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A DIRECT PROCEDURE TO COMPUTE ZERO–CURVATURE REPRESENTATIONS. THE CASE $\mathfrak{sl}_2$

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Abstract. This paper is an introduction to a direct procedure to compute zero-curvature representations. Focus is on the simplest case of coefficients in $\mathfrak{sl}_2$.

1. Introduction

Zero-curvature representations (ZCR) were introduced in complete integrability theory of nonlinear partial differential equations (Zakharov and Shabat [15]). Knowing a ZCR depending on a "non-removable" (spectral) parameter opens a way to Bäcklund transformations, infinite series of conservation laws, and solution via a Riemann–Hilbert problem ([15, 12]). However, so far the only widely used method to find a ZCR for a given equation is the "prolongation procedure" of Wahlquist and Estabrook [14]. In the course of computation, an "incomplete" Lie algebra $\mathfrak{g}$ (prolongation algebra) arises and is being "closed." A linear representation of the algebra is a ZCR. The procedure is considered algorithmic for a significant class of equations (Dodd and Fordy [4]), but still some cases remain difficult if not impossible: cases of dependence on higher order derivatives as well as general classification problems (such as those solved by symmetry analysis [10]). Also the case when the ZCR explicitly depends on coordinates needs a special handling (Molino [11]).

Within the category of diffieties (geometric objects corresponding to partial differential equations) Krasil'shchik and Vinogradov [6] introduced the general concept of a covering, of which Wahlquist–Estabrook structures are a special case. One is able to compute isomorphism classes of coverings effectively after unique representatives are found in each class.

In [8], a new procedure was suggested in the case of ZCR's, based on the analogy with the $\mathcal{C}$-spectral sequence and geometric theory of conservation laws [13]. It turns out to be better adapted to classification problems and there is no restriction as for what variables the ZCR may depend on. The problem is reduced to solving a particular system of equations in total derivatives, similar to that for generating functions of conservation laws, but nonlinear. The

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bad news is that the Lie algebra $\mathfrak{g}$ must be given on the input side, which means that the computation may have to be repeated for different algebras until one of them suits. As one is usually interested in simple algebras and every simple algebra embeds into an appropriate $\mathfrak{sl}_n$, this practically means going through first members of the $\mathfrak{sl}_n$ series (cf. [4, p. 408]).

This paper is intended to gather some basic information on a rather introductory level. In Section 4 we focus on the simplest case of ZCR's with coefficients in $\mathfrak{sl}_2$, leaving the general case to a subsequent paper.

The paper [8] is now widely available from the Electronic Library of the European Mathematical Society (ELibEMS); errata attached to the electronic version fix some regrettable typing errors left by the author in the printed book.

2. Basics

Consider a system of N partial differential equations

$$F^\ell(x, y, \dots, u^k_I, \dots) = 0, \tag{1}$$

$\ell = 1, \dots, N$, in two independent variables x, y and an arbitrary number M of dependent variables u^k, $k = 1, \dots, M$. Here and in what follows I stands for a symmetric multiindex in x, y (possibly void).

Let G be a matrix Lie group, let $\mathfrak{g}$ be the corresponding Lie algebra. A *zero-curvature representation* (ZCR) for eq. (1) with coefficients in $\mathfrak{g}$ (or $\mathfrak{g}$-valued) is a pair of $\mathfrak{g}$-matrices A, B such that

$$A_y - B_x + [A, B] = 0 \tag{2}$$

holds as a consequence of eq. (1). When $\mathfrak{g} \subseteq GL(n, \mathbb{C})$, a $\mathbb{C}^n$-valued function Ψ is called a *linear pseudopotential* for eq. (1) whenever it satisfies the linear system

$$\Psi_x = A\Psi, \quad \Psi_y = B\Psi. \tag{3}$$

As is well known, the integrability condition for the system (3) is eq. (2)—and hence also eq. (1).

For any G-matrix H, matrices

$$A^H = H_x H^{-1} + HAH^{-1}, \quad B^H = H_y H^{-1} + HBH^{-1} \tag{4}$$

form a ZCR again. The mapping $(A, B) \mapsto (A^H, B^H)$ is called a *gauge transformation*; it is a group action. The pair (A^H, B^H) is said to be *gauge equivalent* to the pair (A, B). If Ψ is a linear pseudopotential for (A, B), then $H\Psi$ is the linear pseudopotential for (A^H, B^H). A ZCR gauge-equivalent to the pair $(0, 0)$ is said to be *trivial*.

