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CONFORMAL KILLING SPINORS AND SPECIAL GEOMETRIC STRUCTURES IN LORENTZIAN GEOMETRY - A SURVEY

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ABSTRACT. This paper is a survey on special geometric structures that admit conformal Killing spinors based on lectures, given at the "Workshop on Special Geometric Structures in String Theory", Bonn, September 2001 and at ESI, Wien, November 2001. We discuss the case of Lorentzian signature and explain which geometries occur up to dimension 6.

CONTENTS

1. INTRODUCTION

A classical object in differential geometry are conformal Killing fields. These are by definition infinitesimal conformal symmetries i.e. the flow of such vector fields preserves the conformal class of the metric. The number of linearly independent conformal Killing fields measures the degree of conformal symmetry of the manifold. This number is bounded by $\frac{1}{2}(n+1)(n+2)$, where n is the dimension of the manifold. It is the maximal one if the manifold is conformally flat. S. Tachibana and T. Kashiwada (cf. [TK69], [Kas68]) introduced a generalisation of conformal Killing fields, the conformal Killing forms (or twistor forms). Conformal Killing forms are solutions of a conformally invariant twistor type equation on differential forms. They were studied in General Relativity mainly from the local viewpoint in order to integrate the equation of motion (e.g. [PW70]), furthermore they were used to obtain symmetries of field equations ([BC97], [BCK97]). Recently, U. Semmelmann ([Sem01]) started to discuss global properties of conformal Killing forms in Riemannian geometry. Another generalisation of conformal Killing vectors is that of conformal Killing spinors (or twistor spinors), which are solutions of the conformally invariant twistor equation on spinors introduced by R. Penrose in General Relativity (cf. [PR86]). Whereas conformal Killing fields are classical symmetries, conformal Killing spinors define infinitesimal symmetries on supermanifolds (cf. [ACDS98]). Special kinds of such spinors, parallel and special Killing spinors, occur in supergravity and string theories. In 1989 A. Lichnerowicz and T. Friedrich started a systematic study of twistor spinors in conformal Riemannian geometry. Whereas the global structure of Riemannian manifolds admitting conformal Killing spinors is quite well understood (cf. e.g. [Lic88b], [Lic88a], [Lic89], [Fri89] [Lic90], [BFGK91], [Hab90], [Hab93], [Hab94], [Hab96], [KR94], [KR96], [KR97], [KR98]), the state of art in its origin, Lorentzian geometry, is far from being satisfactory. J. Lewandowski ([Lew91]) described the local normal forms of 4-dimensional space-times with zero free twistor spinors. His results indicated that there are interesting relations between conformal Killing spinors, different global contact structures and Lorentzian geometry, that should be discovered. In the present survey we discuss some of these structures. Since there is some interest in several kinds of Killing spinors on Lorentzian manifolds by physicists working in string theory, our investigations and methods, motivated mainly from geometry, could be, perhaps, of some use in physics.

2. Conformally invariant operators on spinors

In this section we will define the kind of spinor fields we are interested in and discuss two applications in Riemannian geometry.

Let (M^n, g) be a semi-Riemannian spin manifold of dimension $n \geq 3$. We denote by S the spinor bundle and by $\mu : T^*M \otimes S \to S$ the Clifford multiplication. The

1-forms with values in the spinor bundle decompose into two subbundles

$$T^*M \otimes S = V \oplus Tw,$$

where V, being the orthogonal complement to the "twistor bundle" $Tw := Ker\mu$, is isomorphic to S. Usually, we identify TM and T^*M using the metric g.
We obtain two differential operators of first order by composing the spinor derivative ∇^S with the orthogonal projections onto each of these subbundles,
the *Dirac operator* D

$$D : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) = \Gamma(S \oplus Tw) \xrightarrow{pr_S} \Gamma(S)$$

and the *twistor operator* P

$$P : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) = \Gamma(S \oplus Tw) \xrightarrow{pr_{Tw}} \Gamma(Tw).$$

Locally, these operators are given by the following formulas

$$\begin{aligned} D\varphi &= \sum_{i=1}^{n} \sigma^i \cdot \nabla^S_{s_i}\varphi \\ P\varphi &= \sum_{i=1}^{n} \sigma^i \otimes (\nabla^S_{s_i}\varphi + \frac{1}{n} s_i \cdot D\varphi), \end{aligned}$$

where $(s_1, \ldots, s_n)$ is a local orthonormal basis, $(\sigma^1, \ldots, \sigma^n)$ its dual and $\cdot$ denotes the Clifford multiplication.

