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WDVV EQUATIONS AND THEIR SOLUTIONS FOR A $(3,2)$ AND $(3,4)$ SUPERMANIFOLD; THE ISOTROPIC AND NONISOTROPIC CASE

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Abstract. The WDVV equations for the isotropic and nonisotropic case on a supermanifold of dimension $(3,2)$ and $(3,4)$ will be derived. By the method of characteristics, the general solution is constructed after transforming the system of WDVV equations to appropriate coordinates.

Introduction

Recently a remarkable system of differential equations appeared in the physics of two-dimensional topological field theory. Elaborating axioms describing a topological field theory led in a natural way to these very interesting but complicated and overdetermined system of equations [5].

Mathematically the system describes the conditions for a function $\Phi = \Phi(x)$ of the variable $x = (x^1, \ldots, x^n)$ such that the third order derivatives

$$A_{abc}(x) = \frac{\partial^3 \Phi(x)}{\partial x^a \partial x^b \partial x^c}$$

for any x are the structure constants of an associative algebra. Appropriately this system is known in two-dimensional topological field theory as the equations of associativity or the Witten-Dijkgraaf-E.Verlinde-H.Verlinde (WDVV) system since it is first found in the papers [1,4].

In the physical setting solutions of the WDVV system describe moduli spaces of topological conformal field theories, whereas in the mathematical context the system applies to enumerative problems in algebraic geometry. For instance solutions of the system may be interpreted as generating functions of combinatorial identities counting numerical characteristics of spaces of algebraic curves in some complex manifold or algebraic variety.

In these applications solutions of the WDVV system determine an associative algebra structure on some cohomology space. Generally (and also for convenience) one is dealing

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only with even-dimensional cohomology spaces. So the resulting algebra structure turns out to be commutative.

However following Kontsevich-Manin [3] one may consider also odd-dimensional cohomology spaces. This automatically leads to $\mathbb{Z}^2$-graded (super) spaces, equations and geometry.

For a comprehensive introduction to the WDVV-system we refer the very well-written survey of Dubrovin [2], where all necessary physical motivations and mathematical details can be found. In this paper [2] only the commutative non-graded case is considered and the WDVV-system is classified according to two essential different types. The first and second type correspond to the case where the identity of the algebra is respectively isotropic, or nonisotrpic.

In this paper we will study and solve the WDVV-system in three even and two odd variables $(3, 2)$ and three even and four odd variables $(3, 4)$. Both the isotropic and nonisotropic case are investigated.

1 General

In this section we shall recapitulate a number of notions which lead in the classical case [2] to the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations and for the super or graded situation to the super WDVV-equations [3].

Starting point in the considerations is a supermanifold M which is endowed with a tensor g of rank 2 and a tensor A of rank 3. Local coordinates are given by $\{x^a\}$, the basis of vector fields by $\partial_a = \partial_{x^a}$, and the basis of 1-forms by dx^a. Tensor components are g_{ab} and A^c_{ab} respectively. The degree of x^a i.e., $|x^a|$ which will only appear in expressions of type p.e. $(-1)^{|x^a|\,|x^b|}$ will be written as a i.e. $(-1)^{|x^a||x^b|} = (-1)^{ab}$. The tensor g, A are taken to be even i.e.

$$|g_{ab}| = (-1)^{a+b} \quad |A^c_{ab}| = (-1)^{a+b+c}. \tag{1}$$

We further impose a number of requirements.

1. The existence of a <u>pairing</u>

   $$< \partial_a, \partial_b >= g_{ab}$$

   which is symmetric, i.e.,

   $$g_{ab} = (-1)^{ab} g_{ba}. \tag{2}$$

   Furthermore g_{ab} is <u>constant</u>, and <u>invertible</u> $(g^{ab}) = (g_{ab})^{-1}$.