We call a $\mathfrak{g}$-valued ZCR (A, B) *reducible* if either A, B or any of the gauge-equivalent pairs A^H, B^H fall entirely into a subalgebra of $\mathfrak{g}$. Otherwise it is called *irreducible*. Recall that of special interest in complete integrability theory are irreducible ZCR's with coefficients in a semisimple Lie algebra.

In the sequel we shall use some terminology related to diffieties (see [7] for coordinate free exposition of the diffiety theory). We regard $x, y, \dots, u^k_I, \dots$ as coordinates on the infinite-dimensional *jet space* J^∞. Total derivatives, that is, operators $D_x = \partial/\partial x + \sum_{k,I} u^k_{Ix} \cdot \partial/\partial u^k_I$,

$D_y = \partial/\partial y + \sum_{k,I} u^k_{Iy} \cdot \partial/\partial u^k_I$ are vector fields on J^∞. They commute: $D_x D_y = D_y D_x$. Equations (1) and their differential consequences $D_I F^\ell = 0$ naturally determine a submanifold $\mathcal{E} \subset J^\infty$, called a *diffiety*. The total derivatives have a well-defined restriction to $\mathcal{E}$.

In this framework, matrices A, B above are $\mathfrak{g}$-valued smooth maps defined on $\mathcal{E}$, and A_y, B_x stand for total derivatives $D_y A$, $D_x B$ applied componentwise. Similarly for other matrices and derivatives.

For the notion of a *covering* see [6].

3. The characteristic element

The results of [8] acquire a particularly simple form as the number of independent variables in eq. (1) is only two. The fact most important for us is that to every ZCR (A, B) there is assigned a *characteristic N-tuple* $\chi = (\chi_1, \dots, \chi_N)$ of $\mathfrak{g}$-matrices (called a characteristic element in [8]) with remarkable properties (see below). Recall that N is the number of equations in the system (1).

In [8] we established the existence of the characteristic element. Here we give a practical explicit procedure to compute it. We omit the proof, since it is similar to that of [8, Prop. 2.7].

Proposition 1. *Let (A, B) be a ZCR. Assume that there exist $\mathfrak{g}$-matrices C^I_ℓ satisfying*

$$D_y A - D_x B + [A, B] = \sum_{\ell, I} D_I F^\ell \cdot C^I_\ell \quad \text{on } J^\infty.$$

Put

$$K_\ell = \sum_I (-1)^{|I|} \widehat{D}_I C^I_\ell.$$

Then $K_1|_\mathcal{E}, \dots, K_N|_\mathcal{E}$ is the characteristic N-tuple for (A, B).

Here and in what follows, $\widehat{D}_I$ are operators defined by $\widehat{D}_x = D_x - \mathrm{ad}_A$, $\widehat{D}_y = D_y - \mathrm{ad}_B$ with $\mathrm{ad}_U V = [U, V]$, and $\widehat{D}_{x \cdots xy \cdots y} = \widehat{D}_x \cdots \widehat{D}_x \widehat{D}_y \cdots \widehat{D}_y$. Note that operators $\widehat{D}_x, \widehat{D}_y$ commute by virtue of eq. (2).

For ZCR's coming from the Wahlquist–Estabrook procedure we have typically $C^I_\ell = 0$ for every non-void I, and then simply $K_\ell = C^\emptyset_\ell$.

Note that for $\mathfrak{g} = \mathbb{C}$ a ZCR reduces to a conservation law, while the characteristic N-tuple is just the N-tuple of generating functions in the sense of [13].

Proposition 2. *Let $\chi = (\chi_1, \dots, \chi_N)$ be the characteristic N-tuple for a zero-curvature representation (A, B). Then*

1° ([8, Prop. 3.9]) *χ transforms by conjugation, $\chi \mapsto H \chi H^{-1}$, when A, B transform by gauge;*

2° ([8, eq. 2.11, Prop. 2.7]) *χ satisfies*

$$\sum_{\ell, I} (-1)^{|I|} \widehat{D}_I \Big(\frac{\partial F^\ell}{\partial u^k_I} \chi_\ell \Big) = 0, \quad k = 1, \dots, M.$$

3° ([8, Prop. 4.2, 4.3]) *For non-overdetermined systems, $\chi \neq 0$ unless (A, B) is trivial.*

The operator of 2° may be called a "covariant formal adjoint to the operator of universal linearization" of the system (1). Note that for $\mathfrak{g} = \mathbb{C}$, eq. 2° is exactly the determining condition for generating functions of conservation laws.