Both operators are conformally covariant. More exactly, if $\tilde{g} = e^{2\sigma} g$ is a conformal change of the metric, the Dirac and the twistor operator satisfy

$$\begin{aligned} D_{\tilde{g}} &= e^{-\frac{n+1}{2}\sigma} D_g e^{\frac{n-1}{2}\sigma} \\ P_{\tilde{g}} &= e^{-\frac{\sigma}{2}} P_g e^{-\frac{\sigma}{2}}. \end{aligned}$$

The spinor fields we are interested in are the solutions of the conformally invariant equations $D\varphi = 0$ and $P\varphi = 0$. A spinor field $\varphi \in \Gamma(S)$ is called

harmonic spinor	$\Longleftrightarrow$	$D\varphi = 0$
twistor spinor (conformal Killing spinor)	$\Longleftrightarrow$	$P\varphi = 0$
parallel spinor	$\Longleftrightarrow$	$\nabla^S\varphi = 0$
geometric Killing spinor with Killing number λ	$\Longleftrightarrow$	$\nabla^S_X\varphi = \lambda X \cdot \varphi,\ \lambda \in \mathbb{C}$

The local formula for the twistor operator shows another characterisation of a twistor spinor: A spinor field $\varphi \in \Gamma(S)$ is a twistor spinor if and only if

$$\boxed{\nabla^S_X\varphi + \tfrac{1}{n} X \cdot D\,\varphi = 0.}$$

Obviously, parallel and Killing spinors are special kinds of twistor spinors. Each twistor spinor that satisfies the Dirac equation $D\varphi = \mu\varphi$ is a Killing spinor. On Riemannian manifolds each twistor spinor without zeros can be transformed by a conformal change of the metric into a Killing spinor.
Twistor spinors appeared naturally in Riemannian geometry. For a motivation, let me explain two well known examples, where it was usefull to know structure results for manifolds with special kinds of twistor spinors.

1. Eigenvalue estimates for the Dirac operator on closed Riemannian manifolds with positive scalar curvature R (limiting case)

The Dirac operator D on a closed Riemannian spin manifold is elliptic and essentially selfadjoint. Hence, its spectrum contains only real eigenvalues of finite multiplicity. One is interested in lower estimates for the smallest eigenvalue. It is easy to prove that the Dirac operator D and the twistor operator P are related by

$$D^2 = \frac{n}{n-1}(P^*P + \frac{1}{4}R) \ .$$

If λ is an eigenvalue of D and $D\varphi = \lambda\varphi$, integration of

$$\langle D^2\varphi, \varphi\rangle = \frac{n}{n-1}\left(\langle P^*P\varphi, \varphi\rangle + \frac{1}{4}\langle R\varphi, \varphi\rangle\right)$$

yields the estimate ([Fri80])

$$\lambda^2 \geq \frac{1}{4}\frac{n}{n-1} \cdot \min_{x\in M} R(x), \tag{1}$$

where equality holds if and only if the eigenspinor φ is a twistor spinor. Hence, the discussion of the limiting case (i.e. equality holds in (1)) leads to the problem to find all Riemannian structures which admit real Killing spinors (i.e. Killing spinors with real Killing number). This problem was studied by several authors [1] between 1980 and 1993 and is now essentially solved. The key observation in the understanding of geometric Killing spinors was made by Ch. Bär in 1993 ([Bär93]). He remarked, that a Riemannian manifold (M^n, g) has real Killing spinors if and only if the metric cone $C_+(M) = (\mathbb{R}^+ \times M, dt^2 + t^2 g)$ has parallel spinors. The cone is irreducible in case M^n is not the standard sphere. Then, to the cone, one can apply a result of McK. Wang ([Wan89]) that describes the possible reduced holonomy groups Hol_0 of irreducible manifolds with parallel spinors and is able to derive the corresponding geometries of M. The result is given in the following

