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ON SOME HOCHSCHILD COHOMOLOGY CLASSES OF FUSION ALGEBRAS

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Abstract. The obstructions for an arbitrary fusion algebra to be a fusion algebra of some semisimple monoidal category are constructed. Those obstructions lie in groups which are closely related to the Hochschild cohomology of fusion algebras with coefficients in the K-theory of the ground (algebraically closed) field.
The special attention is devoted to the case of fusion algebra of invariants of finite group action on the group ring of abelian group.

Contents

1. Introduction

Recent activity in the theory of fusion rings (a special kind of semiring) was inspired by the development of quantum field theory [13, 8, 14], although this notion appeared before in different branches of mathematics, for example in the representation theory (semiring of irreducible representations). Most fusion rings considered so far can be represented as fusion rings of simple objects of some semisimple monoidal category [3] (so-called monoidal fusion rings). As we shall see, not all fusion rings admit such a representation.

The article is devoted to the construction of obstructions for an arbitrary fusion ring to be the monoidal. These obstrucions are elements of the Hochschild cohomology of the fusion ring with coefficients in the algebraic K-theory of the ground field. The construction is based on the notion of A_n-space (homotopy associative space) [10, 6, 7, 2]. The obstruction to monoidality is a special case of the obstruction to the extension of A_n-structure to A_{n+1}.

The first obstruction to monoidality, which is an element of the fourth Hochschild cohomology can be expressed explicitly by means of structural constants of a given fusion ring (section 5). In section 6 the triviality of the first obstruction in case of two element fusion ring with identity is analysed.

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2. Fusion rings

Bt definition, a *fusion ring* is a set S with a collection of non-negative integers $\{m^x_{x_1,x_2}, x, x_1, x_2 \in S\}$ (*structural constants*) which satisfy the (*associativity*) condititon

$$m^x_{x_1,x_2,x_3} = \sum_{t\in S} m^x_{x_1,t} m^t_{x_2,x_3} = \sum_{s\in S} m^s_{x_1,x_2} m^x_{s,x_3}, \quad \forall x, x_1, x_2, x_3 \in S.$$

An element e of the fusion algebra S is an *identity* if $m^s_{t,e} = m^s_{e,t} = \delta_{s,t}$ for all $s, t \in S$.

A *morphism* of a fusion ring S to a fusion ring S' is a collection $\{n^s_t, s \in S, t \in S'\}$ of non-negative integers which satisfy the following condition:

$$\sum_{s\in S} m^s_{s_1,s_2} n^t_s = \sum_{t_1,t_2\in S'} {m'}^t_{t_1,t_2} n^{t_1}_{s_1} n^{t_2}_{s_2}$$

for any $s_1, s_2 \in S$ and $t \in S'$.

The *enveloping ring* $A(S)$ of a fusion algebra S is the free $\mathbf{Z}$-module with the basis $\{[s], s \in S\}$ labeled by the elements of S and with the multiplications $[i][j] = \sum_{s\in S} m^s_{i,j}[s]$. A morphism of fusion rings defines a homomorphism of their enveloping rings $f : A(S) \to A(S')$ where $f([s]) = \sum_{t\in S'} n^t_s[t]$.

3. Semisimple monoidal categories

This section is devoted to the most important case of fusion algebras, namely fusion algebras corresponding to semisimple monoidal categories (monoidal fusion algebras).

Let $\mathcal{G}$ be a semisimple monoidal k-linear category over the field k with the set S of isomorphism classes of simple objects. The collection of dimensions $m^x_{y,z} = dim_k Hom_{\mathcal{G}}(X, Y \otimes Z)$ form a fusion ring structure on the set S. Here X, Y and Z are some representatives of the classes $x, y, z \in S$. This fusion ring structure (together with the map from S to the set of simple k-algebras, $x \mapsto End_{\mathcal{G}}(X)$) contains all the information about the tensor product in the category $\mathcal{G}$. But as we shall see, it is not sufficient to reconstruct associativity constraint ϕ of the category. To define this constraint in terms of simple objects it is necessary to fix representative of any class from S. If we do this we can construct a semisimple monoidal category in which isomorphic simple objects coincides and which is equivalent to the category $\mathcal{G}$. We call such a category a *model* $\mathcal{M}(S)$ of the semisimple monoidal category $\mathcal{G}$ with the set S of simple objects. Objects of the model are maps from the set S to the set of finite dimensional vector spaces with only finitely many nonzero values (or formal finite sums $\oplus_{x\in S}V_x x$ of vector spaces). The morphism space $Hom(\oplus_{x\in S}V_x x, \oplus_{y\in S}U_y y)$ coincides with $\oplus_{x\in S}Hom_k(V_x, U_x)\otimes End_{\mathcal{G}}(X)$. By definition, the tensor product of two simple objects $x\otimes y$ is the sum $\oplus_{z\in S}H^z_{x,y}$, where $H^z_{x,y} = Hom_{\mathcal{G}}(Z, X\otimes Y)$. The associativity constraint for the model can be given by the collection of isomorphisms of vector spaces

$$\Phi^x_{x_1,x_2,x_3} : \oplus_{u\in S}H^x_{x_1,u}\otimes H^u_{x_2,x_3} \to \oplus_{v\in S}H^v_{x_1,x_2}\otimes H^x_{v,x_3}, \tag{1}$$

which are defined by the associativity constraint ϕ of the category $\mathcal{G}$

$$\begin{aligned}&\Phi^x_{x_1,x_2,x_3} =\\ &Hom_{\mathcal{G}}(X, \phi_{X_1,X_2,X_3}) : Hom_{\mathcal{G}}(X, X_1\otimes(X_2\otimes X_3))\\ &\to Hom_{\mathcal{G}}(X, (X_1\otimes X_2)\otimes X_3).\end{aligned}$$

The pentagon axiom for the constraint ϕ consists of commutativity of the pentagon diagram [3] for any $x, x_2, x_3, x_3, x_4 \in S$. This condition can be written in a form of the equation:

$$(\oplus_{b\in S}(\Phi^x_{b,x_3,x_4})_{23})(\oplus_{a\in S}(\Phi^t_{x_1,x_2,a})_{12}) =$$

$$(\oplus_{f\in S}(\Phi^f_{x_1,x_2,x_3})_{12})(\oplus_{d\in S}(\Phi^t_{x_1,d,x_4})_{13})(\oplus_{c\in S}(\Phi^c_{x_2,x_3,x_4})_{23}),$$

which is denoted by $A^x_{x_1,x_2,x_3,x_4}$.

