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OPTIMAL CONTROL PROBLEMS ON THE LIE GROUP SP(1)

MIRCEA PUTA

ABSTRACT. An optimal control problem on the Lie group SP(1) is discussed and some of its dynamical and geometrical properties are pointed out.

1. INTRODUCTION

Recent work in nonlinear control has drawn attention to dritf-free, left invariant control systems on matrix Lie groups. We can remind here the case of the matrix Lie group SO(3) studied in connection with the spacecraft dynamics [4], [8], the case of the matrix Lie group SE(3) studied in connection with the control tower problem [6], the case of the matrix Lie group SO(n) studied in connection with the electrical circuits [11] and the case of the matrix Lie group $U(n)$ studied in connection with the molecular dynamics [1]. The goal of our paper is to make a similar study for the matrix Lie group SP(1).

2. THE LIE GROUP SP(1)

Let $\mathbf{H}$ be the noncommutative field of quaternions, i.e.,

$$\mathbf{H} = \{q_0 + iq_1 + jq_2 + kq_3 \mid q_0, q_1, q_2, q_3 \in \mathbf{R}\}.$$

Then we have the usual identification:

$$\mathbf{H} \simeq \mathbf{R}^4.$$

We denote by $GL_1(\mathbf{H})$ the group of automorphisms of $\mathbf{H}$, that is the group of transformations $t : \mathbf{H} \to \mathbf{H}$, of the form:

$$\xi' = a\xi, \tag{2.1}$$

where $\xi, \xi' \in \mathbf{H} \simeq \mathbf{R}^4$ and $a \in \mathbf{H} \setminus \{0\}$, and by SP(1) the subgroup of $GL_1(\mathbf{H})$ consisting of unitary automorphisms of $\mathbf{H}$ with respect to the canonical Hermitian product:

$$\xi \cdot \eta = \xi\overline{\eta}, \quad \forall\, \xi, \eta \in \mathbf{H} \simeq \mathbf{R}^4,$$

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that is, $t \in SP(1)$ iff it is of the form (2.1) with

$$a \cdot \overline{a} = 1.$$

On the other hand, the canonical scalar product on $\mathbf{R}^4$ is expressed in quaternionic form by:

$$< \xi, \eta >= Re(\xi, \overline{\eta}),$$

where $\xi, \eta \in \mathbf{H} \simeq \mathbf{R}^4$. It follows that

$$a\overline{a} = a_0^2 + a_1^2 + a_2^2 + a_3^2,$$

and then one get the identification:

$$\mathrm{SP}\,(1) \simeq S^3 \subset \mathbf{R}^4.$$

Let $\mathrm{Im}\,\mathbf{H} \simeq \mathbf{R}^3$ be the space of imaginary quaternions. Since each element of SO (3) can be expressed in quaternionic form by:

$$\xi' = q\xi\overline{q},$$

where $q \in \mathbf{H}$, $q\overline{q} = 1$, $\xi, \xi' \in \mathrm{Im}\,\mathbf{H}$, one has the isomorphism:

$$\mathrm{SO}\,(3) \simeq SP(1)/\mathbf{Z}_2,$$

where

$$\mathbf{Z}_2 = \{1, -1\}.$$

Correspondingly we have an identification of the Lie algebra so (3) of SO (3) with the Lie algebra sp (1) of SP (1). In terms of matrices one can get also the following identification:

$$\mathrm{sp}\,(1) \cong \left\{ \begin{bmatrix} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & -a_3 & a_2 \\ a_2 & a_3 & 0 & -a_1 \\ a_3 & -a_2 & a_1 & 0 \end{bmatrix} \;\middle|\; a_1, a_2, a_2 \in \mathbf{R} \right\}.$$

Let $\{A_1, A_2, A_3\}$ be the canonical basis of sp (1) given by:

$$A_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix};\ A_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix};\ A_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then the Lie algebra structure of sp (1) is given by the following table:

$[\cdot,\cdot]$	A_1	A_2	A_3
A_1	0	$2A_3$	$-2A_2$
A_2	$-2A_3$	0	$2A_1$
A_3	$2A_2$	$-2A_1$	0

Let us consider now on SP (1) the left invariant system:

$$\dot{g} = g(A_1u_1 + A_2u_2). \tag{2.2}$$

THEOREM 2.1 *The system* (2.2) *is controllable and it is a single bracket one.*
Proof. Indeed, the proof is a consequence of the fact that sp (1) is generated by $A_1, A_2, [A_1, A_2]$. ∎
REMARK 2.1 It is not hard to see that the controlled system (2.2) can be put in the equivalent form:

$$\begin{cases} \dot{x} = zu_2 \\ \dot{y} = zu_1 \\ \dot{z} = xu_2 - yu_1 \\ xu_1 + yu_2 = 0. \end{cases} \tag{2.3}$$