[1] Friedrich, Grunewald, Kath, Hijazi, Baum, Bär, Duff, Nillson, Pope, Nieuwenhuizen and many others

table. A compact simply connected Riemannian manifold (M^n, g) admits real Killing spinors if and only if

$\mathrm{Hol}(C_+(M))$	$\mathrm{SU}(\frac{n}{2})$	$\mathrm{Sp}(\frac{n}{4})$	G_2	Spin7	1
M^n	Einstein-Sasaki	3-Sasaki	nearly Kähler non Kähler	nearly parallel G_2-structure	S^n

2. Rigidity Theorems

As a second example I want to explain the Rigidity Theorem for asymptotically locally hyperbolic (ALH) manifolds, proved by Andersson and Dahl ([AD98]): An ALH manifold is a complete Riemannian manifold with an asymptotically locally hyperbolic end $\mathbb{H}^n/\Gamma$. Andersson and Dahl proved that each ALH manifold of dimension n with scalar curvature R bounded by $R \geq -n(n-1)$ is isometric to the hyperbolic space $\mathbb{H}^n$. The hyperbolic space $\mathbb{H}^n$ admits imaginary Killing spinors (i.e. the Killing number is purely imaginary). Using such Killing spinor on the asymptotic end of the ALH manifold one can construct an imaginary Killing spinor on M^n itself. A complete Riemannian spin manifold (M^n, g) admits imaginary Killing spinors if and only if (M^n, g) is isometric to a warped product $(\mathbb{R} \times F, dt^2 + e^{t\mu} g_F)$, where (F, g_F) is a complete Riemannian manifold with parallel spinors ([Bau89]). But, being asymptotically locally hyperbolic, such a warped product has to be the hyperbolic space $\mathbb{H}^n$.

Whereas twistor spinors on Riemannian manifolds are rather well studied [2], the state of art in Lorentzian geometry, where the twistor equation originally came from, is far from being satisfactory. For that reason, in this survey we want to consider twistor spinors in Lorentzian geometry and discuss the following problems:

(1) *Which Lorentzian geometries admit twistor spinors?*
(2) *How the properties of twistor spinors are related to the geometric structures where they can occur?*

3. Basic facts on twistor spinors

There is a fundamental difference between the two conformally invariant equations $D\varphi = 0$ and $P\varphi = 0$. Whereas the dimension of the space of harmonic spinors

[2] e.g., by Lichnerowicz, Friedrich, K.Habermann, Kühnel, Rademacher

can become arbitrary large, the dimension of the space of twistor spinors is always bounded by $2\,\mathrm{rank}S$. It holds even more: the twistor equation $P\varphi = 0$ can be viewed as a parallelity equation with respect to a suitable covariant derivative in a suitable bundle. Let us explain this fact (well known for a long time, cf. [PR86], [BFGK91]) more in detail. Let $\mathcal{R}$ and W denote the curvature tensor and the Weyl tensor of (M^n, g), respectively, considered as selfadjoint maps on the space of 2-forms $\mathcal{R}, W : \Lambda^2 M \to \Lambda^2 M$. Ric denotes the Ricci tensor of (M^n, g) considered here as $(1,1)$-tensor or as $(2,0)$-tensor as it is needed. In conformal geometry there are 2 further curvature tensors that play an important role, the Rho tensor K

$$K(X) := \frac{1}{n-2}\Big(\frac{R}{2(n-1)}X - \mathrm{Ric}(X)\Big) \quad , \quad X \in TM$$

and the Cotton-York tensor

$$C(X,Y) := (\nabla_X K)(Y) - (\nabla_Y K)(X), \quad X, Y \in TM.$$

Let us consider the double spinor bundle $E = S \oplus S$ with the following covariant derivative

$$\nabla^E_X := \begin{pmatrix} \nabla^S_X & \frac{1}{n}X\cdot \\ -\frac{n}{2}K(X)\cdot & \nabla^S_X \end{pmatrix}$$