Below, in case of simplicity we consider k-linear semisimple categories for which $End(X) = k$ for any simple object X. For example, it is true for algebraically closed field k.

If we start from arbitrary fusion ring S we also can construct the semisimple category $\mathcal{G}(S)$ with a tensor product whose simple objects are parametrized by the elements of S. Since the vector spaces

$$\oplus_{u\in S}\, H^x_{x_1,u}\otimes H^u_{x_2,x_3} \quad \text{and} \quad \oplus_{v\in S}\, H^v_{x_1,x_2}\otimes H^x_{v,x_3} \tag{2}$$

have the same dimension $m^x_{x_1,x_2,x_3}$, they are isomorphic, i.e. the tensor product is *quasiassociative*. It is associative if it is possible to choose the isomorphisms between vector spaces (2) satisfying the pentagon axiom.

In case of the fusion ring consisting of one element x with the fusion rule $x*x = nx$ (the structural constant $m^x_{x,x}$ equals n) the quasiassociative structure is an automorphism Φ of vector space $H\otimes H$. Here $H = H^x_{x,x}$ is an n-dimensional vector space. The pentagon axiom for this structure is equivalent to the following (*pentagon*) equation

$$\Phi_{12}\Phi_{13}\Phi_{23} = \Phi_{23}\Phi_{12}. \tag{3}$$

Here Φ_{ij} denotes the automorphism of $H^{\otimes 3}$ acting on i-th and j-th components.

With any structure of Hopf algebra on the vector space H we can associate a solution of the equation (3):

$$\Phi(g\otimes h) = \sum_{(g)} g_{(0)}\otimes h_{(0)} \quad \forall g,h\in H.$$

General solutions of the equation (3) are similar to these ones. The proof of this fact will be published elsewhere.

The pair consisting of the identity e and the element x with fusion rule $x*x = ne$ provides an example of a fusion ring whose category does not admit a monoidal structure (there is no quasiassociative structure satisfying the pentagon axiom). To simplify the verification of this fact let us prove the following lemma.

Lemma 1 *Let S be a fusion ring with identity e and Φ be an associativity constraint for a corresponding semisimple k-linear category. Then there are identifications $\rho_x : H^x_{x,e} \to k$ and $\lambda_x : H^x_{e,x} \to k$ (for any x) such that $\Phi^z_{x,y,e}, \Phi^z_{x,e,y}$ and $\Phi^z_{e,x,y}$ coincide with the compositions:*

$$(I \otimes \rho_z^{-1})(I \otimes \rho_y) : H^z_{x,y} \otimes H^y_{y,e} \to H^z_{x,y} \to H^z_{x,y} \otimes H^z_{z,e}, \tag{4}$$

$$(\rho_x^{-1} \otimes I)(I \otimes \lambda_y) : H^z_{x,y} \otimes H^y_{e,y} \to H^z_{x,y} \to H^x_{x,e} \otimes H^z_{z,e}, \tag{5}$$

$$(\lambda_x^{-1} \otimes I)(\lambda_z \otimes I) : H^z_{e,z} \otimes H^z_{x,y} \to H^z_{x,y} \to H^x_{e,x} \otimes H^z_{z,e}. \tag{6}$$

Proof:

It follows from the equations $A^x_{x,e,e,e}, A^x_{e,x,e,e}, A^x_{e,e,x,e}$ and $A^x_{e,e,e,x}$ that one can choose the isomorphisms $\rho_x : H^x_{x,e} \to k$ and $\lambda_x : H^x_{e,x} \to k$ such that $\Phi^x_{x,e,e}, \Phi^x_{e,x,e}$ and $\Phi^x_{e,e,x}$ have the form 5,4 and 6 respectively.

Conversely, equations which involve three identity elements imply the statement of the lemma. □

Let us return to the fusion ring $\{e, x\}$ with fusion rule $x * x = ne$. Identifying $H^x_{x,e}$ and $H^x_{e,x}$ with k we can see that unique nontrivial component $\phi^x_{x,x,x} : H^x_{x,e} \otimes H^e_{x,x} \to H^e_{x,x} \otimes H^x_{e,x}$ of a monoidal structure for the category $\mathcal{G}(S)$ (if it exists) corresponds to automorphism Φ of n-dimensional vector space $H = H^e_{x,x}$. The pentagon condition $A^e_{x,x,x,x}$ is equivalent to the equation $\Phi^2 \otimes I = t$, where I is the identity automorphism of the space H and $t : H^{\otimes 2} \to H^{\otimes 2}$ is the permutation of tensor factors (the isomorphisms $(\Phi^e_{x,x,e})_{12}$ and $(\Phi^e_{e,x,x})_{23}$ are identical by the lemma, $(\Phi^x_{x,x,x})_{23}$ and $(\Phi^x_{x,x,x})_{12}$ correspond to $I \otimes A$ and $A \otimes I$ respectively and $(\Phi^e_{x,e,x})_{13}$ to t). It is easy to see that this equation has solutions iff $n = 1$.

This example motivates the following definition. A fusion ring is called *monoidal* if it coincides with the fusion ring of simple objects of some semisimple monoidal category.