∎

3. AN OPTIMAL CONTROL PROBLEM

Let $\mathcal{J}$ be the cost function given by:

$$\mathcal{J}(u_1, u_2) = \frac{1}{2}\int_0^{t_f} [c_1 u_1^2(t) + c_2 u_2^2(t)]dt; \quad c_1 > 0,\ c_2 > 0. \tag{3.1}$$

Then we can prove:
THEOREM 3.1 *The controls that minimize* $\mathcal{J}$ *and steer the system* (2.2) *from* $X = X_0$ *at* $t = 0$ *to* $X = X_f$ *at* $t = t_f$ *are given by:*

$$u_1 = \frac{1}{c_1}P_1; \qquad u_2 = \frac{1}{c_2}P_2,$$

where the functions P_i *are solutions of:*

$$\begin{cases} \dot{P}_1 = -\dfrac{2}{c_2}P_2P_3 \\ \dot{P}_2 = \dfrac{2}{c_1}P_1P_3 \\ \dot{P}_3 = \left(\dfrac{2}{c_2} - \dfrac{2}{c_1}\right)P_1P_2. \end{cases} \tag{3.2}$$

Proof. Simply apply Krishnaprasad's theorem, [3]. It follows that the optimal Hamiltonian is given by:

$$H_o = \frac{1}{2}\left(\frac{P_1^2}{c_1} + \frac{P_2^2}{c_2}\right). \tag{3.3}$$

It is in fact the controlled Hamiltonian H given by

$$H = P_1u_1 + P_2u_2 - \frac{1}{2}(c_1u_1^2 + c_2u_2^2),$$

which is reduced to $(\mathrm{sp}\,(1))^*_-$ via the Poisson reduction. Here $(\mathrm{sp}\,(1))^*_-$ is $(\mathrm{sp}\,(1))^* \simeq \mathbf{R}^3$ together with the minus Lie-Poisson structure given by the matrix:

$$\prod = \begin{bmatrix} 0 & -2P_3 & 2P_2 \\ 2P_3 & 0 & -2P_1 \\ -2P_2 & 2P_2 & 0 \end{bmatrix}. \tag{3.4}$$

Then the optimal controls are given by:

$$u_1 = \frac{1}{c_1}P_1; \qquad u_2 = \frac{1}{c_2}P_2,$$

where the functions P_i are solutions of the reduced Hamilton's equations (or momentum equations) given by:

$$\left[\dot{P}_1, \dot{P}_2, \dot{P}_3\right]^t = \prod \cdot \nabla H_o,$$

which are nothing else but the required equations. ■

REMARK 3.1 The function C given by

$$C = P_1^2 + P_2^2 + P_3^2 \tag{3.5}$$

is a Casimir of our configuration $((\mathrm{sp}\,(1))^*, \prod) \simeq (\mathbf{R}^3, \prod)$, i.e.,

$$(\nabla C)^t \cdot \prod = 0.$$

■

REMARK 3.2 The phase curves of our system (3.2) are the intersections of the elliptic cylinders

$$\frac{P_1^2}{c_1} + \frac{P_2^2}{c_2} = 2H_o$$

with the spheres

$$P_1^2 + P_2^2 + P_3^2 = C.$$

■

THEOREM 3.2 *The dynamics* (3.2) *is equivalent to the pendulum dynamics.*

Proof. Indeed, H_o is a constant of motion, so

$$\frac{P_1^2}{c_1} + \frac{P_2^2}{c_2} = l^2.$$

Let us take now

$$\begin{cases} P_1 = l\sqrt{c_1}\cos\theta \\ P_2 = l\sqrt{2}\sin\theta. \end{cases}$$

Then

$$\begin{aligned} \dot{P}_1 &= -\sqrt{c_1}\dot{\theta}\sin\theta \\ &= -\sqrt{\frac{c_1}{c_2}}\sqrt{c_2}\dot{\theta}\sin\theta \\ &= -\sqrt{\frac{c_1}{c_2}}\dot{\theta}P_2 \end{aligned}$$

or equivalently,

$$\begin{aligned}\dot{\theta} &= -\sqrt{\frac{c_1}{c_2}}\frac{\dot{P}_1}{P_2} \\ &= -\sqrt{\frac{c_1}{c_2}}\left(-\frac{P_2P_3}{P_2}\right)\frac{2}{c_2} \\ &= \frac{2}{\sqrt{c_1c_2}}P_3.\end{aligned}$$