2. The existence of the structure of an $\underline{C_M}$-<u>algebra</u>

   $$\partial_a \circ \partial_b = \sum_c A^c_{ab} \partial_c. \tag{3}$$

   (Super)<u>commutativity</u> results in

   $$A^c_{ba} = (-1)^{ab} A^c_{ab}. \tag{4}$$

In order to have associativity i.e.,

$$\partial_a \circ (\partial_b \circ \partial_c) = (\partial_a \circ \partial_b) \circ \partial_c \tag{5}$$

we require

$$\sum_e A^e_{ab} A^d_{ec} = (-1)^{a(b+c)} \sum_e A^e_{bc} A^d_{ea}. \tag{6}$$

Moreover we demand

$$\partial_d A^c_{ab} = (-1)^{ad} \partial_a A^c_{db}. \tag{7}$$

3. We now introduce

$$A_{abc} = \sum_e A^e_{ab} g_{ec} \tag{8}$$

and impose

$$A_{abc} = (-1)^{a(b+c)} A_{bca}. \tag{9}$$

Condition (9) reflects that

$$< \partial_a \circ \partial_b, \partial_c > = < \partial_a, \partial_b \circ \partial_c > \tag{10}$$

4. Existence of coordinate vector field ∂_0, is required, which is even and

$$A^c_{ob} = \delta_{bc} \tag{11}$$

or equivalently

$$A_{obc} = g_{bc}. \tag{12}$$

An even function Φ defined on M is called a potential for the (A, g)-structure if

$$A_{abc} = \partial_a \partial_b \partial_c \Phi. \tag{13}$$

Conditions (2), (4), (7), (9) are immediate from (13). Crucial in this formulation is (6) which results in the remarkable WDVV equations

$$\sum_{e,f} \partial_a \partial_b \partial_e \Phi g^{ef} \partial_f \partial_c \partial_d \Phi = (-1)^{a(b+c)} \sum_{e,f} \partial_b \partial_c \partial_e \Phi g^{ef} \partial_f \partial_a \partial_d \Phi. \tag{14}$$

It should be noted that (2), (3), (4), (6), (7), (9), (12), (14) hold for all appropriate values of indices $a, \ldots$.

In section 2 and 3 we shall discuss the (A, g)-structures and the potential function Φ for the $(3, 2)$-case i.e. the super manifold M has local coordinates $\{x_0, x_1, x_2, \xi_1, \xi_2\}$ where x_0, x_1, x_2 are even and ξ_1, ξ_2 odd, whereas in sections 4 and 5 we consider the (3,4) case.

We shall make particular choices for the (constant) g matrix, related to the isotropic and nonisotropic case.

In the classical case, no odd variables and dimension 3, the WDVV equations read

$$-f_{xyy} f_{xxx} + f^2_{xxy} - f_{yyy} = 0$$

for the isotropic and

$$-f_{xyy} f_{xxy} + f_{xxx} f_{yyy} - 1 = 0$$

for the nonisotropic case.

2 The isotropic case: $n = 3, m = 2$

In this section we derive the graded or super WDVV equations in the case of 3 even variables x_0, x_1, x_2 and two odd variabales ξ_1, ξ_2; $|x_i| = 0, |\xi_i| = 1$, where $|x|$ denotes the degree of x. We shall restrict here to the isotropic case. In the classical situation this results in the canonical form of the (3×3) matrix g

$$g = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{1}$$

In the graded case, due to the requirements (1), (2) that g_{ab} is constant and g being (graded) symmetric

$$g_{ab} = (-1)^{ab} g_{ba}$$

where a refers to the element x^a, ξ^a, we are led to a (5×5) matrix g

$$g = \left[\begin{array}{ccc|cc} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & -\alpha & 0 \end{array}\right]. \tag{2}$$

Starting from an even function Φ of variables $x_0, x_1, x_2, \xi_1, \xi_2$ we have

$$\Phi = F_1(x_0, x_1, x_2) + F_2(x_0, x_1, x_2)\xi_1\xi_2. \tag{3}$$

From the definition of g by

$$g_{ab} = \partial_0 \partial_a \partial_b \Phi \tag{4}$$

we derive

$$\begin{aligned} \Phi \; &= G(x_1, x_2) + F(x_1, x_2)\xi_1\xi_2 + \tfrac{1}{2}x_0^2 x_2 + \tfrac{1}{2}x_0 x_1^2 - \alpha x_0 \xi_1 \xi_2 \\ &\quad + c_1 x_0^2 + c_2 x_0 x_1 + c_3 x_0 x_2 + c_4 x_0. \end{aligned} \tag{5}$$