Any monoidal k-linear functor [3] $F : \mathcal{G} \to \mathcal{G}'$ between semisimple monoidal k-linear categories provides the morphism of fusion algebras S and S' of simple objects. This morphism is given by the collection of dimensions $n^x_y = dim_k Hom_{\mathcal{G}'}(X, F(Y))$. Here $X \in \mathcal{G}'$ and $Y \in \mathcal{G}$ are some representatives of the classes $x \in S'$,$y \in S$. This morphism structure contains all information about the functor F (without its monoidal structure). Let us denote by H^x_y the n^x_y-dimensional vector space $Hom_{\mathcal{G}'}(X, F(Y))$.

The monoidal constraint for this functor can be given by the collection of isomorphisms of vector spaces

$$\Psi^x_{y,z} : \oplus_{u \in S} H^u_{y,z} \otimes H^x_u \to \oplus_{v,w \in S'} H^v_y \otimes H^w_z \otimes H^x_{v,w},$$

which are defined by the monoidal constraint ψ of the functor F:

$$\Psi^x_{y,z} = Hom_{\mathcal{G}'}(X, \phi_{Y,Z}) : Hom_{\mathcal{G}'}(X, F(Y \otimes Z)) \to Hom_{\mathcal{G}'}(X, F(Y) \otimes F(Z)).$$

The compatibility axiom for the constraint ϕ states the commutativity of hexagon diagrams [3] for any $x \in S', y, z, w \in S$. This condition can be written in a form of the equation:

$$(\oplus_{t_1,t_2,t_3 \in S'}(\Phi^t_{t_1,t_2,t_3})_{13})(\oplus_{y \in S'}(\Psi^y_{s_1,s_2})_{12})(\oplus_{x \in S} t_{12}(\Psi^t_{s,x})_{23})t_{12}) =$$
$$t_{34}(\oplus_{w \in S'}(\Psi^w_{s_1,s_2})_{12})(\oplus_{z \in S}(\Psi^t_{z,s_3})_{23})(\oplus_{s \in S}(\Phi^s_{s_1,s_2,s_3})_{12}).$$

The condition of an existence of a monoidal functor between semisimple monoidal categories $\mathcal{M}(S)$, $\mathcal{M}(S')$ which induce a given morphism of fusion rings S, S' is called *monoidality* of the morphism (with respect to the given monoidal structures on the categories $\mathcal{M}(S)$, $\mathcal{M}(S')$).

The simplest example of a nonmonoidal morphism of fusion rings is provided by the identity map of the one-element fusion ring $S = \{x, x * x = nx\}$ with respect to monoidal structures on the category $\mathcal{M}(S)$ defined by two non-isomorphic n-dimensional Hopf algebras.

4. A_n-SPACES

Let us denote by T^k the set of planar trees with k ends. This set is a disjoint union $\bigsqcup_{i=0,...,k-2} T^k_i$ of subsets T^k_i, which consists of trees with $k-i-1$ vertices. Junction of two neighboring vertices defines the correspondence $R^k \subset T^k \times T^k$. Namely,

$$R^k = \{(t, t')\} \in T^k \times T^k,$$

where t' is a result of the contraction of an internal edge of t.

This correspondence also decomposes into a disjoint union $\bigsqcup_{i=0,...,k-3} R^k_i$, where R^k_i is contained in $T^k_i \times T^k_{i+1}$. The unique element of T^k_{k-2} is denoted by $t(k)$.

This data permits us to define cell complexes BT^k (Stasheff's complexes K_k, [10]): elements of the set T^k_i parametrizes the i-dimensional cells of BT^k, which are glued by means of the correspondence R.

The operation of gluing trees (roots of the trees $t_1, ..., t_k$ are glued to the ends of the tree $t \in T^k$) defines the maps

$$c^{k,l_1,...,l_k} : T^k_i \times T^{l_1}_{i_1} \times ... \times T^{l_k}_{i_k} \to T^{l_1+...+l_k}_{i_1+...+i_k+i}.$$

These maps induce a (non-symmetric) operad structure [6]:

$$BT^k \times BT^{l_1} \times ... \times BT^{l_k} \to BT^{l_1+...+l_k}.$$

Let us denote by $\partial^i_k(t) \in T^{k+1}$ the result of gluing the root of the tree $t(2)$ to i-th end of the tree t for $t \in T^k$ and $i = 1, ..., k$. For $i = 0, k+1$ the expression $\partial^i_k(t)$ denotes the result of gluing the root of the tree t to the second and first ends of the tree $t(2)$ respectively.

Let us shoose the orientations of the complexes BT^k such that the maps $\partial^1_k : BT^k \to BT^{k+1}$ preserve the orientations.

It should be noted that for any space X the collection of spaces $Hom(X^k, X)$ also forms an operad.

An A_n-*space* structure [10, 2, 6] is a collection of continuous maps $BT^k \to Hom(X^k, X)$ for $k \leq n$ that are compatible with compositions.

It is easy to see that the cells which correspond to the trees from the image of the composition map
$$c^{k,l_1,\dots,l_k}(T_i^k \times T_{i_1}^{l_1} \times \dots \times T_{i_k}^{l_k})$$
are contractible in $Hom(X^{l_1+\dots+l_k}, X)$ if $i_p, i_q > 0$ for some p, q.

For example, an A_3-space structure on X consists of a product $\mu : X \times X \to X$ (the image of $pt = BT^2 \to Hom(X^2, X)$) and a homotopy between $\mu(I \wedge \mu)$ and $\mu(\mu \wedge I)$ (the image of $I = BT^3 \to Hom(X^3, X)$).

An A_{n+i}-structure on the space X for which the restriction on the k-cells for $k \leq n$ coincides with the given A_n-structure is called an *extension*. Two A_n-extensions of a given A_{n-1}-structure on the space X define a map $X^n \to \Omega^{n-2}X$. Indeed, these A_n-structures differ only by the maps $I^{n-2} = BT^n \to Hom(X^n, X)$ of the $n-2$-cell. In particular, these maps coincide on the boundary $S^{n-3} = I^{n-2}$. Hence we can glue them together along the boundary and create a map $S^{n-2} = I^{n-2} \coprod_{\partial I^{n-2}} I^{n-2} \to Hom(X^n, X)$ or $X^n \to \Omega^{n-2}X$.

