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PARTIALLY-FLAT GAUGE FIELDS ON MANIFOLDS OF DIMENSION GREATER THAN FOUR

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Abstract. We describe two extensions of the notion of a self-dual connection in a vector bundle over a manifold M from $\dim M{=}4$ to higher dimensions. The first extension, Ω-self-duality, is based on the existence of an appropriate 4-form Ω on the Riemannian manifold M and yields solutions of the Yang-Mills equations. The second is the notion of half-flatness, which is defined for manifolds with certain Grassmann structure $T^{\mathbb{C}}M{\cong}E\otimes H$. In some cases, for example for hyper-Kähler manifolds M, half-flatness implies Ω-self-duality. A construction of half-flat connections inspired by the harmonic space approach is described. Locally, any such connection can be obtained from a free prepotential by solving a system of linear first order ODEs.

1. Self-duality and half-flatness

The Yang-Mills self-duality equations have played a very important role in field theory and in differential geometry. This talk is about ongoing work on two generalisations of the notion of *self-duality* from four to higher dimensional manifolds M. The first generalisation is the notion of

Ω-self-duality.

This is based on the existence of an appropriate 4-form Ω on a pseudo-Riemannian manifold (M,g). For $\Omega\in\Omega^4(M)$ we define a traceless symmetric endomorphism field $B_\Omega:\wedge^2T^*M\to\wedge^2T^*M$ by

$$B_\Omega\omega := *(*\Omega\wedge\omega)\ , \tag{1}$$

where $\omega\in\wedge^2T^*M$.

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Definition 1. *A 4-form $\Omega \in \Omega^4(M)$ on a pseudo-Riemannian manifold M is called* **appropriate** *if there exists a non-zero real constant eigenvalue λ of the endomorphism field B_Ω.*

We note that on a Riemannian manifold the eigenvalues of B_Ω are real for any 4-form Ω. We can now define a generalisation of the four dimensional notion of self-duality thus:

Definition 2. *Let Ω be an appropriate 4-form on a pseudo-Riemannian manifold (M,g) and $\lambda \neq 0 \in \mathbb{R}$. A connection ∇ in a vector bundle $\nu : W \to M$ is* **Ω-self-dual** *if its curvature F^∇ satisfies the linear algebraic system*

$$B_\Omega F^\nabla = \lambda F^\nabla \tag{2}$$

$$(d * \Omega) \wedge F^\nabla = 0\ . \tag{3}$$

It is easy to see that an Ω-self-dual connection ∇ is a Yang-Mills connection, i.e. it satisfies the Yang-Mills equation:

$$d^\nabla * F^\nabla = \pm\tfrac{1}{\lambda} d^\nabla \left(*\Omega \wedge F^\nabla \right) = \pm\tfrac{1}{\lambda} \left((d * \Omega) \wedge F^\nabla + *\Omega \wedge d^\nabla F^\nabla \right) = 0$$

in virtue of eq. (3) and the Bianchi identity $d^\nabla F^\nabla = 0$. This extension of self-duality is based on early work of [CDFN] on gauge fields in flat spaces. Interesting examples of manifolds with appropriate (and parallel) 4-forms are provided, for instance by manifolds with special holonomy groups. So for gauge fields on these manifolds, Ω-self-duality may be defined. Various examples have been discussed in the literature, e.g. in [CS, BKS, DT, T].

The second generalisation of self-duality is the notion of

Half-flatness.

This is defined for manifolds with certain Grassmann structure. In what follows M denotes a complex manifold. A Grassmann structure on M is an isomorphism of the (holomorphic) tangent bundle, $TM \cong E \otimes H$, where E and H are two holomorphic vector bundles over M. This is just a generalised spinor decomposition, allowing representation of vector fields using two types of spinor indices, viz. $X_{a\alpha} := e_a \otimes h_\alpha$, where (e_a) and (h_α) are frames of E and H respectively. If M is four dimensional, both E and H have rank 2 and a, α are the familiar 2-spinor indices. Manifolds with Grassmann structure provide interesting generalisations of four dimensional manifolds. A Grassmann connection ∇ is a linear connection which preserves the