We now impose the associativity conditions (4) on (5) and we arrive at a system of partial differential equations for F, G i.e. (lower index i referring to differentiation with respect to $x^i (i = 1, 2)$)

$$\begin{aligned} -G_{122}G_{111} + G_{112}^2 - G_{222} &= 0 \\ -F_1^2 + 2\alpha F_2 &= 0 \\ -\alpha F_1 G_{122} + \alpha^2 G_{222} - F_2^2 &= 0 \\ -\alpha F_1 G_{112} + \alpha^2 G_{122} - F_1 F_2 &= 0 \\ -\alpha F_1 G_{111} + \alpha^2 G_{112} - F_1^2 - \alpha F_2 &= 0 \end{aligned} \tag{6}$$

Remark: The first equation (6) is a consequence of the other four equations.

We first solve the second equation in (6) by the method of characteristics and then substitute the general solution into the other ones to obtain G after transformation of the system to apropriate coordinates. In terms of $x = x_1, y = x_2, u = F$ the equation reads

$$u_y = \frac{u_x^2}{2\alpha}. \tag{7}$$

The contact ideal I associated to (7) is generated by the contact form ω and its exterior derivative i.e.

$$\begin{aligned} \omega &= du - u_x dx - \frac{u_x^2}{2\alpha} dy \\ d\omega &= -du_x dx - \frac{u_x}{\alpha} du_x dy. \end{aligned} \tag{8}$$

Now, a vector field X

$$X = X^x \partial_x + X^y \partial_y + X^u \partial_u + X^{u_x} \partial_{u_x} \tag{9}$$

is a Cauchy characteristics of I, generated by (8), in case

$$X \lrcorner I \subset I \tag{10}$$

where $\lrcorner$ denotes contraction, from which we derive

$$X^u - u_x X^x - \frac{u_x^2}{2\alpha} X^y = 0$$

and

$$-X^{u_x}(dx + \frac{u_x}{\alpha} dy) + du_x (X^x + \frac{u_x}{\alpha} X^y) = 0.$$

(10) leads to the Cauchy characteristic

$$X = -\frac{u_x}{\alpha}\partial_x + \partial_y - \frac{u_x^2}{2\alpha}\partial_u + 0\partial_{u_x}. \tag{11}$$

Starting from the initial value $u(x, o) = f(x)$ where f is an arbitrary function of x we obtain the parametric solution

$$\begin{aligned} x_1 = x &= -\frac{f'(s)}{\alpha} t + s \\ x_2 = y &= t \\ F = u &= -\frac{f'(s)^2}{2\alpha} t + f(s) \\ F_1 = u_x &= f'(s) \end{aligned} \tag{12}$$

$$\tag{13}$$

Equations (12) define a local $1-1$ transformation $(x_1, x_2) \to (t, s)$ We now substitute (12) into (6) and obtain the following remarkable result i.e.,

$$\begin{aligned} G(x_1, x_2) & \Rightarrow G(t, s) \\ G_{ttt} & = \tfrac{3}{4}\tfrac{f'(s)^4}{\alpha^4} \end{aligned} \tag{14}$$

So $G(t, s)$ is a polynomial of degree 3 in t. Finally, after a short computer calculation we have the final result

$$G = \frac{1}{8}\frac{f'(s)^4}{\alpha^4}t^3 + g_3(s)t^2 + g_2(s)t + g_1(s) \tag{15}$$

where $g_1(s)$ is an arbitrary function i.e., the initial condition of G at $x_2 = 0, g_1(s) = G(x_1, 0)$. The functions $g_2(s), g_3(s)$ satisfy the following equations

$$\begin{aligned} g_2''(s) & = -2g_1'' \cdot \tfrac{f''}{\alpha} - g_1' \tfrac{f'''}{\alpha} + \tfrac{3}{2}\tfrac{(f')^2}{\alpha^2} \\ g_3'(s) & = -g_2' \tfrac{f''}{\alpha} - g_1' \tfrac{(f'')^2}{\alpha^2} - \tfrac{(f')^3}{2\alpha^3} \end{aligned} \tag{16}$$

Equations (16) can be integrated straightforwardly, leading to expressions for $g_3(s)$ and $g_2(s)$.

The even function Φ (3) can be obtained from (5) (12) and (15), (16). The general solution can be written as an expression depending on the intitial data of F, G at $(x_1, 0)$.