Now let us define the cohomology of A_3-spaces with coefficients in some bimodule.

An A_3-*bimodule* over an A_3-space X is a space Y together with the continuous maps
$$\nu : Y \wedge X \to Y, \qquad \upsilon : X \wedge Y \to Y$$
and homotopies
$$\nu(I \wedge \mu) \to \nu(\nu \wedge I), \quad \upsilon(I \wedge \nu) \to \nu(\upsilon \wedge I), \quad \upsilon(I \wedge \upsilon) \to \upsilon(\mu \wedge I).$$
For example, the space of maps to the A_3-space is a A_3-bimodule over this space.

Denote by $[X, Y]$ the set of homotopy classes of continuous maps from X to Y. The *topological cobar* complex of a A_3-space X with coefficients in a A_3-bimodule space Y is a cosimplicial complex of sets
$$C_*(X, Y), \qquad C_n(X, Y) = [X^n, Y]$$
with the coface maps $\partial_n^i : C_{n-1}(X, Y) \to C_n(X, Y)$ defined as follows:
$$\partial_n^i(f) = \begin{cases} \upsilon(I \times f), & i = 0 \\ f(I \times \dots \times \mu \times \dots \times I), & 1 \leq i \leq n \\ \nu(f \times I), & i = n+1 \end{cases}$$
If Y is a loop space (with a loop structure which is not connected with the given A_3-structure) then $C.(X, Y)$ is a complex of groups and these groups are abelian if Y is a double loop space. In the second case the cohomology of the cochain complex associated with $C_*(X, Y)$ is called the *topological* cohomology ($H^*(X, Y)$) of the A_3-space X with coefficient in the A_3-bimodule Y.

Let us note that an A_n-space structure on X (and the operad structures on BT^* and $Hom(X^*, X)$) allows to define maps $BT^l_{\leq n-2} \to Hom(X^l, X)$ for any l. Here $BT^l_{\leq n-2}$ denotes the union of i-dimensional cells in BT^l with $i \leq n-2$.

For example, the map $S^{n-2} = \partial BT^{n+1} = BT^{n+1}_{\leq n-1} \to Hom(X^{n+1}, X)$ defines an element in $Hom(X^{n+1}, \Omega^{n-2}X)$.

Theorem 1 *Let X be an A_n-space for $n \geq 3$ (and independently a loop space if $n = 3$). Then the class in $[X^{n+1}, \Omega^{n-2}X]$ of the map $BT^{n+1}_{\leq n-2} \to Hom(X^{n+1}, X)$ is a cocycle. Its class $\alpha(X)$ in cohomology $H^{n+1}(X, \Omega^{n-2}X)$ does not depend on the choice of the $n-2$-dimensional component of an A_n-structure on the space X and it is trivial iff there is a $n-2$-deformation of the given A_n-structure which can be extended to an A_{n+1}-structure.*

Proof:

Let us consider the map $BT^{n+2}_{\leq n-2} \to Hom(X^{n+2}, X)$. The complex $BT^{n+2}_{\leq n-2}$ is a union of $n-2$-spheres (boundaries of $n-1$-cells), which correspond to planar trees with $n+2$ ends and two vertices. Hence this map provides an equation on the classes of these spheres in $[X^{n+2}, \Omega^{n-2}X]$. The cells labeled by trees, whose vertices have valences (the number of incident, e.g., incoming and outcoming edges) greater then three, are contractible in $Hom(X^{n+2}, X)$. So (potentially) non-trivial components correspond to the trees that can be presented as glueing of trees $t(n+1)$ and $t(2)$, e.g., trees of the form $\partial^i_{n+1}(t)$ for $i = 0, ..., n+2$. Their classes in $[X^{n+2}, \Omega^{n-1}X]$ are equal to $\partial^i_{n+1}(\alpha(X))$ respectively. Hence the map $BT^{n+2}_{\leq n-2} \to Hom(X^{n+2}, X)$ defines the equation $\partial(\alpha(X)) = 0$ in $[X^{n+2}, \Omega^{n-2}X]$ which means that $\alpha(X)$ is a $n+1$-cocycle.

Another choice of the n-component of an A_n-structure provides the cocycle which differs from the previous one by coboundary $\partial(f)$ of the map f which is defined by two A_n-extensions of the given A_{n-1}-structure.

Triviality of the cohomology class of the cocycle α means that we can choose the n-component of A_n-structure such that it can be extendable to an A_{n+1}-structure. $\square$

To define A_n-maps between A_n-spaces we need the following notion. A *tricolored* tree is a planar tree which vertexes are labeled by one of three ordered indices $0 < \epsilon < 1$. The index of vertix which lies over are not less then index of given vertix; moreover the index of the vertix which lies over ϵ-labeled vertix is 1.

As in the non-colored case the set CT^k of tricolored trees with k ends can be presented as the disjoint union $\bigsqcup_{i=0,...,k-1} CT^k_i$ of subsets, which consist of trees with $k - i - 1 + n_\epsilon$ vertices. Here n_ϵ is the number of ϵ-labeled vertices.

We also define the color version CR of the correspondence R by the condition that $(t, t') \in CR$ if the tree t' can be obtained from the tree t by contraction of an edge between equally labeled vertices or between vertices one of which is labeled by ϵ.

This allows us to define cell complexes BCT^k whose i-cells are parametrized by elements of CT^k_i and are glued by means of the correspondence CR.

Two collection of inclusions $T^k_i \to CT^k_i$ which label all vertices by 0 and 1 respectively provides two collections of compositions:

$$u^{k,l_1,...,l_k} : T^k_i \times CT^{l_1}_{i_1} \times ... \times CT^{l_k}_{i_k} \to CT^{l_1+...+l_k}_{i_1+...+l_k+i},$$

and

$$d^{k,l_1,...,l_k} : CT^k_i \times T^{l_1}_{i_1} \times ... \times T^{l_k}_{i_k} \to CT^{l_1+...+l_k}_{i_1+...+l_k+i},$$

which satisfy the usual operad axioms and define two collections of cell complex maps:

$$u^{k,l_1,...,l_k} : BT^k \times BCT^{l_1} \times ... \times BCT^{l_k} \to BCT^{l_1+...+l_k},$$

and

$$d^{k,l_1,\dots,l_k} : BCT^k \times BT^{l_1} \times \dots \times BT^{l_k} \to BCT^{l_1+\dots+l_k}.$$

An A_n-*map* between A_n-spaces X and Y is a collection of continuous maps $BCT^k \to Hom(X^k, Y)$ for $k \le n-1$, which are compatible with the A_n-structures.