Differentiating again, we get

$$\ddot{\theta} = 2l^2\sqrt{c_1c_2}(c_1 - c_2)\sin 2\theta.$$

Thus, pendulum mechanics as required. ■

THEOREM 3.4 *The system* (3.2) *may be realized as a Hamilton-Poisson system in an infinite number of different ways, i.e., there exists infinitely many different, in general nonisomorphic Poisson structures on* $\mathbf{R}^3$ *such that the system* (3.2) *is induced by an appropiate Hamiltonian.*

Proof. Indeed, to begin with, let us observe that our system can be put in an equivalent form:

$$\dot{P} = \nabla C \times \nabla H_o,$$

where $P = [P_1, P_2, P_3]^t$ and C and H_o are respectively given by (3.5) and (3.3). Now, an easy computation shows us that the system (3.3) may be realized as a Hamilton-Poisson system with the phase space $\mathbf{R}^3$, the Poisson bracket $\{\cdot,\cdot\}_{ab}$ given by

$$\{f, g\}_{ab} = -\nabla C' \cdot (\nabla f \times \nabla g),$$

where $a, b \in \mathbf{R}$,

$$C' = aC + bH_o,$$

and the Hamiltonian H' defined by

$$H' = cC + dH_o,$$

where $c, d \in \mathbf{R}$, $ad - bc = 1$. ■

THEOREM 3.5 *The system* (3.2) *has a Lax formulation.*

Proof. Let us take:

$$L = \begin{bmatrix} 0 & -P_3 & P_2 \\ P_3 & 0 & -P_1 \\ -P_2 & P_1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & \frac{2}{c_1}P_3 & \left(\frac{2}{c_2} - \frac{2}{c_1}\right)P_2 \\ -\frac{2}{c_1}P_3 & 0 & 0 \\ \left(\frac{2}{c_1} - \frac{2}{c_2}\right)P_2 & 0 & 0 \end{bmatrix}.$$

Then a direct computation shows us that the system (3.2) can be put in the equivalent form

$$\dot{L} = [L, B],$$

as required. ■

REMARK 3.3 As a consequence of the above theorem it follows that the flow of the system (3.2) is isospectral. ■

THEOREM 3.6 *The system* (3.2) *may be explicitely integrated by elliptic functions.*

Proof. It is known that:

$$P_1^2 c_2 + P_2^2 c_1 = 2H_o c_1 c_2 = l,$$

and

$$P_1^2 + P_2^2 + P_3^2 = C,$$

are constants of motion. Then an easy computation shows us that:

$$P_2^2 = \frac{c_2}{c_2 - c_1}\left[\frac{Cc_2 - l}{c_2} - P_3^2\right]$$

and

$$P_1^2 = \frac{c_1}{c_1 - c_2}\left[\frac{Cc_1 - l}{c_1} - P_3^2\right].$$

Using now the third equation from (3.2) we get

$$(\dot{P}_3)^2 = \frac{4}{c_1 c_2}\left(P_3^2 - \frac{Cc_2 - l}{c_2}\right)\left(\frac{Cc_1 - l}{c_1} - P_3^2\right)$$

that is:

$$t = \int_{P_3(0)}^{P_3} \frac{dt}{\sqrt{\frac{4}{c_1 c_2}\left(P_3^2 - \frac{Cc_2 - l}{c_2}\right)\left(\frac{Cc_1 - l}{c_1} - P_3^2\right)}}$$

which shows that P_3, and hence P_1, P_2 are elliptic functions of time. ■

4. NUMERICAL INTEGRATION OF THE SYSTEM (3.2)

In this section we shall discuss the numerical integration of the system (3.2) via the Lie–Trotter integrator and we shall point out some of their geometrical properties. To begin with, let us observe that the Hamiltonian vector field X_{H_o} splits as follows:

$$X_{H_o} = X_{H_1} + X_{H_2},$$

where

$$H_1 = \frac{1}{2}\frac{P_1^2}{c_1}; \qquad H_2 = \frac{1}{2}\frac{P_2^2}{c_2}.$$

