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REDUCED SYSTEM OF BETHE EQUATIONS FOR THREE-BOSON INTERACTION

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Abstract. The reduced system of Bethe equations is found for three-boson interaction. The invariants and surfaces in the space of Bethe parameters, connected with this system, are constructed. It is shown that at these surfaces the eigenvectors can be presented in power form. Connection between Bethe ansatz and polynomial extensions of Lie algebras is briefly discussed. The computational procedures corresponding to the original Bethe equations and the method of polynomial algebras are compared.

1. Introduction

Bethe ansatz plays an important role in the theory of integrable quantum models [1]-[4]. Initially, it has been applied to the lattice models such as isotropic and anisotropic spin chain [1, 2] and Hubbard model [5, 6]. It was also applied to the quantum versions of sine-Gordon [7, 8] and nonlinear Schrödinger equations [9].

The other class of models was introduced by Gaudin [10]. The possibility of functional Bethe ansatz for them was shown by Sklyamin [11]. These models have been investigated in many works [12]-[15]. The algebraic Bethe ansatz was applied by Jurĉo to the Gaudin's models related to nonlinear quantum optics [16]. They describe such processes as second harmonic generation, three-boson interaction, the Dike model [17] and some cases of four-boson interaction.

The dynamics of three-boson interaction was investigated in our work [18]. It turned out, that in this case the Bethe ansatz formalism has some specific properties distinguishihg the

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three-boson interaction from other models. The point is that the eigenvectors of Hamiltonian in this approach have the form

$$|\lambda_1,\dots,\lambda_N\rangle = (N!)^{-1/2}B(\lambda_1)\cdots B(\lambda_N)|0\rangle. \tag{1}$$

Here $|0\rangle$ is a pseudovacuum vector and $B(\lambda_i)$ are some specific operators depending on the parameters λ_i. In most models vectors (1), corresponding to different sets $\{\lambda_1,\dots,\lambda_N\}$, $\{\lambda'_1,\dots,\lambda'_N\}$, are linearly independent. This condition together with the demand of (1) to be the eigenvector for Hamiltonian gives us the equations for the parameters $\{\lambda_i,\ i = 1,\dots,N\}$. And in fact, in the spin chain models, in quantum sine-Gordon and nonlinear Schrödinger equations these vectors are linearly independent [19]-[21], but in case of three-boson interaction it is not so. In this model eigenvectors (1) with different $\{\lambda_i, i = 1,\dots,N\}$ can be linearly dependent or even coincide.

Therefore the admissible parameters defining an eigenvector (1) occupy some surface S in space C^N. Equations of these surfaces have been found in Ref.[22]. In this work we show how the existence of such dependencies can change the form of equations for the Bethe ansatz parameters. In more exact terms, the structure of equations for the Bethe ansatz parameters is found to be more closely connected with the algebraic structure of the model. The Hamiltonians of Gaudin's models have the form of the sum of products of operators of two algebras. The original Bethe equations are connected only with the form of these algebras. But it turns out that if one considers the concrete representations of these algebras, the structure of corresponding equations can be simplified. We call them reduced Bethe equations.

We define the coordinate system at the parametric surfaces S and with it's help compare the computational procedures corresponding to the original Bethe equations, the reduced Bethe equations and the method of polynomial algebras [23, 24]. The algebraic sense of the parameters λ_i is also clarified.

2. Algebraic Bethe ansatz scheme

According to the usual Bethe ansatz scheme in terms of Gaudin models [16] one must consider the 2×2 matrix $L(\lambda)$ with operator matrix elements

$$L(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{2}$$

This matrix must be a solution of the classical Yang-Baxter equation

$$\Big[L(\lambda)\bigotimes I,\ I\bigotimes L(\mu)\Big] \ + \ \Big[r(\lambda-\mu), L(\lambda)\bigotimes I \ + \ I\bigotimes L(\mu)\Big] = 0. \tag{3}$$

Here $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $r(\lambda)$ is 4×4 matrix

$$r(\lambda) = \begin{pmatrix} f(\lambda) & 0 & 0 & 0 \\ 0 & 0 & g(\lambda) & 0 \\ 0 & g(\lambda) & 0 & 0 \\ 0 & 0 & 0 & f(\lambda) \end{pmatrix}. \tag{4}$$

It follows from equation (3) that elements of matrix (2) must satisfy the following commutation relations

$$[A(\lambda), B(\mu)] = g(\lambda-\mu)B(\lambda) - f(\lambda-\mu)B(\mu), \tag{5}$$
$$[A(\lambda), C(\mu)] = f(\lambda-\mu)C(\mu) - g(\lambda-\mu)C(\lambda),$$
$$[D(\lambda), B(\mu)] = f(\lambda-\mu)B(\mu) - g(\lambda-\mu)B(\lambda),$$
$$[D(\lambda), C(\mu)] = g(\lambda-\mu)C(\lambda) - f(\lambda-\mu)C(\mu),$$
$$[B(\lambda), C(\mu)] = g(\lambda-\mu)(A(\lambda) - A(\mu) - D(\lambda) + D(\mu)),$$
$$[A(\lambda), A(\mu)] = [B(\lambda), B(\mu)] = [C(\lambda), C(\mu)] = [D(\lambda), D(\mu)] =$$
$$= [A(\lambda), D(\mu)] = 0.$$