Example 2.1 *Take* $f(s) = \frac{\alpha}{2}s^2$ *in (12).*

Then

$$\begin{aligned} x &= -st + s \quad or \quad & t &= y \\ y &= t & s &= \tfrac{x}{1-y} \end{aligned}$$

from which we have $F = \frac{\alpha}{2}\frac{x^2}{1-y}$ *and* $G(x, y)$ *is given by (2.14), (2.15).*

If we choose

$$g_1(s) = \frac{c}{8}s^4$$

where c is some constant, then we arrive after a short computation at

$$G(x, y) = \frac{1}{8}\frac{x^4(y + c)}{(1 - y)^2}.$$

In the last case Φ, F, G *become graded with respect to the following*

$$\deg x_0 = 1, \deg x_1 = \frac{1}{2}, \deg x_2 = 0, \deg(\xi_1) + \deg(\xi_2) = 1$$

and we see that

$$\deg \Phi = \deg G = 2, \deg F = 1.$$

3 The nonisotropic case: $n = 3, m = 2$

We shall now discuss the WDVV-equations (2) and their solutions for the nonisotropic situation. Here the matrix $g_{ab}(a, b = 0, \ldots, 4)$

$$g_{ab} = \partial_0 \partial_a \partial_b \Phi \tag{1}$$

can be chosen in the following form

$$g = \left[\begin{array}{ccc|cc} c_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & -c_2 & 0 \end{array}\right] \tag{2}$$

From (1) we already have, starting from

$$\Phi = \phi_1(x_0, x_1 x_2) + \phi_2(x_0, x_1, x_2)\xi_1\xi_2 \tag{3}$$

$$\begin{aligned} \Phi &= \tfrac{c_1}{6} x_0^3 + x_0 x_1 x_2 - c_2 x_0 \xi_1 \xi_2 + k_1 x_0 x_2 + k_2 x_0 + k_3 x_0 x_1 + k_4 x_0^2 \\ &\quad + F(x_1, x_2)\xi_1\xi_2 + G(x_1, x_2). \end{aligned} \tag{4}$$

Substitution of (4) into the associativity conditions (14) leads to the following system of nonlinear partial differential equations for F and G, i.e.

$$\begin{aligned} -c_1 G_{122} G_{112} + c_1 G_{111} G_{222} - 1 &= 0 \\ c_2^2 + 2c_1 F_1 F_2 &= 0 \\ c_2 F_2 G_{122} + c_2 F_1 G_{222} + (F_2)^2 &= 0 \\ c_2 F_2 G_{122} + c_2 F_1 G_{222} + F_1 F_2 - \frac{c_2^2}{c_1} &= 0 \\ c_2 F_2 G_{111} + c_2 F_1 G_{112} + F_1^2 &= 0 \end{aligned} \tag{5}$$

It should be noted that the classical WDVV equations in the nonisotropic case is a consequence of the other four partial differential equations in (5). In a way, similar as in the previous case we construct the general solution of (5) and determine Φ (3).

Start from the second equation of (5) ($F = u$) i.e.

$$u_1 u_2 = \frac{-c_2^2}{2c_1} = \alpha.$$

Contact ideal I is generated by

$$\begin{aligned} \omega &= du - u_1 dx_1 - u_2 dx_2 = du - u_1 dx_1 - \tfrac{\alpha}{u_1} dx_2 \\ d\omega &= du_1 dx_1 + \alpha (u_1)^{-2} du_1 dx_2. \end{aligned} \tag{6}$$

A Cauchy characteristic X

$$X = X^1 \frac{\partial_x}{\partial x_1} + X^2 \frac{\partial}{\partial x_2} + X^u \frac{\partial}{\partial u} + X^{u_1} \frac{\partial}{\partial u_1} \tag{7}$$

has to satisfy

$$X \lrcorner \omega \in I \ \text{ and } \ X \lrcorner d\omega \in I$$

i.e.

$$X^u - X^1 u_1 - X^2 \alpha u_1^{-1} = 0$$

and

$$-X^{u_1} dx_1 + du_1 X^1 + \alpha (u_1)^{-2} X^{u_1} dx_2 - \alpha (u_1)^{-2} du_1 X^2 \in I$$

from which we derive

$$X^{u_1} = 0 \ , \ X^1 = \alpha (u_1)^{-2} X^2, X^u = u_1 X_1 + \alpha u_1^{-1} X^2.$$