Grassmann structure:

$$\nabla(e \otimes h) = \nabla^E e \otimes h + e \otimes \nabla^H h\,,$$

where ∇^E, ∇^H are connections in the bundles E, H and e, h are local sections of E, H respectively. A Grassmann structure with Grassmann connection ∇ is called half-flat if the connection ∇^H in H is flat. A manifold with such a half-flat Grassmann structure is called a half-flat Grassmann manifold. Here we assume that rank $H = 2$ and that a ∇^H-parallel non-degenerate fibre-wise 2-form $\omega_H \in \Gamma(\wedge^2 H^*)$ in the bundle H is fixed, so these manifolds are generalisations of hyper-Kähler manifolds. The torsion of the Grassmann connection lives in the space

$$\begin{aligned}
TM \otimes \wedge^2 T^*M &= TM \otimes \left(S^2H^* \otimes \wedge^2 E^* \ \oplus\ \wedge^2 H^* \otimes S^2 E^*\right) \\
&= EH\left(S^2 H^* \wedge^2 E^* \ \oplus\ \omega_H S^2 E^*\right) \\
&\cong \left(S^3 H \ \oplus\ \omega_H H\right) E \wedge^2 E^* \ \oplus\ \omega_H HES^2E^*\,,
\end{aligned}$$

where we omit the $\otimes$'s and we identify H^* with H using ω_H. Notice that since the bundle H has rank 2, the line bundle $\wedge^2 H$ is generated by the 2-form ω_H. (Choosing a frame (h_1, h_2) for H, we may write $\omega_H(h_\alpha, h_\beta) := \epsilon_{\alpha\beta}$, a skew matrix with $\epsilon_{12} = 1$.)

We now consider gauge fields on these manifolds. Let ∇ be a connection in a holomorphic vector bundle $\nu : W \to M$. (If ν has structure group G, we may choose a frame for it, in which the vector potential takes values in the gauge Lie algebra $\mathfrak{g}$). The curvature of the connection ∇ is a 2-form with values in the gauge algebra, $F^\nabla \in \wedge^2 T^*M \otimes \mathrm{End}\, W$. Now, as above, in virtue of the Grassmann structure the space of bivectors decomposes as

$$\wedge^2 TM = S^2E \otimes \wedge^2 H \oplus \wedge^2 E \otimes S^2 H \ = S^2 E \otimes \omega_H \oplus \wedge^2 E \otimes S^2 H\,.$$

If $e, e' \in \Gamma(E)$, we have the decomposition

$$F(e \otimes h_\alpha\,,\, e' \otimes h_\beta) = \omega_H(h_\alpha, h_\beta) F^{(e,e')} + F^{[e,e']}_{(\alpha\beta)}\,,$$

where $F^{(e,e')}$ is symmetric in e, e' and $F^{[e,e']}_{(\alpha\beta)}$ is skew in e, e' and symmetric in $\alpha\beta$. In 4 dimensions both E and H have rank 2, so this corresponds to the familiar decomposition into self-dual and anti-self-dual parts of the curvature.

Definition 3. *Let M be a manifold with Grassmann structure, not necessarily half-flat. A connection ∇ in a vector bundle $W \to M$ is called* **half-flat** *if its curvature*

$$F^\nabla \in \wedge^2 H \otimes S^2 E \otimes \mathrm{End}\, W = \omega_H \otimes S^2 E \otimes \mathrm{End}\, W\ .$$

Quaternionic Kähler and hyper-Kähler manifolds are special cases of manifolds with (a locally defined) Grassmann structure, the Levi-Civita connection being the canonical Grassmann connection. In the hyper-Kähler case the Grassmann structure is half-flat. For these examples, half-flatness of a gauge connection implies Ω-self-duality with respect to the canonical parallel 4-form Ω. In other words, a half-flat curvature is one of the eigenstates of the endomorphism field B_Ω, as in 4 dimensions. Therefore, in these cases, the two generalisations of the idea of self-duality mentioned above coincide.