The Hamiltonian of a three-boson interaction has the form

$$H = H_0 + gH_I = \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + \omega_3 a_3^+ a_3 + g(a_1^+ a_2 a_3 + a_3^+ a_2^+ a_1), \tag{6}$$

$$\omega_1 = \omega_2 + \omega_3.$$

In this case the elements of matrix $L(\lambda)$ can be taken as

$$A(\lambda) = -\frac{1}{2}\lambda - \frac{1}{\lambda}S_3, \qquad B(\lambda) = a_3^+ + \frac{1}{\lambda}S_+, \tag{7}$$

$$C(\lambda) = a_3 + \frac{1}{\lambda}S_-, \qquad D(\lambda) = \frac{1}{2}\lambda + \frac{1}{\lambda}S_3 = -A(\lambda).$$

Here the 2-boson realization of $SU(2)$ algebra is used

$$S_+ = a_1^+ a_2, \quad S_- = a_2^+ a_1, \quad S_3 = \frac{1}{2}(a_1^+ a_1 - a_2^+ a_2). \tag{8}$$

It is easy to verify that if
$$f(\lambda) = g(\lambda) = \frac{1}{\lambda} \tag{9}$$
operators (7) satisfy commutation relations (5). The eigenfunctions of Hamiltonian (6) can be constructed in the form (1) with the pseudovacuum vector

$$|0\rangle_M = |0, M, 0\rangle. \tag{10}$$

This vector satisfies the equations

$$C(\lambda)|0\rangle_M = 0, \quad A(\lambda)|0\rangle_M = a(\lambda)|0\rangle_M, \quad D(\lambda)|0\rangle_M = d(\lambda)|0\rangle_M$$

$$a(\lambda) = -d(\lambda) = \frac{1}{2\lambda}(M - \lambda^2). \tag{11}$$

Now we find the equations for the parameters $\{\lambda_i, i = 1, \ldots, N\}$. For this, one must consider the trace of the matrix $L^2(\lambda)$.

$$T(\lambda) = \frac{1}{2} Tr L^2(\lambda). \tag{12}$$

This operator has the important property

$$[T(\lambda), T(\mu)] = 0, \tag{13}$$

$$[T(\lambda), B(\mu)] = g(\lambda-\mu)(A(\mu)B(\lambda) - B(\lambda)D(\mu)) + \tag{14}$$
$$+ f(\lambda-\mu)(B(\mu)D(\mu) - A(\lambda)B(\mu)).$$

The pseudovacuum vector (10) is the eigenvector for operator (12)

$$T(\lambda)|0\rangle_M = t_{M,0}(\lambda)|0\rangle_M, \tag{15}$$

$$t_{M,0}(\lambda) = \frac{1}{2}(a^2(\lambda) + d^2(\lambda) + a'(\lambda) - d'(\lambda)).$$

In case of three-boson interaction, i.e., for (7),(8),(11) we have

$$T(\lambda) = \frac{1}{4}\lambda^2 + (a_3^+ a_3 + \frac{1}{2}) + S_3 + \frac{1}{\lambda^2}(S_3^2 + \frac{1}{2}(S_+S_- + S_-S_+)) +$$
$$+ \frac{1}{\lambda}(a_3 S_+ + a_3^+ S_-), \tag{16}$$

$$t_{M,0}(\lambda) = \frac{1}{4}\lambda^2 - \frac{M}{2} + \frac{1}{2} + \frac{1}{\lambda^2}\frac{M}{2}\left(\frac{M}{2} + 1\right).$$

It is obvious that the operator $T(\lambda)$ commutes both with the whole Hamiltonian (6) and with its interacting part H_I

$$[T(\lambda), H] = [T(\lambda), H_I] = 0 \tag{17}$$

and
$$H_I = res\ T(\lambda), \qquad \lambda \to 0. \tag{18}$$

We are lookihg for the eigenvectors of the operator $T(\lambda)$ in the form (1)

$$\Psi_{M,N}(\lambda_1, \ldots, \lambda_N) \equiv |\lambda_1, \ldots, \lambda_N\rangle_M = (N!)^{-\frac{1}{2}} B(\lambda_1)\cdots B(\lambda_N)|0\rangle_M. \tag{19}$$

Acting on it by operator (12) and using the commutation relations (5) and (14) one can find the equation [16]

$$T(\lambda)|\lambda_1, \ldots, \lambda_N\rangle_M = \tag{20}$$

$$= \left\{ \sum_{i=1}^N f^2(\lambda-\lambda_i) - g^2(\lambda-\lambda_i) + \sum_{j\neq i} f(\lambda-\lambda_i) f(\lambda-\lambda_j) + t_{M,o}(\lambda) + \right.$$

$$\left. + \sum_{i=1}^N f(\lambda-\lambda_i)(d(\lambda) - a(\lambda)) \right\} |\lambda_1, \ldots, \lambda_N\rangle_M +$$

$$+ \sum_{i=1}^N g(\lambda-\lambda_i) \left\{ a(\lambda_i) - d(\lambda_i) - 2\sum_{j\neq i} f(\lambda_i - \lambda_j) \right\} |\lambda_1, ..., \lambda_{i-1}, \lambda, \lambda_{i+1}, ..., \lambda_N\rangle_M.$$