For example, an A_3-map between A_2-spaces X, Y is a continuous map $f : X \to Y$, which is the image of $pt = BCT^1 \to Hom(X,Y)$ and a homotopy $\alpha : f\mu \to \mu(f \times f)$, which is defined by the map $I = BCT^2 \to Hom(X^2, Y)$ ($f\mu$ is the image of the 0-cell which corresponds to the 1-labeled tree $t(2)$, $\mu(f \times f)$ is the image of the 0-cell, corresponding to the 0-labeled tree $t(2)$ and homotopy α is the image of the 1-cell, corresponding to the ϵ-labeled tree $t(2)$). We also call this structure an A_3-structure on the map f. Similarly, A_n-structure on the map $f : X \to Y$ between A_n-spaces is an A_n-map structure such that the image of $pt = BCT^1 \to Hom(X,Y)$ coincides with f.

Note that A_3 map between A_3-spaces X, Y allows to define the structure of A_3-bimodule over an A_3-space X on a space Y. In particular, if Y is a loop, the space we have defined is the topological cohomology space $H^*(X, \Omega^* Y)$.

Theorem 2 *Let $(X, \mu_X), (Y, \mu_Y)$ be A_n-spaces (and Y also is independently a loop spaces if $n = 3$) and let $f : X \to Y$ be a A_n-map. Then the images $f_*(\alpha(X)), f^*(\alpha(Y))$ of their canonical classes in $H^{n+1}(X, \Omega^{n-2}Y)$ coincide.*

If the A_n-structures on X, Y can be extended to an A_{n+1}-structure, then the class $\alpha(f)$ in the cohomology $H^n(X, \Omega^{n-2}Y)$ is defined. It does not depend of the choice of n-dimesional component of A_n-structure on f and it is trivial iff this structure can be extended to an A_{n+1}-structure.

Proof:

Let us note that an A_n-map structure on f (and composition operations on BCT^* and $Hom(X^*, Y)$) allows us to define maps $BCT^l_{\le n-2} \to Hom(X^l, Y)$ for any l. Here $BCT^l_{\le n-2}$ denotes the union of i-dimensional cells in BCT^l with $i \le n-2$.

Let us consider the element $\alpha(f)$ in $[X^n, \Omega^{n-2}Y]$ which corresponds to the map of $n-2$-sphere $\partial BCT^n = BCT^n_{\le n-2} \to Hom(X^n, Y)$. This element satisfies the equation which is defined by the map $BCT^{n+1}_{\le n-2} \to Hom(X^{n+1}, Y)$. The cell complex $BCT^{n+1}_{\le n-2}$ is a union of boundaries of $n-1$-cells, which correspond to tricolored trees with $n+1$ ends and $n_\epsilon + 1$ vertices (it means that all except one vertices of the tree are ϵ-labeled). Nondegenerated components correspond to the trees with $n_\epsilon \le 1$. The case $n_\epsilon = 0$ consists of two $n-2$-spheres (two tricolored trees of the form $t(n+1)$: one is 1-labeled and another 0-labeled) whose classes in $[X^{n+1}, \Omega^{n-2}Y]$ are $f_*(\alpha(X))$ and $f^*(\alpha(Y))$ respectively. In case of $n_\epsilon = 1$ we have a trees of the form $\partial^i_n(t(n))$ for $i = 0, \dots, n+1$. Nondegenerate colored trees correspond to the case where the tree $t(n)$ is ϵ-labeled. Their classes in $[X^{n+2}, \Omega^n X]$ are equal to $\partial^i_{n+1}(\alpha(X))$ respectively. Hence the map $BT^{n+2}_{\le n-2} \to Hom(X^{n+2}, X)$ defines the equation $\partial(\alpha(X)) = 0$ in $[X^{n+2}, \Omega^{n-2}X]$. $\square$

Let us note that for an A_4-space (X, μ_X) and A_4-bimodule (Y, μ_Y) the collection of abelian groups $H^p(X, \Omega^q Y)$ is equipped with a differential $d^{p,q} : H^p(X, \Omega^q Y) \to H^{p+2}(X, \Omega^{q+1} Y)$. Indeed, let χ be a representative for some cocycle $x \in C^p(X, \Omega^q Y)$. We can choose the homotopy τ between the map $\partial(\chi) : X^{p+1} \to \Omega^q Y$ and the trivial map 1_{p+1} (the map into a fixed point). Since $\partial(1_{p+1})$ and $\partial^2(\chi)$ can be canonically identified with the trivial map $1_{p+2} : X^{p+2} \to \Omega^q Y$ the homotopy $\partial(\tau)$ can be regarded as an autohomotopy of 1_{p+2} or as a map $K^{p+2} \to \Omega^{q+1}Y$. It is easy to see that the class of this map in $[X^{p+2}, \Omega^{q+1}Y]$ is a cocycle and the class of this cocycle in $H^{p+2}(X, \Omega^{q+1}Y)$ does not depend of the choice of χ and τ and defines a homomorphism $d : H^p(X, \Omega^q Y) \to H^{p+2}(X, \Omega^{q+1}Y)$.

Direct checking shows that the canonical class of an A_n-space (A_n-map) lies in the kernel of the differential d and the image of the canonical class in $E_3^{p,q} = ker(d^{p,q})/im(d^{p-2,q-1})$ does not depend on the choice of not only the n-component of the A_n-structure (theorems 1,2) by also on the choice of the $n-1$-component.