There is an interesting special case of Grassmann structure, considered by [AG]:

Definition 4. *A* **spin $\frac{m}{2}$ Grassmann structure** *on a (complex) manifold M is a holomorphic Grassmann structure of the form $TM \cong E \otimes F = E \otimes S^m H$, with a holomorphic Grassmann connection $\nabla^{TM} = \nabla^E \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^F$, where H is a rank 2 holomorphic vector bundle over M with symplectic form ω_H and holomorphic symplectic connection ∇^H, and ∇^F is the connection in $F = S^m H$ induced by ∇^H. M is called* **half-flat spin $\frac{m}{2}$ Grassmann manifold** *if the connection ∇^F is flat.*

The bundle $S^m H$ is associated with the spin $\frac{m}{2}$ representation of the group $\mathrm{Sp}(1, \mathbb{C})$. Any frame (h_1, h_2) for H defines a frame for $S^m H$, $(h_A := h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_m})$, where the multi-index $A := \alpha_1 \alpha_2 \ldots \alpha_m$, $\alpha_i = 1, 2$. The symplectic form ω_H induces a bilinear form ω_H^m on $F = S^m H$ given by

$$\omega_H^m(h_A, h_B) := \mathfrak{S}_A \mathfrak{S}_B \omega_H(h_{\alpha_1}, h_{\beta_1}) \omega_H(h_{\alpha_2}, h_{\beta_2}) \cdots \omega_H(h_{\alpha_m}, h_{\beta_m}) \ ,$$

where $\mathfrak{S}_A$ denotes the sum over all permutations of the α's. This form is skew-symmetric if m is odd and symmetric if m is even. To any section $e \in \Gamma(E)$ and multi-index A we associate the vector field $X_A^e := e \otimes h_A$ on M and we put $X_{aA} := X_A^{e_a}$.

Let $(M, \nabla^{\mathrm{TM}})$ be a half-flat spin $\frac{m}{2}$ Grassmann manifold and ∇ a connection in a holomorphic vector bundle $W \to M$. The space of bivectors $\wedge^2 TM$ has the following decomposition into $GL(E) \otimes \mathrm{Sp}(1, \mathbb{C})$-submodules:

$$\wedge^2 TM = \wedge^2 \Big(E \otimes S^m H \Big) = \wedge^2 E \otimes S^2 S^m H \ \oplus \ S^2 E \otimes \wedge^2 S^m H \, ,$$

where

$$\begin{aligned} S^2 S^m H &= S^{2m} H \ \oplus \ \omega_H^2 S^{2m-4} H \ \oplus \ \cdots \oplus \omega_H^{2[\frac{m}{2}]} S^{2m-4[\frac{m}{2}]} H \\ \wedge^2 S^m H &= \omega_H S^{2m-2} H \ \oplus \ \omega_H^3 S^{2m-6} H \oplus \cdots \oplus \omega_H^{2[\frac{m}{2}]+1} S^{2m-4[\frac{m}{2}]-2} H \, . \end{aligned}$$

Here we use the convention that $S^l H = 0$ if $l < 0$. The corresponding decomposition of the curvature of ∇ is:

$$
\begin{aligned}
&F\big(X^e_A, X^{e'}_B\big)\\
&= \mathfrak{S}_A \mathfrak{S}_B \sum_{k=0}^{[m/2]} \Big(\omega_H(h_{\alpha_1}, h_{\beta_1}) \cdots \omega_H(h_{\alpha_{2k}}, h_{\beta_{2k}}) \overset{(2k)}{F}{}^{[ee']}_{\alpha_{2k+1}\dots\alpha_m\beta_{2k+1}\dots\beta_m}\\
&\qquad + \omega_H(h_{\alpha_1}, h_{\beta_1}) \cdots \omega_H(h_{\alpha_{2k+1}}, h_{\beta_{2k+1}}) \overset{(2k+1)}{F}{}^{(ee')}_{\alpha_{2k+2}\dots\alpha_m\beta_{2k+2}\dots\beta_m}\Big) ,
\end{aligned}
$$

where the tensors $\overset{(2k)}{F} \in \Gamma(\wedge^2 E^* \otimes S^{2m-4k} H^*)$ and $\overset{(2k+1)}{F} \in \Gamma(S^2 E^* \otimes S^{2m-4k-2} H^*)$ are the curvature components in irreducible $GL(E)\cdot\mathrm{Sp}(1,\mathbb{C})$-submodules.

For manifolds with spin $\frac{m}{2}$ Grassmann structure it is natural to define half-flat connections as those satisfying

$$\overset{(2i)}{F} = 0 \ , \quad \text{for all } i \in \mathbb{N}.$$

However, for these manifolds it is useful to consider a more refined notion:

Definition 5. *A connection ∇ in the vector bundle $\nu : W \to M$ is called* **k-partially flat** *if $\overset{(i)}{F} = 0$ for all $i \leq 2k$. Here $0 \leq k \leq [\frac{m+2}{2}]$.*

Clearly, $[\frac{m+2}{2}]$-partially flat connections are simply flat connections. We note that for $m{=}1$, 0-partially flat connections are precisely half-flat connections. For general odd $m{=}2p{+}1$, 0-partially flat connections in a vector bundle ν over flat spaces with spin $\frac{m}{2}$ Grassmann structure were considered in [W]. Some other k-partially flat connections on flat spaces were considered in [DN].