From the above computation we derive the Cauchy characteristic

$$X = \alpha (u_1)^{-2} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + 2\alpha (u_1)^{-1} \frac{\partial}{\partial u} + 0 \frac{\partial}{\partial u_1}. \tag{8}$$

Integration of this vectorfield leads to

$$\begin{aligned} \overline{x}_1 &= \alpha ((u_1)_0)^{-2} t + (x_1)_0 \\ \overline{x}_2 &= t \\ \overline{u} &= 2\alpha ((u_1)_0)^{-1} t + u_0 \end{aligned} \tag{9}$$

$$\overline{u}_1 = (u_1)_0 \tag{10}$$

We now choose initial condition at $y = 0$ i.e.

$$u(s, 0) = f(s). \tag{11}$$

Combination of (9), (11) then gives the general local parametric solution

$$\begin{aligned} x_1 &= \alpha (f'(s))^{-2} t + s \\ x_2 &= t \\ u &= 2\alpha (f'(s))^{-1} t + f(s) \\ u_1 &= f'(s). \end{aligned} \tag{12}$$

Remarkably (12) induces a transformation $(x_1, x_2) \leftrightarrow (t, s)$ which is of course locally one to one, but very interesting for solving the other equations of (5)

$$\begin{aligned} (x_1)_t &= \alpha (f')^{-2}, (x_1)_s = 1 - 2\alpha t (f')^{-3} f'', (x_2)_t = 1, (x_2)_s = 0 \\ &\text{and} \\ (t)_{x_1} &= 0, \ (t)_{x_2} = 1 \\ (s)_{x_1} &= [1 - 2\alpha t (f')^{-3} f''], \ (s)_{x_2} = -\alpha (f')^{-2} [1 - 2\alpha t (f')^{-3} f'']^{-1} \end{aligned} \tag{13}$$

Substitution of (12), (13) into the other equations of (5) yields after some calculations $G \equiv G(t,s)$

$$G_{ttt} = \frac{-2c_2^3}{c_1^2}(f')^{-3}$$

from which we have that G is a polynomial of degree 3 in t. From this we obtain the final result

$$G = \frac{-c_2^3}{3c_1^2}(f')^{-3}t^3 + g_3(s)t^2 + g_2(s)t + g_1(s) \tag{14}$$

where $g_1(s)$ is an arbitrary function of s and g_2, g_3 are obtained from the following equations by quadrature

$$\begin{array}{ll} (g_3)' & = \frac{-c_2^2}{c_1^2} g_1'(f'')^2(f')^{-6} + \frac{c_2^2}{c_1} g_2' f''(f')^{-3} + \frac{c_2}{c_1}(f')^{-1} \\ (g_2)'' & = 2\frac{c_2^2}{c_1} g_1'' f''(f')^{-3} + \frac{c_2^2}{c_1} g_1' f'''(f')^{-3} - 3\frac{c_2^2}{c_1} g_1'(f'')^2(f')^{-4} - \frac{1}{c_2} f' \end{array}$$

Summarizing the computations so far, we obtain the general solutions of (5) and by this function $\tilde{F}$ by the transformation in (x_1, x_2)

$$\begin{array}{lll} x_1 & = & -\frac{c_2^2}{c_1}(f')^{-2} \cdot t + s \\ x_2 & = & t^2 \end{array}$$

where $F \equiv F(t,s), G \equiv G(t,s)$ are given by (13), (15)

while $f(s)$ is the initial condition $f(s) = F(s,0)|(x = s, y = 0)$ and $g_1(s)$ is an arbitrary function of course $g_1(s) = G(s,0)|(x = s, y = 0)$.