As we want the vector (19) to be the eigenvector of operator (12), it follows from the last equation that the additional terms in the right-hand part of (20), containing vectors

$$\Psi^i_{M,N} = |\lambda_1, \dots, \lambda_{i-1}, \lambda_0, \lambda_{i+1}, \dots, \lambda_N\rangle, \tag{21}$$

must be equal to zero. If all these vectors are linearly independent and ortogonal to vector (19) we have the system

$$a(\lambda_i) - d(\lambda_i) - 2\sum_{j\neq i} f(\lambda_i - \lambda_j) = 0, \qquad i = 1, \dots, N. \tag{22}$$

In the case of (9),(11) this system takes the form

$$\lambda_i + \sum_{j\neq i} \frac{2}{\lambda_i - \lambda_j} = \frac{M}{\lambda_i}, \qquad i = 1, \dots, N. \tag{23}$$

This very system (23) is usually regarded as the system of Bethe equations for the three-boson interaction, see for example [16]. It is clear from (20) that the conditions (23) are necessary and sufficient for vector (19) to be the eigenvector of the operator (16) only if all vectors (21)) are linearly independent, but in case of three-boson interaction these vectors are linearly independent only if $N \leq M$. But if $N > M$ vectors (19),(21) are linearly dependent. As the result of this fact the conditions (23) can be weakened and replaced by the another system. Now we find these equations.

3. Reduced system of Bethe equations

The explicit form of vector (19) was found for the three-boson interaction in [18].

$$\Psi_{M,N}(\lambda_1, \dots, \lambda_N) = \tag{24}$$

$$= \sum_{k=0}^{\min(N,M)} \Big(\frac{k!M!(N-k)!}{(M-k)!N!}\Big)^{\frac{1}{2}} \Big(\sum_{i_1<\dots<i_k} (\lambda_{i_1}\cdots\lambda_{i_k})^{-1}\Big)|k, M-k, N-k\rangle.$$

We see that for every $N > M$ this sum contains $(M+1)$ members, therefore among $(N+1)$ vectors (19),(21) only $(M+1)$ ones are linearly independent. Let us construct the set of linearly independent vectors. As the first member of this set we take vector (24), the other M vectors have the form

$$\Psi_l \equiv |0, M, N\rangle - \Big(\frac{(M-l)!N!}{l!M!(N-l)!}\Big)^{\frac{1}{2}} \Big(\sum_{i_1<\dots<i_l} (\lambda_{i_1}\dots\lambda_{i_l})^{-1}\Big)^{-1} |l, M-l, N-l\rangle.$$

$$l = 1, \dots, M. \tag{25}$$

All vectors (25) are linearly independent and ortogonal to the vector (24). Let us calculate the scalar product

$$c^j_l = \big\langle \Psi_l \mid \Psi^j_{M,N}\big\rangle = 1 - \Big(\sum_{i_1<\dots<i_l}{}^{j} (\lambda_{i_1}\dots\lambda_{i_l})^{-1}\Big)\Big(\sum_{i_1<\dots<i_l} (\lambda_{i_1}\dots\lambda_{i_l})^{-1}\Big)^{-1}. \tag{26}$$

In the sum $\sum^J$ parameter λ_j is replaced by the parameter λ.

Substituting in (20) vectors (25) instead of vectors (21) one gets the system of equations

$$\sum_{i=1}^{N} g(\lambda-\lambda_i)\Big\{a(\lambda_i)-d(\lambda_i)-2\sum_{j\neq i} f(\lambda_i-\lambda_j)\Big\}c_l^i=0, \quad l=1,\dots,M. \tag{27}$$

Taking into account formulas (9), and (11), system (27) can be transformed into the system

$$\sum_{i=1}^{N}(\lambda_i)^{-l}\Big(\lambda_i-\frac{M}{\lambda_i}+2\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j}\Big)=0, \qquad l=1,\dots,M. \tag{28}$$

Thus, if $M<N$ one can use the reduced system (28) instead of usual system (23). System (23) contains N equations and N unknown parameters λ_i. Its solutions are points in N-dimensional complex space. System (28) contains M equations for N unknown parameters, its solutions are $(N-M)$-dimensional complex surfaces in N-dimensional space of parameters. Each surface consists of M connected components. All points from some connected component give the same vector (24), so

$$\Psi_{M,N}(\lambda_1,\dots,\lambda_N)\equiv\Psi_{M,N}(\mu_1,\dots,\mu_N),$$

if $\{\lambda_i\}$ and $\{\mu_i\}$ belong to the same connected component. Equations (23) give us the specific points from each component. For $M=1,\ N=1,2,\dots,14$ these points have been evaluated in [18]. Thus we see that the usual Bethe equations (23) give the correct set of parameters, but this set is not unique. Note that equations (28), in contrast to (23), permit solutions containing equal $\lambda_i=\lambda_j$.