The penultimate case $k = [\frac{m}{2}]$ is particularly interesting for odd m.

Theorem 1. *Let M be a half-flat spin $\frac{m}{2}$ Grassmann manifold M. If m is odd and the vector bundle $E \to M$ admits a ∇^E–parallel symplectic form ω_E, then M has canonical* Sp(E)·Sp(H)*–invariant metric $g = \omega_E \otimes \omega_H^m$ and 4-form $\Omega \neq 0$. If Ω is co-closed with respect to the metric g, then any $\frac{m-1}{2}$-partially flat connection ∇ in a vector bundle W over M is Ω-selfdual and hence it is a Yang-Mills connection, i.e. it satisfies the Yang-Mills equation.*

Sketch of proof: To describe Ω we use the following notation: e_a is a basis of E, h_α is a basis of H, h_A is the corresponding basis of $S^m H$ and $X_{aA} := e_a \otimes h_A$ the corresponding basis of $TM = E \otimes S^m H$. With respect to these bases, the skew symmetric forms ω_E, ω_H and ω_H^m are represented by the matrices ω_{ab}, $\omega_{\alpha\beta}$ and ω_{AB}

respectively. We define Ω by

$$\Omega := \sum \omega_{ab}\omega_{cd}\omega_{AC}\omega_{BD} X^{aA} \wedge X^{bB} \wedge X^{cC} \wedge X^{dD} ,$$

where X^{aA} is the dual basis of the basis X_{aA}. The connection ∇ is $\frac{m-1}{2}$-partially flat if and only if its curvature F belongs to the space $S^2E^* \otimes \omega_H^m \otimes \mathrm{End}\,(W)$. Contracting a tensor $S = S_{ab}\omega_{AB} e^a e^b \otimes h^A h^B$ in $S^2E^* \otimes \omega_H^m$ with Ω, after some algebra, yields λS with $\lambda = -4(m+1) \neq 0$. Hence any $\frac{m-1}{2}$-partially flat connection is Ω-selfdual and is a Yang-Mills connection. $\square$

2. Construction

We now briefly sketch a construction of half-flat connections on half-flat Grassmann manifolds using a variant of the *harmonic space* approach (c.f. [GIOS1]). The method affords application to a construction of partially flat connections on a half-flat spin $\frac{m}{2}$ Grassmann manifold as well. Details and proofs will appear in [ACD].

We denote by S_H the $\mathrm{Sp}(1,\mathbb{C})$-principal holomorphic bundle over M consisting of symplectic bases of $H_m \cong \mathbb{C}^2$, $m \in M$,

$$S_H = \{ s = (h_+, h_-) \mid \omega_H(h_+, h_-) = 1 \} .$$

The bundle $S_H \to M$ is called **harmonic space**. A fixed parallel symplectic frame (h_1, h_2) of H, such that $m \mapsto s_m = (h_1(m), h_2(m)) \in S_H$ yields a trivialisations $M \times \mathrm{Sp}(1,\mathbb{C}) \cong S_H$ defined by

$$(m, \mathfrak{U}) \mapsto s_m \mathfrak{U} = \Big(h_+ = \sum h_\alpha u_+^\alpha \ , \ h_- = \sum h_\alpha u_-^\alpha \Big) \, , \, \mathfrak{U} = \begin{pmatrix} u_+^1 & u_-^1 \\ u_+^2 & u_-^2 \end{pmatrix} ; \, \det \mathfrak{U} = 1 \, .$$

This is precisely the harmonic space of [GIKOS, GIOS2]. The matrix coefficients $u_\pm^\alpha$ of $\mathrm{Sp}(1,\mathbb{C})$ are considered as holomorphic functions on S_H and together with local coordinates of M, we obtain a system of local coordinates on S_H constrained by the relation $u_+^1 u_-^2 - u_-^1 u_+^2 = 1$. We denote by $\partial_{++}, \partial_{--}, \partial_0$ the fundamental vector fields on S_H generated by the standard generators of $\mathrm{Sp}(1,\mathbb{C})$, satisfying the relations

$$[\partial_{++} \, , \, \partial_{--}] = \partial_0 \ , \quad [\partial_0 \, , \, \partial_{++}] = 2\partial_{++} \ , \quad [\partial_0 \, , \, \partial_{--}] = -2\partial_{--} \ .$$