The procedure to compute ZCR's of non-overdetermined systems (1) follows immediately. The input is eq. (1) and the Lie algebra $\mathfrak{g}$.

Procedure. *On the diffiety $\mathcal{E}$, solve the determining system*

$$
\begin{aligned}
0 &= D_y \bar{A} - D_x \bar{B} + [\bar{A}, \bar{B}], &(5)\\
0 &= \sum_{\ell,I} (-1)^{|I|} \widehat{D}_I \Big(\frac{\partial F^\ell}{\partial u^k_I} \bar{\chi}_\ell \Big) &(6)
\end{aligned}
$$

on unknowns $\bar{\chi}, \bar{A}, \bar{B}$, for $(\bar{\chi}, \bar{A}, \bar{B})$ running through triples (χ, A, B), $\chi \ne 0$, in normal form with respect to the group action

$$(\chi, A, B) \mapsto (H \chi H^{-1}, A^H, B^H).$$

Some comments are due. The unknowns $\bar{A}, \bar{B}$ enter eqs. (5–6) via the coefficients of $\widehat{D}_x, \widehat{D}_y$. Components of $\bar{\chi}$ come as auxiliary unknowns; note that (6) is linear in $\bar{\chi}$.

The number of equations in system (5,6) is $\dim \mathfrak{g} + N \dim \mathfrak{g} = (N+1) \dim \mathfrak{g}$. The number of unknowns is $2 \dim \mathfrak{g}$ for the components of A, B and $N \dim \mathfrak{g}$ for the components of χ, this amount is usually reduced by $\dim \mathfrak{g}$ in the process of taking the normal form. Thus, in case of general position we get $(N+1) \dim \mathfrak{g}$ equations on the same number of unknowns.

Possible normal forms $\bar{\chi}, \bar{A}, \bar{B}$ depend only on the algebra $\mathfrak{g}$. Once found, they may serve equally well for any eq. (1). Let us present some general considerations.

Recall that χ is an N-tuple of matrices, $\chi = (\chi_1, \dots, \chi_N)$. An algorithm to find a normal form of an N-tuple of matrices was given in [2] (the Belickij algorithm). As the first step, χ_1 is reduced to its Jordan normal form J. Matrices H commuting with J form a subgroup G_J, the stabilizer of J. For each particular form of J, the algorithm finds possible normal forms K for χ_2 with respect to the action of G_J. The case $N > 2$ is reduced to $N = 2$ at the cost of enlarging the size of the matrices (with severe impact on the complexity). Needless to say, it may also turn out that the common stabilizer $G_J \cap G_K$ is already trivial (consists of multiples of the identity matrix) so that the normal form is achieved at level $N = 2$.

That said, let us turn back to our setting. In case that the normal form $\bar{\chi}$ still has a non-trivial stabilizer (in particular, when $N = 1$), another step is needed, namely finding a normal form of the matrix A (or B) with respect to the remaining gauge freedom. Here [2] is of little help, but this task may be quite elementary if the stabilizer G_J is simple enough (cf. Proposition 3 below).

The number of normal forms grows rather dramatically with the dimension. However, one observes a significant reduction when the ZCR's are required to be irreducible. In the next section we analyse the simplest case of $\mathfrak{sl}_2$.

4. Case $\mathfrak{g} = \mathfrak{sl}_2$, $N = 1$

The case $\mathfrak{sl}_2$ is important enough to deserve priority. We let

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{pmatrix}$$

so that equation (2) becomes

$$\begin{aligned} 0 &= D_x b_1 - D_y a_1 + b_2 a_3 - b_3 a_2, \\ 0 &= D_x b_2 - D_y a_2 + 2b_1 a_2 - 2b_2 a_1, \\ 0 &= D_x b_3 - D_y a_3 + 2b_3 a_1 - 2b_1 a_3. \end{aligned} \tag{7}$$