4. Spectrum formula and invariants of the surfaces

The eigenvalue of the operator $T(\lambda)$ was found in [16].

$$t_{M,N}(\lambda)=\sum_{i=1}^{N}\Big[f^2(\lambda-\lambda_i)-g^2(\lambda-\lambda_i)+f(\lambda-\lambda_i)(d(\lambda)-a(\lambda))+$$
$$+\sum_{i\neq j} f(\lambda-\lambda_i)f(\lambda-\lambda_j)]+t_{M,0}(\lambda). \tag{29}$$

In the case of (9),(11),(15) it takes the form

$$t_{M,N}(\lambda)=\sum_{i=1}^{N}\Big\{\frac{\lambda^2-M}{\lambda(\lambda-\lambda_i)}+\sum_{j\neq i}\frac{1}{(\lambda-\lambda_i)(\lambda-\lambda_j)}\Big\}+$$
$$+\frac{1}{4}\lambda^2-\frac{M}{2}+\frac{1}{2}+\frac{1}{\lambda^2}\frac{M}{2}\Big(\frac{M}{2}+1\Big). \tag{30}$$

This formula is valid if $N\leq M$, but if $N>M$ it is valid only for those $\{\lambda_i,\ i=1,\dots,N\}$ which satisfy equations (23); but if parameters $\{\lambda_i,\ i=1,\dots,N\}$ satisfy only the weaker system (28) the formula (30) is not valid. We must add to it some terms, because now vectors $\Psi_{M,N}$ and $\Psi^i_{M,N}$ are linearly dependent. Such addition, $\Delta_{M,N}$, to eigenvalue (30) has the form

$$\Delta_{M,N}(\lambda) = C^{-2}_{M,N} \sum_{i=1}^{N} \Big\langle \Psi^i_{M,N} \Big| \Psi_{M,N} \Big\rangle F_i(\lambda). \tag{31}$$

Here

$$C^2_{M,N} = \sum_{k=0}^{M} \Big(\frac{k!(N-k)!M!}{(M-k)!N!} \Big) \Big(\sum_{i_1<\cdots<i_k} (\lambda_{i_1} \ldots \lambda_{i_k})^{-1} \Big)^2$$

— is the square of the norm of vector (23) and

$$F_i(\lambda) = \frac{1}{\lambda - \lambda_i} \Big(\frac{M}{\lambda_i} - \lambda_i - 2 \sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j} \Big).$$

Using the fact

$$\Big\langle \Psi^i_{M,N} \Big| \Psi_{M,N} \Big\rangle =$$
$$= \sum_{k=0}^{M} \Big(\frac{k!(N-k)!M!}{(M-k)!N!} \Big) \Big(\sum_{i_1<\cdots<i_k}{}^{i} (\lambda_{i_1} \ldots \lambda_{i_k})^{-1} \Big) \Big(\sum_{i_1<\cdots<i_k} (\lambda_{i_1} \ldots \lambda_{i_k})^{-1} \Big)$$

we get

$$\Delta_{M,N}(\lambda) = \sum_{i=1}^{N} F_i(\lambda). \tag{32}$$

Therefore, for arbitrary set of parameters $\{\lambda_i,\ i = 1, \ldots, N\}$, satisfing the system (28), the eigenvalue of the operator $T(\lambda)$ has the form

$$t_{M,N}(\lambda) = \sum_{i=1}^{N} \Bigg\{ \frac{\lambda^2 - M}{\lambda(\lambda - \lambda_i)} + \sum_{j\neq i} \frac{1}{(\lambda-\lambda_i)(\lambda-\lambda_j)} + \frac{M - \lambda_i^2}{\lambda_i(\lambda-\lambda_i)} -$$
$$-2 \sum_{j\neq i} \frac{1}{(\lambda-\lambda_i)(\lambda_i-\lambda_j)} \Bigg\} + \frac{1}{4}\lambda^2 - \frac{M}{2} + \frac{1}{2} + \frac{1}{\lambda^2}\frac{M}{2}\Big(\frac{M}{2}+1\Big) =$$
$$= \frac{1}{\lambda} \sum_{i=1}^{N} \frac{M}{\lambda_i} + N + \frac{1}{4}\lambda^2 - \frac{M}{2} + \frac{1}{2} + \frac{1}{\lambda^2}\frac{M}{2}\Big(\frac{M}{2}+1\Big). \tag{33}$$

Comparing formulas (16) and (33) we have

$$E_I = \sum_{i=1}^{N} \frac{M}{\lambda_i} \quad , \qquad H_I \Psi_{M,N} = E_I \Psi_{M,N}. \tag{34}$$