Moreover, for an A_n-space (X, μ_X) and A_n-bimodule (Y, μ_Y) there is a "restricted spectral sequence" consisting of abelian groups $E_k^{p,q}$ $(k \leq n)$ and differentials

$$d_k^{p,q} : E_k^{p,q} \to E_k^{p+k,q+k-1}$$

$(k \leq n-1)$ such that $E_k^{p,q} = ker(d_{k-1}^{p,q})/im(d_{k-1}^{p-k,q-k+1})$. This restricted spectral sequence can be constructed using filtrations on the groups $[X^*, Y]$ which are defined as follows.

An A_k-*cocycle* of an A_n-space X with coefficients in an A_n-bimodule Y $(k \leq n)$ is the class in $[X^m, Y]$ of a map $f : X^m \to Y$ with the homotopies $\tau_1 : \partial(f) \to 1$, $\tau_i : \partial(\tau_{i-1}) \to 1$ for $2 \leq i \leq k$.

It is useful for applications to change the group of values of canonical classes. Since an A_3-space (bimodule) structure on a loop space induces a graded ring (bimodule) structure on its homotopy groups, the natural map from $C_*(X, Y)$ to the Hochschild complex $C_*(\pi.(X), \pi.(Y))$ of the ring $\pi_*(X)$ with coefficients in the bimodule $\pi_*(Y)$ induces the homomorphism of cohomology

$$H^*(X, Y) \longrightarrow HH_*(\pi_*(X), \pi_*(Y)).$$

So we have the class $\alpha(X) \in HH_{n+3}(\pi_0(X), \pi_n(X))$ for any A_n-space X.

5. Cohomological obstruction for monoidality

In [9] Quillen had associated to an abelian (exact) category $\mathcal{A}$ a topological space $BQ\mathcal{A}$ such that exact functors between categories define continuous maps between the corresponding spaces and isomorphisms of functors define homotopies between the corresponding maps. In other words, Quillen's space is a 2-functor from the 2-category of abelian (exact) categories (with isomorphisms of functors as 2-morphisms) to the 2-category of topological spaces. Waldhausen [15] proved that the 2-functor $K = \Omega BQ$ is permutative (in some sense) with the natural product operations. Namely, he constructed a continuous map $K(\mathcal{A}) \wedge K(\mathcal{B}) \to K(\mathcal{C})$ for any biexact functor $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$. He also proved that $K(\mathcal{A})$ is an infinite loop space for any abelian category $\mathcal{A}$. The homotopy groups $K_*(\mathcal{A}) = \pi_*(\mathcal{A})$ of the Waldhausen space $K(\mathcal{A})$ are called the *algebraic K-theory* of the category $\mathcal{A}$. The 2-categorical nature of the functor K implies that the Waldhausen space $K(\mathcal{G})$ is an A_∞-space for any abelian monoidal category $\mathcal{G}$ with respect to the product corresponding to the monoidal product in $\mathcal{G}$. In particular, the space $K(\mathcal{M}_k(S))$ is A_∞-space for any monoidal (over the field k) fusion algebra S.

The structure of the space $K(\mathcal{M}_k(S))$ motivates the following construction. If S is a fusion ring and K is a homotopy associative space (which is also a loop space), then we can construct a new space $K(S)=\bigvee_{s\in S}K$ with the product defined by the fusion rule of S. Namely, the component of $\mu_{K(S)}$ from the i and j-labeled components of $K(S)$ to the k-labeled has the form $\mu_{K(S)}{}_{i,j}^{k}=m_{i,j}^{k}\mu_K$ where the multiplication by the non-negative integer $m_{i,j}^{k}$ is defined by means of internal loop-space structure.

We say that the fusion ring S is of *A_n-type* with respect to an A_∞-space K if the natural A_3-structure on $K(S)$ is extendeble to an A_n-structure. The morphism of fusion rings S,S' is of *A_n-type* with respect to A_∞-space K if the corresponding map $K(S)\to K(S')$ has an A_n-structure.

The *canonical* class $\alpha(S)$ of the fusion ring S with respect to the A_∞-space K is a class of the homotopy associative space $K(S)$ in the Hochschild cohomology $HH^*(A(S),A(S)\otimes\pi_*(K))$. The following theorem is a direct corollary of the definitions.

Theorem 3 *For the monoidal (over the field k) fusion ring S the canonical cohomology class $\alpha_*(S)\in HH^*(A(S),A(S)\otimes K_*(k))$ is trivial.*

Properties of canonical classes of A_n-spaces proved in the previous section (theorem 2) imply the following theorem.

Theorem 4 *Let the fusion rings S,S' be of A_n-type with respect to A_∞-space K. Let $f: A(S)\to A(S')$ be the homomorphism of enveloping rings which corresponds to the morphism of fusion rings S and S' of A_n-type. Then the canonical classes of these fusion rings satisfy the following condition:*

$$f_*(\alpha(X))=f^*(\alpha(Y))\in HH^{n+3}(A(S),A(S')\otimes\pi_n(K)).$$

If the fusion rings S,S' are of A_{n+1}-type then the class $\alpha(f)$ in cohomology $HH^{n+2}(A(S),A(S')\otimes\pi_n(K))$ is defined. It is trivial if the map f admits a A_{n+1}-structure.

For example, any fusion ring is of A_3-type with respect to arbitrary A_∞-space K. Hence the class $\alpha(S)\in HH^4(A(S),A(S')\otimes\pi_1(K))$ is defined. Any morphism of fusion rings is also of A_3-type; hence the images $f_*(\alpha(X)),f^*(\alpha(Y))$ in $HH^4(A(S),A(S')\otimes\pi_1(K))$ of the first classes of the fusion rings S,S' (connected by the morphism f) coincide.