The curvature of this derivative is given by

$$\mathcal{R}^{\nabla^E}(X,Y)\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \frac{1}{2}\begin{pmatrix} W(X\wedge Y)\cdot\varphi \\ W(X\wedge Y)\cdot\psi - nC(X,Y)\cdot\varphi \end{pmatrix} \tag{2}$$

(cf. [BFGK91], chap. 1). Using integrability conditions for twistor spinors one obtains the following correspondence between twistor spinors and parallel sections in (E, ∇^E):

Proposition 3.1. *Let (M^n, g) be a semi-Riemannian spin manifold and $\varphi \in \Gamma(S)$. Then*

(1) *If φ is a twistor spinor, then* $\nabla^E \begin{pmatrix} \varphi \\ D\varphi \end{pmatrix} = 0.$

(2) *If* $\nabla^E \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = 0$, *then φ is a twistor spinor and $\psi = D\varphi$.*

A semi-Riemannian manifold is (locally) conformally flat if and only if $W = 0$ (hence $C = 0$) in dimension $n \geq 4$ or $C = 0$ in dimension $n = 3$, where $W = 0$ is always satisfied. Therefore, by formula (2) the spin manifold (M^n, g) is conformally flat if and only if the curvature $\mathcal{R}^{\nabla^E}$ of ∇^E vanishes. Together with Proposition 3.1 one obtains the maximal possible number of linearly independent twistor spinors, which is attained on conformally flat manifolds as in the case of conformal vector fields.

Proposition 3.2.

(1) *The dimension of the space of twistor spinors is a conformal invariant and bounded by*
$$\dim\ker P \leq 2\,\mathrm{rank} S = 2^{[\frac{n}{2}]+1} =: d_n.$$

(2) *If* $\dim\ker P = d_n$*, then* (M^n, g) *is conformally flat.*

(3) *If* (M^n, g) *is simply connected and conformally flat, then* $\dim\ker P = d_n$.

For example, all simply connected space forms $\mathbb{R}^n_k, \tilde{\mathbb{H}}^n_k, \tilde{S}^n_k$ admit the maximal number of linearly independent twistor spinors.

4. Twistor spinors on Lorentzian spin manifolds

Now, we restrict our attention to the case of Lorentzian signature $(-,+\ldots+)$. Let (M^n, g) be an oriented and time-oriented Lorentzian spin manifold. On the spinor bundle S there exists an indefinite non-degenerate inner product $\langle\cdot,\cdot\rangle$ such that

$$\begin{aligned} \langle X\cdot\varphi,\psi\rangle &= \langle\varphi, X\cdot\psi\rangle \qquad \text{and} \\ X(\langle\varphi,\psi\rangle) &= \langle\nabla^S_X\varphi,\psi\rangle + \langle\varphi,\nabla^S_X\psi\rangle, \end{aligned}$$

for all vector fields X and all spinor fields φ, ψ. Each spinor field $\varphi \in \Gamma(S)$ defines a vector field V_φ on M, the so-called Dirac current, by

$$g(V_\varphi, X) := -\langle X\cdot\varphi,\varphi\rangle. \tag{3}$$

A motivation to call twistor spinors conformal Killing spinors is the

Proposition 4.1. *Let* φ *be a spinor field on a Lorentzian spin manifold* (M^n, g) *with Dirac current* V_φ*. Then*

(1) V_φ *is causal and future-directed.*

(2) *The zero sets of* φ *and* V_φ *coincide.*

(3) *If* φ *is a twistor spinor,* V_φ *is a conformal Killing field.*

Now, let us discuss 3 types of special Lorentzian geometries that admit conformal Killing spinors.