In the work [16] for E_I the following expression was found

$$E_I = \sum_{i=1}^{N} \lambda_i. \tag{35}$$

But this formula (35) is valid only for those $\{\lambda_i\}$ which satisfy equations (23), while formula (33) is valid for all sets $\{\lambda_i\}$, belonging to surfaces (28). So we see that the quantity

$$S_1 = \sum_{i=1}^{N} \lambda_i^{-1} = \frac{1}{M} E_I$$

is an invariant of the surface (to be more exact — of its connected component). The other invariants have the form

$$S_k = \sum_{i_1<\cdots<i_k} (\lambda_{i_1}\ldots\lambda_{i_k})^{-1} = (-1)^k \frac{N!}{(N-k)!}\frac{A_{M-k}}{A_M}, \quad k = 1,\ldots,M. \tag{36}$$

Here A_l — is a $l\times l$ -lower principal minor of the matrix A_{MN}

$$A_{MN} = \| H_I - EI \| = \| h_{ij} - E\delta_{ij} \| = \| a_{ij} \|, \qquad i,j = 1,\ldots,M+1. \tag{37}$$

$$h_{ij} = \delta_{i+1,j}h_i + \delta_{i,j+1}h_j, \qquad h_i = ((M+1-i)(N+1-i))^{1/2}.$$

$$A_l = \| a_{ij}^l \|, \quad i,j = 1,\ldots,l, \quad a_{ij}^l = a_{M+1-l+i,M+1-l+j}. \tag{38}$$

Equation $\det A_{MN} = 0$ gives us the eigenvalues of vectors (24). Formulas (36) can be found by comparing (24) with the usual structure of eigenvectors of the matrix (37).

It was shown in [18] that the space D of the eigenvectors of Hamiltonian (6) is the direct sum of subspaces $D_{M,N}$

$$D = \bigoplus_{M,N=0}^{\infty} D_{M,N}, \quad D_{M.N} = \{|K, M-K, N-K\rangle, \quad K = 0,1,\ldots\min(M,N)\}, \tag{39}$$

$$\dim D_{M,N} = \min(M,N) + 1.$$

The vectors from these subspaces are the eigenvectors of two operators

$$K_1 = S_3^2 + \frac{1}{2}(S_+S_- + S_-S_+), \qquad K_2 = (a_3^+a_3 + \frac{1}{2}) + S_3. \tag{40}$$

with the following eigenstates (33)

$$K_1\Big|_{D_{M,N}} = \frac{M}{2}\Big(\frac{M}{2}+1\Big), \qquad K_2\Big|_{D_{M,N}} = N - \frac{M}{2} + \frac{1}{2}. \tag{41}$$

In case $N \geq M$ $D_{M,N}$ is the space of $(M+1)$-dimensional representation of the algebra $SU(2)$. If $N < M$, the space $D_{M,N}$ forms only part of the space of the representation. Different vectors (24) from the space $D_{M,N}$ have different energy $E = \sum_{i=1}^{N}\frac{M}{\lambda_i}$. The operators K_1, K_2, H_I are the coefficients attached to different degrees of the parameter λ in formula (16) for the operator $Tr(\lambda)$ ($\lambda^{-2}, \lambda^0, \lambda^{-1}$ -correspondingly).

The original equations for the Bethe ansatz parameters (23) correspond to the case $N \leq M$, and $\dim S = 0$, therefore S is a set of points in the space C^N. The reduced Bethe equations (28) correspond to the case $N > M$ and $\dim S = N - M > 0$. The number

N of equations of the original Bethe system (23) can be reduced now by the dimension of surface S

$$N - \dim S = M.$$

M is the number of equations of reduced system (28). Hamiltonian H_I (6) can be presented by the generators of algebra $SU(2)\{S_3, S_+, S_-\}$ and Heisenberg's algebra $\{a_3, a_3^+\}$

$$H_I = a_3 S_- + a_3^+ S_+. \tag{42}$$

They are the same for all N, M. The structure of reduced system (28) is more closely connected with concrete values of these numbers. In Table 1., Ref.[18], the spaces of the representations of algebra $SU(2)$ are given by rows, the infinite columns correspond to the representation of Heisenberg's algebra $\{a_3, a_3^+\}$, and the spaces $D_{M,N}$ are the diagonals. The structure of these diagonals and of the reduced Bethe equations is defined by the intersection of the spaces of representation of these two algebras.

5. Asymptotics and generating functions for the eigenvectors

The fact that for $N > M$ series (24) contains only $(M+1)$ members can be used for calculating the asymptotics of the vectors $\Psi_{M,N}$ for large N. For this, let us consider the $(M+1)$-dimensional irreducible representation of algebra $SU(2)$. In the space of this representation one can find the eigenvectors of the operator $S_+ + S_-$

$$(S_+ + S_-)\Psi_M^m = E_M^m \Psi_M^m, \qquad m = 0, 1, \ldots, M. \tag{43}$$

The solutions of this equation have the form

$$\Psi_M^m = \sum_{k=0}^{M} C_{M,k}^m l_k, \qquad C_{M,0}^m = 1. \tag{44}$$

Here $\{l_k,\ k = 0, \ldots, M\}$ is a basis in the space of the representation, such that

$$S_3 l_k = (k - \frac{M}{2}) l_k.$$

Then it follows from (37) that vector (24)

$$\Psi_{M,N}^m \to \sum_{k=0}^{M} C_{M,k}^m |k, M-k, N\rangle \tag{45}$$

and $$E_{M,n}^m \to \sqrt{N} E_M^m \qquad \text{if} \quad N \to \infty.$$

As a basis $\{l_k,\ k = 0, \ldots, M\}$ we can use $l_k = |k, M-k, 0\rangle$. Then

$$\Psi_M^m = \sum_{k=0}^{M} C_{M,k}^m |k, M-k, 0\rangle$$

and equation (41) takes the form

$$\Psi_{M,N}^m \to (N!)^{-1/2} (a_3^+)^N \Psi_M^m \qquad \text{if} \quad N \to \infty. \tag{46}$$