Example 3.1 *We choose*

$$f(s) = c_2 s^{1/2}$$

and

$$g_1(s) = as^{3/2}$$

a an arbitrary constant, then we obtain from (13)

$$\begin{array}{lll} t & = & x_2 \\ s & = & \frac{c_1 x_1}{c_1 - 2x_2} \end{array} \tag{15}$$

and

$$\begin{array}{lll} g_2(s) & = & -\frac{2}{3}\frac{(c_1+6a)}{c_1} s^{3/2} \\ g_3 & = & \frac{4}{3}\frac{(2c_1+3a)}{c_1^2} s^{3/2}, \end{array}$$

while $F = u, G$ are obtained from (13), (15).

$$F = u = (-\frac{2c_2}{c_1}t + c_2)s^{1/2} \tag{16}$$

$$G = \left[-\frac{8}{3c_1^2}t^3 + \frac{4}{3}\frac{(2c_1+3a)}{c_1^2}t^2 - \frac{2}{3}\frac{(c_1+6a)}{c_1}t + a\right] s^{3/2} \tag{17}$$

t, s being given by (15). □

4 The isotropic case: $n = 3, m = 4$

In sections 2 we discussed the WDVV equations as they arise in the case of two odd variables. Here we shall investigate the influence in coupling and intertwining in the situation for four odd variables. Local coordinates will be $x_0, x_1, x_2, \xi_1, \xi_2, \xi_3, \xi_4$. The matrix $g_{ab}(2)$ is of the form

$$g_{ab} = \left[\begin{array}{ccc|cccc} & & 1 & 0 & 0 & 0 & 0 \\ & 1 & & 0 & 0 & 0 & 0 \\ 1 & & & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & & & & -c_2 \\ 0 & 0 & 0 & & & -c_3 & \\ 0 & 0 & 0 & & +c_3 & & \\ 0 & 0 & 0 & +c_2 & & & \end{array}\right] \tag{1}$$

The potential Φ from which g_{ab} is determined by

$$g_{ab} = \partial_0 \partial_a \partial_b \Phi \tag{2}$$

has the following structure

$$\begin{array}{ccccccccc} \Phi & = & \frac{1}{2}x_0^2 x_2 & + & \frac{1}{2}x_0 x_1^2 & + & c_2 x_0 \xi_1 \xi_4 & + & c_3 x_0 \xi_2 \xi_3 \\ & + & k_{12}\xi_1\xi_2 & + & k_{13}\xi_1\xi_3 & + & k_{24}\xi_2\xi_4 & + & k_{34}\xi_3\xi_4 \\ & + & k_{02}x_0x_2 & + & k_{01}x_0x_1 & + & k_{00}x_0^2 & & \\ & + & F\xi_1\xi_4 & + & \overline{F}\xi_2\xi_3 & + & H\xi_1\xi_2\xi_3\xi_4 & + & G \end{array} \tag{3}$$

where k_{ij} are constant and $F, \overline{F}, H, G$ are functions of x_1, x_2. The associativity conditions lead to the following system of partial differential equations for $F, \overline{F}, H, G$; i.e.,

$$2c_2F_2 - (F_1)^2 = 0 \tag{4}$$
$$2c_3\overline{F}_2 - (\overline{F}_1)^2 = 0 \tag{5}$$
$$c_3F_2 + c_2\overline{F}_2 + F_1\overline{F}_1 = 0 \tag{6}$$
$$c_2^2G_{222} + c_2F_1G_{122} - F_2^2 = 0 \tag{7}$$
$$c_2^2G_{122} + c_2F_1G_{112} - F_1F_2 = 0 \tag{8}$$
$$c_2^2G_{112} + c_2F_1G_{111} + c_2F_2 - F_1^2 = 0 \tag{9}$$
$$c_3c_2H_2 + c_3H_1F_1 + H^2 = 0 \tag{10}$$
$$c_3c_2^2\overline{F}_{112} + c_3c_2H_1G_{111} + c_3c_2H_2 + c_3c_2F_1\overline{F}_{111} - 2c_3H_1F_1 + 2c_2\overline{F}_{11}H = 0 \tag{11}$$
$$c_3^2G_{222} + c_3\overline{F}_1G_{122} - \overline{F}_2^2 = 0 \tag{12}$$
$$c_3^2G_{122} + c_3\overline{F}_1G_{112} - \overline{F}_1\overline{F}_2 = 0 \tag{13}$$
$$c_3^2G_{112} + c_3\overline{F}_1G_{111} + c_3\overline{F}_2 - \overline{F}_1^2 = 0 \tag{14}$$
$$c_3c_2H_2 + c_2H_1\overline{F}_1 + H^2 = 0 \tag{15}$$
$$c_3^2c_2F_{112} + c_3c_2H_1G_{111} + c_3c_2H_2 + c_3c_2\overline{F}_1F_{111} - 2c_2H_1\overline{F}_1 + 2c_3F_{11}H = 0. \tag{16}$$