The first class admits a more direct description with respect to the space $K(\mathcal{M}(k))=BGL(k)^+$ for the field k. Namely, it coincides with the class of the Hochcshild cocycle $A\in Z^4(A(S),A(S)\otimes k^*)$, which is defined by the following formula

$$A(x_1,x_2,x_3,x_4)=\sum_{x\in S}\det(A^x_{x_1,x_2,x_3,x_4}),$$

where k^* is the group of invertible elements of the field k and $det(A^x_{x_1,x_2,x_3,x_4})$ is the determinant of the linear automorphism, which is a (clockwise) circumference of the diagram $A^x_{x_1,x_2,x_3,x_4})$.

It follows from the description that first class of fusion ring S in $HH^4(A(S),A(S)\otimes k^*)$ is the image of the class in $HH^4(A(S),A(S)\otimes\pi_1(B\Sigma^+_\infty))$ where the space $B\Sigma^+_\infty$ is Quillen's plus-construction applyed to the infinite symmetric group (an analog of the Waldhausen space $K(\mathcal{A})$ for the category of finite sets). Barrat-Priddy-Quillen theorem [1, 11, 12] states that this space is homotopically equivalent to the spheric spectrum. In particular, its homotopy groups (stable homotopy groups of spheres) are torsion and

$$\pi_1(B\Sigma_\infty^+) = \mathbf{Z}/2\mathbf{Z}, \quad \pi_2(B\Sigma_\infty^+) = \mathbf{Z}/2\mathbf{Z}, \quad \pi_3(B\Sigma_\infty^+) = \mathbf{Z}/24\mathbf{Z}.$$

6. Calculation of the first obstruction

In this section we give the combinatorial description of the cocycle representing the first obstruction and calculate its class in case of two element fusion ring with identity.

To write the first obstruction in an explicit form, choose a linear ordering $>$ on the set S and linearly ordered sets

$$X_{y,z}^x = \{f_{y,z}^x(i), \quad i = 1, ..., m_{y,z}^x\}$$

for any $x, y, z \in S$. This data allows us to define the 'well' order on the sets

$$\bigsqcup_{u\in S} X_{y,u}^x \times X_{z,w}^u, \quad \bigsqcup_{v\in S} X_{y,z}^v \times X_{v,w}^x$$

and to define the map $\Phi_{y,z,w}^x$ such sets as an (unique) order preserving the bijection. More precisely, elements of the first set

$$f_{y,z,w}^x(u,i,j) = f_{y,u}^x(i) \times f_{z,w}^u(j)$$

are parametrized by the collections

$$(u,i,j), \quad u \in S, i = 1, ..., m_{y,u}^x, j = 1, ..., m_{z,w}^u,$$

which are ordered lexicographically

$$(u,i,j) > (u',i',j') \Longleftrightarrow \left\{ \begin{array}{lll} u > u' & & \\ u = u', & i > i' & \\ u = u', & i = i', & j > j' \end{array} \right. .$$

The elements of the second set

$$g_{y,z,w}^x(v,s,t) = f_{y,z}^v(t) \times f_{v,w}^x(s)$$

are parametrized by the collections

$$(v,s,t), \quad v \in S, s = 1, ..., m_{y,z}^v, j = 1, ..., m_{v,w}^x,$$

are ordered analogously:

$$(v,s,t) > (v',s',t') \Longleftrightarrow \left\{ \begin{array}{lll} v > v' & & \\ v = v', & s > s' & \\ v = v', & s = s', & t > t' \end{array} \right. .$$

The following lemma is useful for the calculation of the first obstruction.

Lemma 2 *1. The sign of the unique order preserving bijection between lexicographically ordered sets $X \times Y$ and $Y \times X$ (the permutation of factors) equals $(-1)^{\binom{|X|}{2}\binom{|Y|}{2}}$.*

2. The sign of the unique order preserving bijection between lexicographically ordered disjoint unions $\bigsqcup_{a,b\in S} X(a,b)$ and $\bigsqcup_{b,a\in S} X(a,b)$ equals

$$(-1)^{\sum_{a_1>a_2,b_1<b_2}|X(a_1,b_1)||X(a_2,b_2)|}.$$

Proof:

The sign the automorphism of the set X sending one well order $>_1$ to another $>_2$ is determined by the evenness of the number of inversions of orders, e.g. the quantity of pairs $x, y \in X$ such that $x >_1 y$ and $x <_2 y$.

1. The inversion of two lexicographic orders on $X \times Y$ is a pair $(x_1, y_1), (x_2, y_2) \in X \times Y$ such that $x_1 > x_2$ and $y_1 < y_2$. The quantity of these pairs is $\binom{|X|}{2}\binom{|Y|}{2}$.

2. The inversion of two lexicographic orders on the disjoint union $\bigsqcup_{a,b\in S} X(a,b)$ is a pair $x \in X(a_1,b_1), y \in X(a_2,b_2)$ such that $a_1 > a_2$ and $b_1 < b_2$. The quantity of these pairs is $\sum_{a_1>a_2,b_1<b_2}|X(a_1,b_1)||X(a_2,b_2)|$. $\square$

To write down the first obstruction, let us fix an order on the set of vertices of any planar binary tree.

First of all, we can regard any planar tree as a partial order on the set of its vertices V. Thus the set V decomposes into the disjoint union of subsets of vertices. Since the tree is planar we can draw all vertices from these subsets lying on the same horizontal line. We can define the well order on V ordering them left to right.

The order on the set of vertices of the planar binary tree T with n ends allows to define an (lexicographic) order on the set

$$X^y_{x_1,\dots,x_n}(T) = \bigsqcup_{f:E\to S} \times_{v\in V} X^{f(e_1)}_{f(e_2),f(e_3)},$$

where the sum is over all markings of the tree T, e.g., the functions from the set of edges E to the fusion ring S, such that the values on the ends are $x_1, \dots, x_n$; $e_1 \in E$ is the unique edge with the end v and e_2, e_3 are edges which begins at v.