Example 1. Suppose that

$$a_2 b_3 = a_3 b_2. \tag{8}$$

Then the first equation of (7) reads $0 = D_x b_1 - D_y a_1$ and therefore $a_1\,dx + b_1\,dy$ is a conservation law for equation (1). Consider the 1-dimensional covering $\mathcal{H}$ over (1) with a nonlocal variable h satisfying $h_x = a_1$, $h_y = b_1$. On this covering h is a local function. Put $a_2' := a_2 \mathrm{e}^{-2h}$, $b_2' := b_2 \mathrm{e}^{-2h}$, $a_3' := a_3 \mathrm{e}^{2h}$, $b_3' := b_3 \mathrm{e}^{2h}$. Then the remaining two equations of system (7) become $0 = \mathrm{e}^{2h}(D_x b_2' - D_y a_2')$, $0 = \mathrm{e}^{-2h}(D_x b_3' - D_y a_3')$ and we see that $a_2'\,dx + b_2'\,dy$ and $a_3'\,dx + b_3'\,dy$ are two conservation laws on $\mathcal{H}$. They are, however, linearly dependent because of the relation $a_2' b_3' = a_3' b_2'$ (by (8)). Thus we got nothing more than a single nonlocal conservation law.

Note that eq. (8) determines a solvable subalgebra in $\mathfrak{sl}_2$ to which the ZCR is reducible.

Proposition 3. *Let (A, B) be an irreducible $\mathfrak{sl}_2$-valued ZCR, let $\chi \ne 0$ be its characteristic matrix. Then we have one of the following two normal forms for χ and A:*
– *Nilpotent case*

$$\bar\chi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \bar A = \begin{pmatrix} 0 & a_2 \\ a_3 & 0 \end{pmatrix}. \tag{9}$$

– *Diagonal case*

$$\bar\chi = \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}, \quad \bar A = \begin{pmatrix} a_1 & 1 \\ a_3 & -a_1 \end{pmatrix}. \tag{10}$$

No further reduction of B is possible, i.e., $\bar B = B$.

Proof. Assume first that $a_2 \ne 0$. For $\chi \ne 0$ we have exactly two possible Jordan normal forms. If $\bar\chi = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, then the stabilizer consists of matrices $H = \left(\begin{smallmatrix} 1 & 0 \\ h & 1 \end{smallmatrix}\right)$ and acts according to

$$A^H = \begin{pmatrix} a_1 - h a_2 & a_2 \\ D_x h + a_3 + 2h a_1 - h^2 a_2 & -a_1 + h a_2 \end{pmatrix}. \tag{11}$$

By setting $h = a_1/a_2$ we get the form (9).

If $\bar{\chi} = \left(\begin{smallmatrix} r & 0 \\ 0 & -r \end{smallmatrix}\right)$, then the stabilizer consists of matrices $H = \left(\begin{smallmatrix} h & 0 \\ 0 & 1/h \end{smallmatrix}\right)$, $h \ne 0$, and acts by

$$A^H = \begin{pmatrix} D_x h/h + a_1 & h^2 a_2 \\ a_3/h^2 & -D_x h/h - a_1 \end{pmatrix}.$$

Put $h = a_2^{-1/2}$ to obtain the form (10).

Observe that in both cases the stabilizer of the pair $(\bar{\chi}, \bar{A})$ is already trivial.

The case $a_2 = 0$, on the other side, cannot occur for irreducible ZCR's. This follows from Proposition 4 below, whose proof does not depend on this one. ■

Proposition 4. *Let (A, B) be an $\mathfrak{sl}_2$-valued ZCR. Assume that at least one of the non-diagonal elements a_2, a_3, b_2, b_3 is zero. Then (A, B) is gauge-equivalent to the ZCR of Example 1.*

Proof. If b_2 or b_3 is zero, then one gets $a_2 = 0$ or $a_3 = 0$ after the symmetry $x \leftrightarrow y$, $A \leftrightarrow B$ of (2). If $a_3 = 0$, then one gets $a_2 = 0$ after $A \mapsto -A^\top$, $B \mapsto -B^\top$, which is another symmetry of (2). Therefore, without loss of generality we may assume that $a_2 = 0$.

If moreover $b_2 = 0$, then we have eq. (8) and the ZCR is that of Example 1. Assume therefore that $b_2 \ne 0$. Consider matrices $A' = A^H$, $B' = B^H$, gauge equivalent to A, B with $H = \left(\begin{smallmatrix} 1 & 0 \\ h & 1 \end{smallmatrix}\right)$, $h = b_1/b_2$. Analogously to formula (9) we get $b_1' = 0$ while $a_2' = a_2$ remains zero.