6. Coordinate systems on the parametric surfaces

It was found that the equations of the parametric surfaces S_{N-M} have the form [22]

$$\sum_{i_1<\cdots<i_k} (\lambda_{i_1}\ldots\lambda_{i_k})^{-1} = I_k = (-1)^k \frac{N!}{(N-k)!}\frac{A_{M-k}}{A_M}, \quad k = 1,\ldots,M. \tag{47}$$

In order to fix a point on the surface S_{N-M} one must define other $(N-M)$ equations

$$\sum_{i_1<\cdots<i_{M+l}} (\lambda_{i_1}\ldots\lambda_{i_{M+l}})^{-1} = J_l, \qquad l = 1,\ldots,N-M. \tag{48}$$

These numbers, J_l, can be considered as the coordinates of point λ on the surface S_{N-M}. Thus each point $\lambda = \lambda_1,\ldots,\lambda_N$ in N-dimensional complex space C^N is defined by two sets of numbers $\{I_k, \quad k = 1,\ldots,M\}$ and $\{J_l, \quad l = 1,\ldots,N-M\}$. Members I_k of the first set are the values of invariants (47) of solutions (24), they define the surface S_{N-M}. Members J_l of the second set are not the invariants of the solutions and define the point on the surface. But in practice it is more convenient to consider the quantities

$$C_n = \sum_{i_1<\cdots<i_n} (-1)^n(\lambda_{i_1}\ldots\lambda_{i_n}) \qquad n = 1,\ldots,N. \tag{49}$$

If $N > M$ the quantities C_n are not the invariants of solutions (24), but with their help one can construct the polynomials

$$P_N = \sum_{i=0}^{N} x^{N-i}C_i, \qquad C_0 = 1 \tag{50}$$

whose roots are the parameters $\lambda_1\ldots\lambda_N$. The quantities C_n can be directly found via matrix elements of the Hamiltonian H.

It was shown in [22], that in case of $N \le M$

$$C_n = \frac{1}{n!}A_n, \qquad n = 1,\ldots,N, \tag{51}$$

where A_n is the lower major n-dimensional minor of $(N+1)\times(N+1)$ matrix A_{MN}.

If $N > M$, one must consider the extended $(N+1)\times(N+1)$ matrix $\tilde{A}_{MN}$ instead of $(M+1)\times(M+1)$ matrix A_{MN} (37). This matrix has a bloc structure

$$\tilde{A}_{MN} = \left\| \begin{matrix} A_{MN} & 0 \\ 0 & G_{MN} \end{matrix} \right\|. \tag{52}$$

The upper bloc coincides with the matrix A_{MN}. The lower $(N-M)\times(N-M)$- dimensional bloc G_{MN} has such a structure, that numerical values of all quantities C_n can be calculated with the help of formula

$$C_n = \frac{1}{n!}\tilde{A}_n, \tag{53}$$

where $\tilde{A}_n$ is the lower major n-dimensional minor of $(N+1)\times(N+1)$ extended matrix $\tilde{A}_{MN}$ (48). Since matrix $\tilde{A}_{MN}$ has a bloc structure the minor $\tilde{A}_n$ can be factorised for all $n \geq N-M$

$$\tilde{A}_n = G \cdot A_{n-N+M}, \qquad G = \det \|G_{MN}\|. \tag{54}$$

Now we can find the connection between I_K, J_l and minors of the matrix $\tilde{A}_{MN}$.

$$I_K = \frac{C_{N-K}}{C_N} = (-1)^K \frac{N!}{(N-K)!}\frac{\tilde{A}_{N-K}}{\tilde{A}_N} = (-1)^K \frac{N!}{(N-K)!}\frac{A_{M-K}}{A_M}. \tag{55}$$

$$K = 1, \ldots, M.$$

$$J_l = \frac{C_{N-M-l}}{C_N} = (-1)^{M+l}\frac{N!}{(N-M-l)!}\frac{\tilde{A}_{N-M-l}}{\tilde{A}_N}, \quad l = 1, \ldots, N-M. \tag{56}$$

We see that the invariants I_K of solution (24) are the functions of the elements of the matrix A_{MN} only, which has clear physical sense. The coordinates J_l of the points on the parametric surface depend on the elements of the lower "unphysical" bloc G_{MN} of matrix $\tilde{A}_{MN}$. More precisely, quantities J_l are functions of the lower major minors of the matrix G_{MN}. There are $N-M$ of such minors, including the determinant G

$$A_{J,1}, \ldots, A_{J,N-M} = G. \tag{57}$$

numbers can be also considered as the coordinates of the point on the surface S_{N-M}. Defining any explicit form for the elements of the matrix G_{MN}, one can use them as the third set of coordinates on the surface S_{N-M}.