In particular, we can define the orders on the (five) sets corresponding to the planar binary trees with four ends which parametrize the vertices of the pentagon diagram $A^x_{x_1,x_2,x_3,x_4}$:

$$\bigsqcup_{a,b\in S} X^x_{x_1,a} \times X^a_{x_2,b} \times X^b_{x_3,x_4}, \quad \bigsqcup_{c,b\in S} X^x_{c,b} \times X^c_{x_2,x_2} \times X^b_{x_3,x_4},$$

$$\bigsqcup_{a,e\in S} X^x_{x_1,a} \times X^a_{e,x_4} \times X^e_{x_2,x_3}, \bigsqcup_{d,e\in S} X^x_{d,x_4} \times X^d_{x_1,e} \times X^e_{x_2,x_3},$$

$$\bigsqcup_{c,d\in S} X^x_{d,x_4} \times X^d_{c,x_3} \times X^c_{x_1,x_2}.$$

These orders define the correspondence between the (order preserving) map and any arrow of the diagram $A^x_{x_1,x_2,x_3,x_4}$. On the other hand, we can define a map that corresponds to any arrow of the diagram $A^x_{x_1,x_2,x_3,x_4}$ by means of the maps Φ. These pairs of maps does not coincide. Using lemma 2 we can calculate the signs of the differences:

$$S^x_{x_1,x_2,x_3,x_4} =$$

$$\sum_{a>a',b<b'} m^x_{x_1,a} m^a_{x_2,b} m^b_{x_3,x_4} m^x_{x_1,a'} m^{a'}_{x_2,b'} m^{b'}_{x_3,x_4} +$$

$$\sum_{c>c',b<b'} m^x_{c,b} m^c_{x_1,x_2} m^b_{x_3,x_4} m^x_{c',b'} m^{c'}_{x_1,x_2} m^{b'}_{x_3,x_4} +$$

$$\sum_{a>a',e<e'} m^x_{x_1,a} m^a_{e,x_4} m^e_{x_2,x_3} m^x_{x_1,a'} m^{a'}_{e',x_4} m^{e'}_{x_2,x_3} +$$

$$\sum_{d>d',e<e'} m^x_{d,x_4} m^d_{x_1,e} m^e_{x_2,x_3} m^x_{d',x_4} m^{d'}_{x_1,e'} m^{e'}_{x_2,x_3} +$$

$$\sum_{d>d',c<c'} m^x_{d,x_4} m^d_{c,x_3} m^c_{x_1,x_2} m^x_{d',x_4} m^{d'}_{c',x_3} m^{c'}_{x_1,x_2} +$$

$$\sum_{c,b} m^x_{c,b} \binom{m^c_{x_1,x_2}}{2} \binom{m^b_{x_3,x_4}}{2}.$$

The cocycle representing the class of the first obstruction can be written as follows

$$\alpha(x_1,x_2,x_3,x_4) = \sum_{x\in S} x \otimes \alpha^x_{x_1,x_2,x_3,x_4},$$

where $\alpha^x_{x_1,x_2,x_3,x_4} = (-1)^{S^x_{x_1,x_2,x_3,x_4}}$.

Example.

Let us consider the case of two elements fusion ring $S = \{e, x\}$ with the fusion rule $x * x = mx + n$. Let us note that the first obstruction lies in $H^4(A(S), A(S) \otimes \mathbf{Z}/2\mathbf{Z}) \cong H^4(A(S), A(S)/2A(S))$.

Lemma 3 *The fourth Hochcshild cohomology group $H^4(A(S), A(S)/2A(S))$ of the above algebra $A(S)$ are isomorphic to*

$$A(S)/(2,m)A(S) \cong \begin{cases} 0, & m \equiv 1(2) \\ \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}, & m \equiv 0(2) \end{cases}.$$

The isomorphism sends the class of the cocyle α of the standard complex to the class of its value $\alpha(x,x,x,x) \in A(S)/(2,m)A(S)$.

Proof: We use the presentation of Hochcshild cohomology groups $H^k(A(S), M)$ with coefficients in the $A(S)$-bimodule M as the groups of extensions $Ext^k_{A(S)-A(S)}(A(S), M)$ in the category of $A(S)$-bimodules. Since $A(S) = \mathbf{Z}[x, x^2 = mx + n]$ we can identify free $A(S)$-bimodule of rank one $A(S) \otimes_{\mathbf{Z}} A(S)$ with $\mathbf{Z}[x_1, x_2, x_i^2 = mx_i + n]$. The $A(S)$-bimodule $A(S)$ has a 2-periodic resolution

$$A(S) \leftarrow^{\mu} A(S) \otimes A(S) \leftarrow^{x_1 - x_2} A(S) \otimes A(S) \leftarrow^{x_1 + x_2 - m}$$

$$A(S) \otimes A(S) \leftarrow^{x_1 - x_2} A(S) \otimes A(S) \leftarrow^{x_1 + x_2 - m} \ldots$$

Applying the functor $Hom_{A(S)-A(S)}(?, M)$ to this resolution we can calculate the cohomology $H^*(A(S), M)$

$$H^{2i}(A(S), M) \equiv \{y \in M, xy = yx\}/\{xz + zx - mz, z \in M\},$$

$$H^{2i+1}(A(S), M) \equiv \{y \in M, xy + yx = my\}/\{xz - zx, z \in M\}.$$

Since in our case the bimodule $M = A(S)/2A(S)$ is symmetric we have

$$H^4(A(S), A(S)/2A(S)) \equiv A(S)/(2, m)A(S).$$

□

The unique nontrivial factor of the expression $\alpha^e_{x,x,x,x}$ is the last one which equals $(-1)^{\binom{n}{2}+n\binom{m}{2}}$. The last factor of the expression $\alpha^x_{x,x,x,x}$ is equal to $(-1)^{m\binom{m}{2}}$ and all the other coincide with $(-1)^{nm}$. Hence

$$\alpha^x_{x,x,x,x} = (-1)^{m\binom{m}{2}+nm}.$$

It follows from the above description that the first obstruction of the fusion ring S is nontrivial iff

$$m \equiv 0(2), n \equiv 2, 3(4), \quad \text{or} \quad m \equiv 2(4), n \equiv 1(4).$$

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