Thus, the problem is reduced to the case when $a_2 = b_1 = 0$. Assuming again that $b_2 \ne 0$ (otherwise we have eq. (8)) one easily finds the general solution of the (now simplified) eq. (7) as $a_1 = \frac{1}{2} D_x b/b$, $a_3 = \frac{1}{2} D_{xy} b/b^2 - \frac{1}{2} D_x b \cdot D_y b/b^3$, $b_2 = b$, $b_3 = \frac{1}{2} D_{yy} b/b^2 - \frac{3}{4}(D_t b)^2/b^3 + \beta/b$, / b, where β is any function satisfying $D_x \beta = 0$. The function b is arbitrary. If we let

$$K = \begin{pmatrix} b^{-1/2} & 0 \\ -\frac{1}{2} D_y b \cdot b^{-3/2} & b^{1/2} \end{pmatrix},$$

then the gauge-equivalent pair A^K, B^K is simply $A^K = 0$, $B^K = \left(\begin{smallmatrix} 0 & 1 \\ \beta & 0 \end{smallmatrix}\right)$, which obviously satisfies eq. (8). ■

Needless to say, both nilpotent and diagonal cases are possible. E.g., KdV is of nilpotent and mKdV is of diagonal type.

5. Remarks and examples

Remark 1. A natural question is whether the normal forms of Proposition 3 fix the gauge or not. The answer is negative in general, at least in the nilpotent class (9): any solution embeds into a "false" 1-parameter family

$$A = \begin{pmatrix} 0 & c a_2 \\ a_3/c & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & c b_2 \\ b_3/c & -b_1 \end{pmatrix}, \quad c \in \mathbb{C}. \tag{12}$$

Here c is removed by conjugation with $\left(\begin{smallmatrix} 1/\sqrt{c} & 0 \\ 0 & \sqrt{c} \end{smallmatrix}\right)$.

Indeed, eq. (6) is clearly $\mathbb{C}$-linear in $\bar{\chi}$. Therefore, along with $\bar{\chi} = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ any multiple $c\bar{\chi} = \left(\begin{smallmatrix} 0 & 0 \\ c & 0 \end{smallmatrix}\right)$, $c \in \mathbb{C}$, satisfies eq. (6) with the same A, B. However, the matrix $c\bar{\chi}$ has the same normal form $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, which then satisfies eq. (6) with the matrices A, B as in (12).

A simple example has been already presented in [8] (equation $u_{xy} = f(u)$). Below we give another, quite unpretentious demonstration of the procedure. Not all ZCR's we obtain admit a parameter, and those that do are not new. However, with an explicit dependence on coordinates, with an equation depending on an arbitrary function, and with a rather loose restriction on the jet order, this example seems to be rather out of the domain of the Wahlquist–Estabrook procedure.

Example 2. To keep things simple, we consider equations of the form

$$u_t = u_{xxx} + F(t, x, u, u_x). \tag{13}$$

and compute irreducible $\mathfrak{sl}_2$-valued ZCR's of nilpotent type. Then $a_1 = 0$, $J = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ by eq. (9) and we get equations (5), (6) in the form

$$\begin{aligned}
0 &= D_x b_1 + b_2 a_3 - b_3 a_2, \\
0 &= D_x b_2 - D_t a_2 + 2b_1 a_2, \\
0 &= D_x b_3 - D_t a_3 - 2b_1 a_3, \\
0 &= -D_{xx} a_2 + b_2 - 4a_2^2 a_3 - a_2 \frac{\partial F}{\partial u_x}, \\
0 &= -6a_2\, D_x a_2, \\
0 &= 4\, a_3\, D_x a_2 + 2a_2\, D_x a_3 - 2b_1 - \frac{\partial F}{\partial u} + D_x \frac{\partial F}{\partial u_x}.
\end{aligned} \tag{14}$$

Recall that restriction on the diffiety means that u_t and its derivatives $u_{tx}, u_{tt}, u_{txx}, u_{ttx}, \dots$ are no more coordinates, but functions on $\mathcal{E}$, computable from eq. (13). The restricted total derivatives then are $D_x = \partial/\partial x + u_x\, \partial/\partial u + u_{xx}\, \partial/\partial u_x + u_{xxx}\, \partial/\partial u_{xx} + \cdots$ and $D_t = \partial/\partial t + u_t\, \partial/\partial u + u_{tx}\, \partial/\partial u_x + u_{txx}\, \partial/\partial u_{xx} + \cdots$, where $u_{tx} = D_x u_t = u_{4x} + D_x F$, $u_{txx} = D_{xx} u_t = u_{5x} + D_{xx} F$, etc.