For example,

$$G_{MN} = \|d_{ij} - E\delta_{ij}\|, \qquad d_{ij} = (\delta_{i,j+1} + \delta_{i+1,j})r_i + \delta_{ij}d. \tag{58}$$

Now parameters $\{d, r_i, \quad i = 1, \ldots, N-M-1\}$ can be considered as the coordinates of the point on the surface S_{N-M}.

It is necessary to note that the invariants I_K are not all independent. Each solution (24) is defined by three parameters M, N, E_I. Correspondingly, among the invariants I_K only three are independent; all others can be found will the help of recurrence condition

$$(N-K+1)I_{K-1} + E_I I_K + \frac{h_{M-K+1}}{N-K} I_{K+1} = 0. \tag{59}$$

As we know, ther is one point marked out on the surface S_{N-M}. This point is defined by the original system (23). Let us find its coordinates in the system (59). It can be done by comparing the $SU(2)$ and $SU(1,1)$ forms of the Bethe ansatz for three-boson interaction, or, more exactly, formulas (24),(37) with formulas (A.2) and (A.4) in [22]. Then it is easy to see that

$$d = 0, \quad r_l = ((N-M-l)(-l)(M+1+l))^{1/2}, \qquad l = 1, \ldots, N-M-1 \tag{60}$$

are the coordinates of this point.

7. The comparison of different computational procedures

Let us compare the computational procedures corresponding to the original and reduced systems for the Bethe ansatz parameters. In case of system (23) one must deal with the extended $(N+1)\times(N+1)$ matrix $\tilde{A}_{MN}$ (52), which contains physical and unphysical blocs A_{MN} and G_{MN}, correspondingly. First, one must solve the equation

$$|A_{MN}| = 0 \tag{61}$$

and find the eigenvalues E_I of the Hamiltonian H_I. Then it is necessary to evaluate the minors $\tilde{A}_n$ of the matrix $\tilde{A}_{MN}$ and to construct a polynomial

$$P_N = \sum_{n=0}^{N} \frac{x^{N-n}}{n!} \tilde{A}_n. \tag{62}$$

The roots of this polynomial are the desired quantities r_n.

In case of the reduced system (28) we must first evaluate the invariants I_K with the help of matrix A_{MN} and formula (47). Then we must divide N parameters λ_n into two parts

$$\begin{aligned} \lambda_i^{-1} &= \mu_i, \qquad i = 1,\dots,M. \\ \lambda_{M+j}^{-1} &= \nu_j, \qquad j = 1,\dots,N-M. \end{aligned} \tag{63}$$

The values of ν_j may be taken arbitrarily. We fix them and evaluate the quantities

$$Q_l = \sum_{j_1<\dots<j_l} (\nu_{j1}\dots\nu_{jl}), \qquad l = 1,\dots,N-M. \tag{64}$$

The parameters μ_i now satisfy the equations

$$\sum_{i_1<\dots<i_K} (\mu_{i_1}\dots\mu_{i_K}) = I_K - Q_K - \sum_{l=1}^{K-1} I_{K-l}Q_l = L_K., \quad K = 1,\dots,M. \tag{65}$$

Therefore, M parameters μ_i are the roots of the polynomial

$$\tilde{P}_M = \sum_{K=0}^{M} x^K L_{M-K}. \tag{66}$$

So, in case of the original Bethe equation (23) one must begin with solving the equation (61) of order $(M+1)$; then solve the equation $P_N = 0$ of order N. In case of the reduced Bethe equations one must solve first the same equation (61), and then the equation $\tilde{P}_M = 0$, whose order is M. If $M << N$ the difference in computations may be quit distinct.

In the next section we consider another approach to the same problem, based on the infinite dimensional polynomial algebras. By this method to find eigenvalues and eigenstates of the Hamiltonian H it is sufficient to solve only equation (61) . Therefore, the knowledge of the Bethe parameters λ is necessary not only for constructing the eigenstates of the Hamiltonian H_I, but also for investigating the algebraic structure of the model. The knowledge of the parameters λ helps us to represent some operators from polynomial algebra $SU_d^{(1,2)}(2)$ in terms of $SU(2)$ and Heisenberg's algebras.

