle 1. *There is a unique point $A_* \in \mathcal{M}_{11}$ for which (M_{A_*}, g) is the gravitational part of a maximally supersymmetric solution of eleven-dimensional supergravity.*

Explicitly, we can write this solution as

$$\begin{aligned} g &= 2dx^+dx^- - \left(\sum_{i=1}^{3}(x^i)^2 + \tfrac{1}{4}\sum_{i=4}^{9}(x^i)^2\right)\left(dx^-\right)^2 + \sum_{i=1}^{9}\left(dx^i\right)^2 \\ F &= 3dx^- \wedge dx^1 \wedge dx^2 \wedge dx^3 \ . \end{aligned}$$

The isometry group of the metric is not just G_A but the larger group

$$G_A \rtimes (\mathrm{SO}(3) \times \mathrm{SO}(6)) \ ,$$

where $\mathrm{SO}(3) \times \mathrm{SO}(6) \subset \mathrm{SO}(9)$ acts on G_A by exponentiating the restriction of the natural action of $\mathrm{SO}(9)$ on $V \oplus V^* = \mathbb{R}^9 \oplus \mathbb{R}^9$. Intriguingly, the dimension of the isometry group is 38, which is the same as the dimension of the isometry groups of the other maximally supersymmetric solutions of AdS-type. This deserves to be better understood.

5. Maximal supersymmetry in type IIA supergravity

Let (N, h, Φ, H, A, G) be a maximally supersymmetric solution of type IIA supergravity. Let (M, g, F), where g and F are given by (3) and (4) respectively, denote the corresponding circle-invariant[2] solution of eleven-dimensional supergravity. Since no supersymmetry is lost in this process, (M, g, F) is also maximally supersymmetric. Moreover, the action of the Killing vector ξ must leave invariant *all* Killing spinors.

It is then a matter of going through the maximally supersymmetric solutions classified in Theorem 1 and checking whether there exists a Killing vector which leaves all Killing spinors invariant. For the AdS solutions, it follows from the semisimplicity of the isometry algebra that no such Killing vector exists. It was shown in [FOP01], albeit in a different context, that neither does the maximally supersymmetric pp-wave solution admit such Killing vectors. Finally, for the flat solution with $F = 0$, we can let ξ be any translation along a spacelike direction.

[2]This is a local result – the action is only infinitesimal, hence we cannot distinguish between circle or $\mathbb{R}$ actions.

The resulting IIA solution is such that (N, h) is flat, the dilaton is constant and all other fields vanish. In summary, we have proven the following.

Theorem 2. *The only maximally supersymmetric solution of type IIA supergravity is a flat spacetime with constant dilaton and vanishing* (A, H, G).

Since only the common sector fields are nonzero, this solution is also a maximally supersymmetric solution of type IIB supergravity. However in this case we know at least another class of maximally supersymmetric solutions, with geometry $\mathrm{AdS}_5 \times S^5$. The classification of maximally supersymmetric solutions of type IIB supergravity is work in progress [FOP].

Acknowledgments

It is a pleasure to thank George Papadopoulos for the ongoing collaboration on this project, and David Calderbank and Michael Singer for conversations. I would like to express my thanks to Dmitri Alekseevsky, Vicente Cortés, Chand Devchand and Toine Van Proeyen for the invitation to participate in the workshop, and to the DFG for their financial support. I am a member of EDGE, Research Training Network HPRN-CT-2000-00101, supported by The European Human Potential Programme.

References

[BFGK90] H Baum, T Friedrich, R Grunewald, and I Kath, *Twistor and Killing spinors on riemannian manifolds*, Seminarberichte, no. 108, Humboldt-Universität, Berlin, 1990.

[CJS78] E Cremmer, B Julia, and J Scherk, *Supergravity in eleven dimensions*, Phys. Lett. **76B** (1978), 409–412.

[CW70] M Cahen and N Wallach, *Lorentzian symmetric spaces*, Bull. Am. Math. Soc. **76** (1970), 585–591.

[FOP] JM Figueroa-O'Farrill and G Papadopoulos, *Maximal supersymmetry in supergravity*, in preparation.

[FOP01] JM Figueroa-O'Farrill and G Papadopoulos, *Homogeneous fluxes, branes and a maximally supersymmetric solution of M-theory*, J. High Energy Phys. **06** (2001), 036, `arXiv:hep-th/0105308`.

[KG84] J Kowalski-Glikman, *Vacuum states in supersymmetric Kaluza-Klein theory*, Phys. Lett. **134B** (1984), 194–196.

[Nah78] W Nahm, *Supersymmetries and their representations*, Nucl. Phys. **B135** (1978), 149–166.

[Wu64] H Wu, *On the de Rham decomposition theorem*, Illinois J. Math. **8** (1964), 291–311.

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