The fifth equation (14) is equivalent to $a_2 = a_2(t)$. To solve the others we impose an upper bound on the jet order of a_3, b_1, b_2, b_3, namely, we assume that they are functions of $t, x, u, u_x, u_{xx}, u_{xxx}, u_{4x}, u_{5x}$. The procedure is then the standard one: by differentiating the expressions (14) with respect to u_{8x}, u_{7x} etc. we get conditions on a_3, b_1, b_2, b_3 which eventually (we omit computations performed with the aid of computer algebra) lead to the final result that

$$\begin{aligned}
a_1 &= 0, \\
a_2 &= m, \\
a_3 &= u + p, \\
b_1 &= m u_x - \frac{1}{2m} \frac{\partial q}{\partial x} + \frac{1}{2m} \frac{\partial m}{\partial t}, \\
b_2 &= -2m^2 u + q, \\
b_3 &= u_{xx} - 2mu^2 + \left(\frac{q}{m} - 2mp\right) u - \frac{1}{2m^2} \frac{\partial^2 q}{\partial x^2} + \frac{pq}{m},
\end{aligned} \tag{15}$$

are components of a ZCR for the equation

$$\begin{aligned}0 &= u_t - u_{xxx} + 6muu_x + \left(4mp - \frac{q}{m}\right) u_x \\ &+ \left(2m\frac{\partial p}{\partial x} - \frac{2}{m}\frac{\partial q}{\partial x} + \frac{1}{m}\frac{\partial m}{\partial t}\right) u \\ &+ \frac{1}{2m^2}\frac{\partial^3 q}{\partial x^3} - \frac{2p}{m}\frac{\partial q}{\partial x} - \frac{q}{m}\frac{\partial p}{\partial x} + \frac{\partial p}{\partial t} + \frac{p}{m}\frac{\partial m}{\partial t}.\end{aligned} \tag{16}$$

Here $m(t)$, $p(t,x)$, $q(t,x)$ are arbitrary functions. The "false" parameter c of eq. (12) has been already removed. In the course of computation we assumed that a_2, b_2, b_3 are nonzero (see Proposition 4).

Note that for $m = 1$, $p = \lambda =$ parameter, $q = 4\lambda$ the class (16) includes the KdV equation $u_t = u_{xxx} - 6uu_x$, with

$$A_{\mathrm{KdV}} = \begin{pmatrix} 0 & 1 \\ u + \lambda & 0 \end{pmatrix},$$

$$B_{\mathrm{KdV}} = \begin{pmatrix} u_x & -2u + 4\lambda \\ u_{xx} - 2u^2 + 2\lambda u + 4\lambda^2 & -u_x \end{pmatrix}.$$

For $m = 1$, $p = S = -\frac{1}{12}xt^{-1} + \lambda t^{-1}$, $q = 4S$, one gets the cylindric KdV equation $u_t = u_{xxx} - 6uu_x + \frac{1}{2}ut^{-1}$, with

$$A_{\mathrm{cKdV}} = \begin{pmatrix} 0 & 1 \\ u + S & 0 \end{pmatrix},$$

$$B_{\mathrm{cKdV}} = \begin{pmatrix} u_x + \frac{1}{6}t^{-1} & -2u + 4S \\ u_{xx} - 2u^2 + cSu + 4S^2 & -u_x - \frac{1}{6}t^{-1} \end{pmatrix}.$$

Both ZCR's are well known ([1] and [9]). Here λ is the true spectral parameter.

Obviously, the ZCR (15) contains a parameter if the functions m, p, q depend on some λ that cancels in eq. (16). Whether this happens is easily checked by differentiating eq. (16) with respect to λ. Detailed investigation (omitted here) shows that all resulting "integrable" equations reduce to KdV by coordinate transformations (as cKdV does). This is, of course, in good agreement with what we already know from [10, 5].

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