In system of equations (4) we presented the equations in a symmetric way in F and $\overline{F}$. It is a straightforward computation that a solution of system (4) $\cdots$ (23) solves the remaining system (13) $\cdots$ (16). In order to solve the system of WDVV equations (4) we proceed in a way similar to that in section 2. First of all, from (4-18) we derive

$$\begin{aligned} F &= c_2(\tfrac{1}{2}(h')^2 t + h) \\ \overline{F} &= c_3(\tfrac{1}{2}(h')^2 t + h + \text{ const }) \end{aligned} \tag{17}$$

where s, t satisfy

$$\begin{aligned} x_1 &= h'(s)t + s \\ x_2 &= t \end{aligned} \tag{18}$$

and h is an arbitrary function of s, $h' = \frac{dh(s)}{ds}$.

We transform the system according to (17), (18) and derive the following result

$$G_{tttt} = 0 \tag{19}$$

i.e., G is a polynomial of degree 3 with respect to t. So

$$G = g_6 t^3 + g_5 t^2 + g_4 t + g_3 \tag{20}$$

where $g_3, \ldots, g_6$ are functions dependent on s. Substitution of (20) leads to the following conditions.

$$H = \frac{c_3 c_2}{c_3 c_2 \overline{h} + t} \tag{21}$$

where $\overline{h}$ is another arbitrary function of s. Finally we arrive at the following expressions for $g_3, \ldots, g_6$

$$\begin{aligned} g_6 &= \tfrac{1}{8}(h')^4 \\ g_5' &= g_4'' h'' - g_3'(h'')^2 + \tfrac{1}{2}(h')^3 \\ g_4'' &= 2g_3'' h'' + g_3' h''' + \tfrac{3}{2}(h')^2 \end{aligned} \tag{22}$$

$$c_3^2 c_2^2 (h'')^2 \overline{h}^2 + c_3 c_2 \overline{h}' g_3''' - 3c_3 c_2 \overline{h}' h' - 2c_3 c_2 h'' \overline{h} + 1 = 0. \tag{23}$$

In case

$$\overline{h}' = 0$$

i.e., $\overline{h} =$ constant; then (23) reduces to

$$h'' = \frac{-1}{c_2 c_3 \overline{h}}$$

from which h is obtained. In this situation there are no restrictions on g_3.

Otherwise, $\overline{h}' \neq 0, g_3, g_4, g_5$ are obtained from (22) , (23) by quadrature.

(17), (20), (21), (22), (23) together with (18) present the general solution of the WDVV equations (4), and so lead to the potential Φ (3)

5 The nonisotropic case: $n = 3, m = 4$

Here we discuss first the graded WDVV equations as they result from the algebraic structure defined in section 1. The number of even variables is $3, (x_0, x_1, x_2)$ and the number of odd variables is $4, (\xi_1, \xi_2, \xi_3, \xi_4)$. The matrix g_{ab} (1) (2) is choosen to be

$$g_{ab} = \left[\begin{array}{ccc|cccc} c_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & 0 & -c_3 & 0 & 0 \\ 0 & 0 & 0 & -c_2 & 0 & 0 & 0 \end{array}\right] \tag{1}$$

Due to the definition of g_{ab} (12) we arrive at the even potential Φ of the form

$$\begin{array}{rcl} \Phi & = & \frac{1}{6}c_1x_0^3 - c_2x_0\xi_1\xi_2 - c_3x_0\xi_2\xi_3 + x_0x_1x_2 \\ & + & k_1x_0 + k_2x_0^2 + k_3x_0x_1 + k_4x_0x_2 \\ & + & f_{12}\xi_1\xi_2 + f_{13}\xi_1\xi_3 + f_{24}\xi_2\xi_4 + f_{34}\xi_3\xi_4 \\ & + & F\xi_1\xi_4 + \overline{F}\xi_2\xi_3 + H\xi_1\xi_2\xi_3\xi_4 + G \end{array} \tag{2}$$