4.1. Brinkman spaces with parallel spinors.

A Lorentzian manifold is called *Brinkman space* if it admits a non-trivial lightlike parallel vector field. Let us consider two examples of such spaces.

Example 1: pp-manifolds.

A *pp-manifold* is a Brinkman space with the additional curvature property

$$\text{Trace}_{(3,5),(4,6)}\mathcal{R}\otimes\mathcal{R}=0.$$

Equivalently, pp-manifolds can be characterised as those Lorentzian manifolds (M^n,g), where the metric has the following local normal form depending only on one function f of $(n-1)$ variables

$$g=dt\,ds+f(s,x_1,\dots,x_{n-2})ds^2+\sum_{i=1}^{n-2}dx_i^2.$$

(cf. [Sch74]). In terms of holonomy, pp-manifolds can be characterised as those Lorentzian manifolds for which the holonomy algebra is contained in the abelian ideal $\mathbb{R}^{n-2}$ of the algebra $(\mathbb{R}\oplus\mathfrak{so}(n-2))\ltimes\mathbb{R}^{n-2}\subset\mathfrak{so}(1,n-1)$, cf. [Lei01a]. Using the latter fact one can easily prove that for each simply connected pp-manifold

$$\dim\ker P\geq\frac{d_n}{4}.$$

Furthermore, on pp-manifolds each twistor spinor is parallel. An important example of geodesically complete pp-manifolds are the Lorentzian symmetric spaces with solvable transvection group which occur also as a special model for a certain string equation in 10 + 1-dimension (cf. the talk of J. Figueroa O'Farrill at this Workshop or [FOP01], [BFOHP01]).

Example 2: Brinkman spaces with special Kähler flag.

Let (M^n,g) be a Brinkman space with the lightlike parallel vector field V. V defines a flag $\mathbb{R}V\subset V^\perp\subset TM$ in TM, where $V^\perp=\{Y\in TM\,|\,g(V,Y)=0\}$. We equip the bundle $E:=V^\perp/\mathbb{R}V$ with the positive definite inner product $\tilde g$ induced by g and the metric connection $\tilde\nabla$ induced by the Levi-Civita connection of g. We call the flag $\mathbb{R}V\subset V^\perp\subset TM$ a *special Kähler flag*, if the bundle E admits an orthogonal almost complex structure $J:E\to E$, $J^2=-id$, such that $\text{Trace}\,(J\circ\mathcal{R}^{\tilde\nabla}(X,Y))=0$ for all $X,Y\in TM$. It was proved by I. Kath in [Kat99] that (M^{2m},g) is a Brinkman space with special Kähler flag if and only if (M^{2m},g) has pure parallel spinors.

4.2. **Twistor spinors on Lorentzian Einstein Sasaki manifolds.** An odd-dimensional Lorentzian manifold $(M^{2m+1},g;\xi)$ equipped with a vector field ξ is called *Lorentzian Sasaki manifold* if

(1) ξ is a timelike Killing field with $g(\xi,\xi)=-1$.

(2) The map $J := -\nabla\xi : TM \to TM$ satisfies

$$\begin{aligned} J^2X &= -X - g(X,\xi)\xi \qquad \text{and} \\ (\nabla_X J)(Y) &= -g(X,Y)\xi + g(Y,\xi)X. \end{aligned}$$

Let us consider the metric cone $C_-(M) := (\mathbb{R}^+ \times M, -dt^2 + t^2 g)$ with timelike cone axis over (M,g). The cone metric has signature $(2,2m)$. Then the following relations between properties of M and those of its cone are easy to verify

$(M^{2m+1}, g; \xi)$		cone $C_-(M)$
Lorentzian Sasaki	$\Longleftrightarrow$	(pseudo) Kähler
Lorentzian Einstein Sasaki ($R<0$)	$\Longleftrightarrow$	Ricci-flat and (pseudo) Kähler
Lorentzian Einstein Sasaki ($R<0$)	$\Longleftrightarrow$	$\mathrm{Hol}_0(C_-(M)) \subset SU(1,m)$