8. Bethe ansatz and polynomial extensions of Lie algebras

The other approach to quantum optical problems based on the polynomial extensions of Lie algebras was put forth in [23, 24]. According to it, the Hamiltonian of the system must be presented in the form

$$H = \sum_{i=1}^{N} \alpha_i Y_i, \qquad \alpha_i = \mathrm{const}, \tag{67}$$

where Y_i are operators from the enveloping algebra of some finite-dimensional algebra G. These operators satisfy the commutation relations

$$[Y_i, Y_j] = f_{ij}(Y_k), \qquad i, j, k = 1, \ldots, N, \tag{68}$$

where $f_{i,j}$ are structural polynomials of finite order. Knowing these polynomials one can study the properties of Hamiltonian (52). In case of three-boson interaction Hamiltonian (6) can be presented in the form

$$H = C + g(Y_+ + Y_-), \tag{69}$$

where

$$C = H_0 = \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + \omega_3 a_3^+ a_3, \quad Y_+ = a_1 a_2^+ a_3^+, \quad Y_- = a_1^+ a_2 a_3. \tag{70}$$

These operators together with the additional operator

$$Y_0 = \frac{1}{3}(-a_1^+ a_1 + a_2^+ a_2 + a_3^+ a_3) \tag{71}$$

satisfy the commutation relations

$$[Y_0, Y_\pm] = \pm Y_\pm; \quad [C, Y_{\pm 0}] = 0, \quad [Y_-, Y_+] = \Psi_{1,2}(Y_0 + 1) - \Psi_{1,2}(Y_0). \tag{72}$$

The structural polynomial $\Psi_{1,2}(Y_0)$ defines the polynomial algebra $SU_d^{(1,2)}(2)$ with commutation relations (72). It has the form [23]

$$\Psi_{1,2}(Y_0) = \frac{1}{4}(R_0 - Y_0 + 1)(2Y_0 + R_0 + R_1)(2Y_0 + R_0 - R_1), \tag{73}$$

$$R_0 = \frac{1}{3}(2a_1^+ a_1 + a_2^+ a_2 + a_3^+ a_3), \qquad R_1 = -a_2^+ a_2 + a_3^+ a_3.$$

We try to find the eigenvectors of Hamiltonian (69) in a form

$$|E\rangle = \sum_{n=1}^{s} Q_n(E)(Y_+)^n |s, k, 0\rangle. \tag{74}$$

The coefficients Q_n satisfy the recurrence relation

$$Q_{n-1} - E_I Q_n + \Psi_{1,2}(l_1 + n + 1) Q_{n+1} = 0. \tag{75}$$

Here E_I is an eigenvalue of the interaction Hamiltonian H_I and l_1 is an eigenvalue of operator Y_0

$$H_I|E\rangle = (Y_+ + Y_-)|E\rangle = E_I|E\rangle,$$

$$Y_0|s,k,0\rangle = \frac{1}{3}(k-s)|s,k,0\rangle = l_1|s,k,0\rangle.$$

Recurrence relations (75) are valid for all polynomial deformations $SU_d^{(p,q)}(2)$ of algebra $SU(2)$ with the specific polynomial $\Psi_{p,q}$ in each case. In case of algebra (70) – (73) we have

$$Q_{n-1} - E_I Q_n + (s-n)(n+1)(n+1-k)Q_{n+1} = 0, \tag{76}$$

$$Q_{-1} \equiv 0, \qquad Q_0 \equiv 1.$$

The condition $Q_{s+1} = 0$ gives us the equation for the eigenvalues E_I of the operator H_I. This condition is equivalent to equation (61). We see that equation (61) and the recurrence relation give us the coefficients Q_n and eigenstate (74) without using the Bethe parameters λ. These parameters are necessary for transforming the vector $|E\rangle$ from the form (59) into the form (1), or, more formally, for splitting the polynomial algebra $SU_q^{(1,2)}(p,q)$ into the composition of algebra $SU(2)$ and Heisenberg's algebra $\{a_3, a_3^+\}$.

By comparing of formulas (24) and (74) for the eigenvectors of Hamiltonian (24) and (74) one can find the connection between polynomials Q_n and invariants S_i (36).

$$Q_i = \left(\frac{(M-i)!(N-M+i)!}{M!(N-M)!i!}\right)^{1/2} \frac{S_{M-i}}{S_M}, \qquad \begin{array}{c} i = 0,\ldots,M. \\ M < N \end{array} \tag{77}$$

$$Q_i = \left(\frac{(N-i)!i!(N-M)!}{N!(M-N+i)!}\right)^{1/2} \frac{S_{N-i}}{S_N}, \qquad \begin{array}{c} i = 0,\ldots,N. \\ M \geq N \end{array}$$

In general, comparing the method of polynomial algebras with the method of Bethe ansatz one can conclude that according to the method of polynomial algebras eigenvectors (74) of Hamiltonian are generated by the operator having a form of a polynomial, while in the method of Bethe ansatz the generating operator (19) has the factorized form. Correspondingly, in the first case we are interested in the coefficients Q_n of this polynomial, and in the second one in the roots λ_i of the related algebraic equation. We shall investigate this connection in more detail elsewhere.

9. Conclusion

We have found that if $N > M$ the system of Bethe equations (23) can be weakened. In fact, for each M only finite number of states are described by the usual system (23), but for infinite number of states reduced system (23) can be used. This system contains M equations for any $N > M$. With its help one can find their asymptotics for large N. Solutions of equation (28) satisfy the invariance conditions (36). The other quantum optical models, such as Dike model and four-boson interaction have similar properties, and we intend to investigate them subsequently. We also intend to investigate connection between equations (23),(28) and the polynomial algebras.

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