For any section $e \in \Gamma(E)$ we define vector fields $X_\pm^e \in \mathfrak{X}(S_H)$ by the formula

$$X_\pm^e|_{(h_+, h_-)} = \widetilde{e \otimes h_\pm} \ ,$$

where $\widetilde{Y}$ is the horizontal lift of a vector field Y on M. They satisfy the relations

$$[\partial_0, X^e_\pm] = \pm X^e_\pm \ , \quad [\partial_{\pm\pm}, X^e_\pm] = 0 \ , \quad [\partial_{\pm\pm}, X^e_\mp] = X^e_\pm \ ,$$

$$[X^e_\pm, X^{e'}_\pm] \ = \ X_\pm^{\nabla_{\pi_* X^e_\pm} e'} - X_\pm^{\nabla_{\pi_* X^{e'}_\pm} e} - \widetilde{T}(\pi_* X^e_\pm, \pi_* X^{e'}_\pm) \, ,$$

$$[X^e_+, X^{e'}_-] \ = \ X_-^{\nabla_{\pi_* X^e_+} e'} - X_+^{\nabla_{\pi_* X^{e'}_-} e} - \widetilde{T}(\pi_* X^e_+, \pi_* X^{e'}_-) \, ,$$

where T is the torsion of the Grassmann connection, $\widetilde{T}(X,Y) := \widetilde{T(X,Y)}$ denotes the horizontal lift of the vector $T(X,Y)$ and we have used the abbreviation $\nabla_X e := \nabla^E_X e$. We denote by $\mathcal{D}_\pm$ the distributions generated by vector fields of the form $X^e_\pm$, $e \in \Gamma(E)$.

The holomorphic Grassmann structure is called admissible if the torsion of the Grassmann connection ∇ has no component in $S^3 H \otimes E \otimes \wedge^2 E^*$. This means, in particular, that the torsion can be written as a sum of tensors linear in ω_H. Now, it is not difficult to prove that if a Grassmann structure is admissible, the distribution $\mathcal{D}_+$ (associated to any parallel frame (h_1, h_2)) on S_H is integrable.

The basic idea of the harmonic space construction is to lift the geometric data from M to S_H via the projection $\pi : S_H \to M$. The pull back $\pi^*\nabla$ of a half-flat connection ∇ in the trivial vector bundle $\nu : W = \mathbb{C}^r \times M \to M$ is a connection in the vector bundle $\pi^*\nu : \pi^* W \to S_H$ which satisfies equations defining the notion of a half-flat gauge connection on S_H. One can also define the weaker notion of an almost half-flat connection in $\pi^*\nu : \pi^* W \to S_H$ by considering only the equations on the curvature involving ∂_0, ∂_{++} and X^e_+ in one of the arguments.

Now, the main result, for which the integrability of $\mathcal{D}_+$ is crucial, can be stated:

Theorem 2. *Let M be a manifold with a half-flat admissible Grassmann structure. Consider a matrix-valued function A_{++} on S_H (the "analytic prepotential"), constant along the leaves of $\mathcal{D}_+$ with the homogeneity property $\partial_0 A_{++} = 2A_{++}$, and an invertible matrix-valued solution of the system of first-order linear ODE's, $\partial_{++}\Phi = -A_{++}\Phi$, $\partial_0 \Phi = 0$. The pair (A_{++}, Φ) defines an almost half-flat connection $\nabla^{(A_{++},\Phi)}$ in the bundle $\pi^*\nu : \pi^* W \to S_H$, which depends only on A_{++}, Φ and their first and second partial derivatives. The potential $A(X^e_+)$ of this connection with respect to the frame $e\Phi = e_i \Phi^i_j$, where $e = (e_i)$ is the standard frame of $W = \mathbb{C}^r \times M$, determines a half-flat connection in the bundle $\nu : W \to M$. Conversely, any half-flat connection over M can be obtained using this construction.*

The proof and further details may be found in [ACD].

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