The standard example for regular Lorentzian Einstein Sasaki manifolds are S^1-bundles over Riemannian Kähler Einstein spaces of negative scalar curvature: Let (X^{2m}, h) be a Riemannian Kähler Einstein spin manifold of scalar curvature $R_X < 0$ and let $(M^{2m+1}, \pi, X; S^1)$ denote the S^1-principal bundle associated to the square root $\sqrt{\Lambda^{m,0}X}$ of the canonical bundle of X given by the spin structure. Furthermore, let A be the connection on M induced by the Levi-Civita connection of (X,h). Then

$$g := \pi^* h - \frac{16m}{(m+1)R_X} A \odot A$$

defines a Lorentzian Einstein Sasaki metric on the spin manifold M^{2m+1}. Lorentzian Einstein Sasaki manifolds admit a special kind of twistor spinors.

Proposition 4.2. *(*[Kat99], [Boh00]*) Let (M,g) be a simply connected Lorentzian Einstein Sasaki manifold. Then M is spin and admits a twistor spinor φ on M such that*

a) *V_φ is a timelike Killing field with $g(V_\varphi, V_\varphi) = -1$*

b) *$V_\varphi \cdot \varphi = -\varphi$*

c) *$\nabla_{V_\varphi}\varphi = -\frac{1}{2} i\varphi$.*

Conversselly, if (M,g) is a Lorentzian spin manifold with a twistor spinor satisfying a), b) and c). Then $\xi := V_\varphi$ is a Lorentzian Einstein Sasaki structure on (M,g).

4.3. **Twistor spinors on Fefferman spaces.** Fefferman spaces are Lorentzian manifolds which appear in the frame work of CR geometry. Let us first explain the necessary notations from CR geometry. Let N^{2m+1} be a smooth oriented manifold of odd dimension $2m+1$. A *CR structure on N* is a pair (H,J) where

1. $H \subset TM$ is a real $2m$-dimensional subbundle.
2. $J : H \to H$ is an almost complex structure on H, $J^2 = -Id$.
3. If $X, Y \in \Gamma(H)$, then $[JX, Y] + [X, JY] \in \Gamma(H)$ and

$$J([JX, Y] + [X, JY]) - [JX, JY] + [X, Y] \equiv 0 \text{ (integrability condition).}$$

Standard examples of CR manifolds are the following:

- Real hypersurfaces N of a complex manifold (M, J_M). The CR structure is given by $H := TN \cap J_M(TN)$, $J := J_M\big|_H$.
- Riemannian Sasaki manifolds (N, g, ξ). The CR structure is given by $H := \xi^\perp$ and $J := -\nabla\xi$.
- Heisenberg manifolds $\mathcal{H}e(m) = He(m)/\Gamma$, where $He(m)$ is the Heisenberg group of matrices

$$He(m) = \left\{ \begin{pmatrix} 1 & x^t & z \\ 0 & I_m & y \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x, y \in \mathbb{R}^m, z \in \mathbb{R} \right\}$$

  and Γ is a discrete lattice in $He(m)$. The Lie algebra of the Heisenberg group is of the form $\mathfrak{he}(m) = \text{span}(X_1, \ldots, X_m, Y_1, \ldots, Y_m, Z)$, where the only non-zero commutators are $[X_i, Y_i] = Z$, $i = 1, \ldots, m$. The CR-structure is given by $H = \text{span}(X_1, \ldots, X_m, Y_1, \ldots, Y_m)$ and $J(X_i) = Y_i$, $J(Y_i) = -X_i$.