where $k_1, \ldots, k_4$ are constants, $f_{12}, f_{13}, f_{24}, f_{34}, F, \overline{F}, H, G$ being functions dependent on x_1, x_2. Then the associativity relations (6), (13) first lead to the conditions that $f_{12}, f_{13}, f_{24}, f_{34}$ are constant and a system of partial differential equations for $F, \overline{F}, H, G$

$$c_2^2 + 2c_1F_1F_2 = 0 \tag{3}$$

$$c_3^2 + 2c_1\overline{F}_1\overline{F}_2 = 0 \tag{4}$$

$$c_2c_3 + c_1\overline{F}_1F_2 + c_1\overline{F}_2F_1 = 0 \tag{5}$$

$$F_1G_{112} + F_2G_{111} + \frac{1}{c_2}(F_1)^2 = 0 \tag{6}$$

$$F_1G_{122} + F_2G_{112} - \frac{c_2}{c_1} + \frac{1}{c_2}F_1F_2 = 0 \tag{7}$$

$$F_1G_{222} + F_2G_{122} + \frac{1}{c_2}(F_2)^2 = 0 \tag{8}$$

$$H_1G_{222} + H_2G_{122} - \overline{F}_{12}F_{22} - \overline{F}_{22}F_{12} + \frac{1}{c_2}(H_2F_2 - HF_{22}) + \frac{1}{c_3}(H_2\overline{F}_2 - H\overline{F}_{22}) = 0 \tag{9}$$

and

$$G_{122}G_{112} - G_{111}G_{222} + \frac{1}{c_1} = 0 \tag{10}$$

It should be noted that (10) as well as all other equations resulting from the associativity-conditions are consequences of (3) - (9) , which is the system of WDVV-equations in the nonisotropic (3,4)-case. We now construct the general solution for system (3) - (16) in order to construct the potential Φ.
First of all we note that from equations (3), (11), (12) we derive in a similar way as in section 3 and 4

$$\begin{array}{rcl} F & = & -\frac{c_2^2}{c_1}(h')^{-1}t + h \\ \overline{F} & = & -\frac{c_2c_3}{c_1}(h')^{-1}t + \frac{c_3}{c_2}h + \mathrm{const.} \end{array} \tag{11}$$

where t, s are defines by

$$\begin{array}{rcl} x_1 & = & -\frac{c_2^2}{2c_1}(h')^{-2}t + s \\ x_2 & = & t \end{array} \tag{12}$$

whereas h is an arbitrary function of s, $h' = \frac{dh(s)}{ds}$.

We now transform the system of equations (3) according to (12) and derive

$$G_{tttt} = 0 \tag{13}$$

i.e. G is a polynomial of degree 3 in t,

$$G(s,t) = g_5 t^3 + g_4 t^2 + g_3 t + g_2 \tag{14}$$

where $g_2, \ldots, g_5$ are functions of s.

Proceeding with the result (14) we arrive at

$$g_5 = -\frac{c_2^3}{3c_1}(h')^{-3}$$

$$\begin{array}{rcl} g_4' & = & \frac{c2}{c_1^2}[-c_2^3 g_2'(h'')^2 + c_2 c_1 g_3' h''(h')^3 + c_1(h')^5](h')^{-6} \\ g_3'' & = & \frac{1}{c_2\, c_1}[2c_2^3 g_2'' h'' h' + c_2^3 g_2' h''' h' - 3c_2^3 g_2'(h'')^2 \\ & - & c_1(h')^5](h')^{-4} \\ g_2''' & = & \frac{c_3}{c_2}[c_3 c_2^2 (h'')^2 \overline{h}^2 - 2c_1 h''(h')^2 \overline{h} - c_1 (h')^3 \overline{h}']/ \\ & & [c_3 c_2^2 \overline{h}' + c_1 h'] \end{array} \tag{15}$$

$$H = -c_3 h'/(c_3 h' \overline{h} + \overline{t}) \tag{16}$$

In (15, (16) $\overline{h}$ is an arbitrary function of s.

(11), ... , (16) represent the general solution for the potential Φ, expressed in two arbitrary function of s; h and $\overline{h}$.

Conclusion:

We derived the WDVV equations for a (3.2) and (3.4) dimensional manifold and constructed the general solutions for the isotropic and nonisotropic cases.

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