Let (N, H, J) be a CR manifold. In order to define Fefferman spaces we fix a contact form $\theta \in \Omega^1(N)$ on N such that $\theta|_H = 0$. Let us denote by T the Reeb vector field of θ which is defined by the conditions $\theta(T) = 1$ and $T \lrcorner\, d\theta = 0$. In the following we suppose that the Levi form $L_\theta : H \times H \to \mathbb{R}$

$$L_\theta(X, Y) := d\theta(X, JY)$$

is positive definite. Then (N, H, J, θ) is called a *strictly pseudoconvex manifold.* The tensor field $g_\theta := L_\theta + \theta \odot \theta$ defines a Riemannian metric on N. There is a special metric covariant derivative on a strictly pseudoconvex manifold, the *Tanaka-Webster connection* $\nabla^W : \Gamma(TN) \to \Gamma(TN^* \otimes TN)$ uniquely defined by the conditions

$$\begin{aligned} \nabla^W g_\theta &= 0 \\ Tor^W(X, Y) &= L_\theta(JX, Y) \cdot T \\ Tor^W(T, X) &= -\frac{1}{2}([T, X] + J[T, JX]) \end{aligned}$$

for $X,Y\in\Gamma(H)$. This connection preserves the subbundle H and satisfies $\nabla^W J = 0$ and $\nabla^W T = 0$ (cf. [Tan75], [Web78]). Let us denote by $T_{10}\subset TN^{\mathbb{C}}$ the eigenspace of the complex extension of J on $H^{\mathbb{C}}$ to the eigenvalue i. Then L_θ extends to a hermitian form on T_{10} by

$$L_\theta(U,V) := -i\,d\theta(U,\bar V), \quad U,V\in T_{10}.$$

For a complex 2-form $\omega\in\Lambda^2 N^{\mathbb{C}}$ we denote by $\mathrm{trace}_\theta\omega$ the θ-trace of ω:

$$\mathrm{trace}_\theta\omega := \sum_{\alpha=1}^{m}\omega(Z_\alpha,\bar Z_\alpha),$$

where $(Z_1,\dots,Z_m)$ is an unitary basis of (T_{10},L_θ). Let $\mathcal{R}^W$ be the $(4,0)$-curvature tensor of the Tanaka-Webster connection ∇^W on the complexified tangent bundle of N

$$\mathcal{R}^W(X,Y,Z,V) := g_\theta(([\nabla^W_X,\nabla^W_Y]-\nabla^W_{[X,Y]})Z,\bar V).$$

and let us denote by

$$\mathrm{Ric}^W := \mathrm{trace}_\theta^{(3,4)} := \sum_{\alpha=1}^{m}\mathcal{R}^W(\cdot,\cdot,Z_\alpha,\bar Z_\alpha)$$

the *Tanaka-Webster Ricci curvature* and by $R^W := \mathrm{trace}_\theta\mathrm{Ric}^W$ the *Tanaka-Webster scalar curvature*. The Ricci curvature Ric^W is a $(1,1)$-form on N with $\mathrm{Ric}^W(X,Y)\in i\mathbb{R}$ for real vectors $X,Y\in TN$. The scalar curvature R^W is a real function.
Now, let us suppose that (N^{2m+1},H,J,θ) is a strictly pseudoconvex spin manifold. The spin structure of (N,g_θ) defines a square root $\sqrt{\Lambda^{m+1,0}N}$ of the canonical line bundle

$$\Lambda^{m+1,0}N := \{\omega\in\Lambda^{m+1}N^C \mid V\lrcorner\,\omega = 0\ \ \forall V\in\bar T_{10}\}.$$

We denote by (F,π,N) the S^1-principal bundle associated to $\sqrt{\Lambda^{m+1,0}N}$. We fix a connection form A on F and consider the corresponding decomposition of the tangent bundle $TF = ThF\oplus TvF = H^*\oplus\mathbb{R}T^*\oplus TvF$ into the horizontal and vertical part, where H^* and T^* denote the horizontal lifts of H and T. Then a Lorentzian metric h is defined by

$$h := \pi^*L_\theta - i\,c\,\pi^*\theta\odot A,$$

where c is a non-zero real number. The Fefferman metric is a metric of the latter type, where the choice of A and c is done in such a way that the conformal class of h does not depend on the contact form θ. Such a choice can be made with the connection

$$A_\theta := A^W - \frac{i}{4(m+1)}R^W\cdot\theta,$$

where A^W is the connection form of F defined by the Tanaka-Webster connection ∇^W. The curvature form of A^W is $\Omega^{A^W} = -\frac{1}{2}\mathrm{Ric}^W$. Then

$$h_\theta := \pi^* L_\theta - i\,\frac{8}{m+2}\pi^*\theta \odot A_\theta$$

is a Lorentzian metric such that the conformal class $[h_\theta]$ is an invariant of the CR-structure (N, H, J). The metric h_θ is S^1-invariant, the fibres of the S^1-bundle are lightlike. We call (F^{2m+2}, h_θ) with its canonically induced spin structure *Fefferman space of the strictly pseudoconvex spin manifold* (N, H, J, θ).

The Fefferman metric was first discovered by C. Fefferman for the case of strictly pseudoconvex hypersurfaces $N \subset \mathbb{C}^{m+1}$ ([Fef76]). The considerations of Fefferman were extended by Burns, Diederich and Snider ([BDS77]) and by Lee ([Lee86]) to the case of abstract (not nessarily embedded) CR-manifolds. A geometric characterization of Fefferman metrics was given by Sparling (cf. [Spa85], [Gra87]). Fefferman spaces admit a special kind of twistor spinors.

Proposition 4.3. *(*[Bau99]*) Let (N, H, J, θ) be a strictly pseudoconvex spin manifold with the Fefferman space (F, h_θ). Then there exist two linearly independent twistor spinors φ on (F, h_θ) such that*

a) *V_φ is a regular lightlike Killing field*

b) $V_\varphi \cdot \varphi = 0$

c) *$\nabla_{V_\varphi}\varphi = i\,c\,\varphi$ where $c \in \mathbb{R}\backslash\{0\}$.*

Conversely, if (M, g) is an even dimensional Lorentzian spin manifold with a twistor spinor satisfying a), b) and c), then there exists a strictly pseudoconvex spin manifold (N, H, J, θ) such that its Fefferman space is locally isometric to (M, g).

4.4. **Twistor spinors inducing lightlike Killing fields.** As we noticed in Proposition 4.1 each twistor spinor φ induces a causal conformal Killing field V_φ. Now we study the situation where V_φ is lightlike and Killing. The following result explains the role Fefferman spaces are playing in the set of all special geometries that admit twistor spinors.

Proposition 4.4. *(*[BL02]*) Let (M, g) be a Lorentzian spin manifold admitting a twistor spinor φ such that V_φ is lightlike and Killing. Then the function* $\mathrm{Ric}(V_\varphi, V_\varphi)$ *is constant and non-negative on M. Furthermore,*

(1) *$\mathrm{Ric}(V_\varphi, V_\varphi) > 0$ if and only if (M, g) is locally isometric to a Fefferman space.*

(2) $\mathrm{Ric}(V_\varphi, V_\varphi) = 0$ *if and only if* (M, g) *is locally conformal equivalent to a Brinkman space with parallel spinors.*

4.5. **Twistor spinors in dimension** $n \leq 6$**.** Finally, we describe all geometric structures of Lorentzian spin manifolds with twistor spinors that appear locally up to dimension 6.

Proposition 4.5. *(*[Lew91], [Lei01b], [BL02]*) Let* (M^n, g) *be a Lorentzian manifold with twistor spinors without zeros. Then* (M^n, g) *is locally conformal equivalent to one of the following kinds of Lorentzian structures.*

$n = 3$	*pp-manifold*
$n = 4$	*pp-manifold* *Fefferman space*
$n = 5$	*pp-manifold* *Lorentzian Einstein Sasaki manifold* $(\mathbb{R}, -dt^2) \times (N^4, h)$, *where* (N^4, h) *is a Riemannian Ricci-flat Kähler manifold*
$n = 6$	*pp-manifold* *Fefferman space* $\mathbb{R}^{1,1} \times (N^4, h)$, *where* (N^4, h) *is a Ricci-flat Kähler manifold* *Brinkman space with special Kähler flag*

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