The integral curves of X_{H_1} and X_{H_2} are given by:

$$P(t) = \exp(tX_{H_1}) \cdot P(0) = \phi_1(t, P(0))$$

and respectively

$$P(t) = \exp(tX_{H_2}) \cdot P(0) = \phi_2(t, P(0)).$$

Now, following [10] (see also [5] and [9]), the Lie-Trotter formula gives rise to an explicit integrator of the equation (3.2) namely:

$$P^{k+1} = \phi_1(t, \phi_2(t, P^{(k)}),$$

or explicitely:

$$\begin{cases} P_1^{k+1} = P_1^k \cos \dfrac{2P_2(0)}{c_2} t - P_3^k \sin \dfrac{2P_2(0)}{c_2} t \\ P_2^{k+1} = P_1^k \sin \dfrac{2P_1(0)}{c_1} t \sin \dfrac{2P_2(0)}{c_2} t + P_2^k \cos \dfrac{2P_1(0)}{c} t \\ +P_3^k \sin \dfrac{2P_1(0)}{c_1} t \cos \dfrac{2P_1(0)}{c_1} t \\ P_3^{k+1} = P_1^k \cos \dfrac{2P_1(0)}{c_1} t \sin \dfrac{2P_2(0)}{c_2} t - P_2^k \sin \dfrac{2P_1(0)}{c_1} t \\ +P_3^k \cos \dfrac{2P_1(0)}{c_1} t \cos \dfrac{2P_2(0)}{c_2} t. \end{cases} \tag{4.1}$$

Some of its properties are sketched in the following theorem:

THEOREM 4.1 *The numerical integrator* (4.1) *has the following properties:* (**i**) *The numerical integrator* (4.1) *preserves the Poisson structure* (3.4). (**ii**) *The numerical integrator* (4.1) *preserves the Casimirs of our configuration* $(\mathbf{R}^3, \prod)$. (**iii**) *Its restriction to each coadjoint orbit* $(P_1^2 + P_2^2 + P_3^2 = k,\ \omega_k = \dfrac{1}{k}(P_2 dP_1 \wedge dP_3 - P_3 dP_1 \wedge dP_2 - P_1 dP_2 \wedge dP_3))$ *gives rise to a symplectic integrator.* (**iv**) *The numerical integrator* (4.1) *does not preserve the Hamiltonian* H_o *given by* (3.3).

Proof. The items (**i**)–(**iii**) hold because ϕ_1 and ϕ_2 are flows of some Hamiltonian vector fields, hence they are Poisson maps. Item (iv) is essentially due to the fact that

$$\{H_1, H_2\} \neq 0.$$

■

5. STABILITY

It is not hard to see that the equilibrium states of our system (3.2) are:

$$e_1 = (M, 0, 0); \qquad e_2 = (0, M, 0); \qquad e_3 = (0, 0, M),$$

where $M \in \mathbf{R}$. Now we shall discuss their nonlinear stability. Recall that an equilibrium point p is nonlinear stable if trajectories starting close to p stay close to p. In other words, a neighborhood of p must be flow invariant.

THEOREM 5.1 *The equilibrium state* e_1 *is:* (**i**) *unstable, if* $c_1 > c_2$; (**ii**) *nonlinear stable if* $c_1 < c_2$.

Proof. First consider the system linearized about e_1. Its eigenvalues are given by solutions of the equation:

$$\lambda\left(\lambda^2 - 4M^2\frac{c_1 - c_2}{c_1^2 c_2}\right) = 0.$$

(**i**) If $c_1 > c_2$ then a root of the characteristic polynomial has positive real part, thus e_1 is unstable as required. (**ii**) If $c_1 < c_2$, then the characteristic polynomial has two imaginary eigenvalues and one zero eigenvalue. Is the system stable? We shall prove that it is, via the energy-Casimir method, [2], [7]. Consider the energy-Casimir function:

$$H_\varphi = \frac{1}{2}\left(\frac{P_1^2}{c_1} + \frac{P_2^2}{c_2}\right) + \varphi\left(\frac{1}{2}(P_1^2 + P_2^2 + P_3^2)\right),$$

where $\varphi : \mathbf{R} \to \mathbf{R}$ is an arbitrary smooth real valued function defined on $\mathbf{R}$. Let φ', φ'' denote its first and second derivatives. Now, the first variation of H_φ is given by:

$$\delta H_\varphi = \frac{P_1}{c_1}\delta P_1 + \frac{P_2}{c_2}\delta P_2 + \varphi'(\cdot)(P_1\delta P_1 + P_2\delta P_2 + P_3\delta P_3).$$

This equals zero at the equilibrium of interest if and only if:

$$\varphi'\left(\frac{1}{2}M^2\right) = -\frac{1}{c_1}. \tag{5.1}$$

The second variation of H_φ at the equilibrium of interest is given via (5.1) by:

$$\delta^2 H_\varphi(e_1) = \varphi''\left(\frac{1}{2}M^2\right)M^2(\delta P_1)^2 + \frac{c_1 - c_2}{c_1 c_2}(\delta P_2)^2 - \frac{1}{c_1}(\delta P_3)^2.$$

Since $c_1 < c_2$; $c_1, c_2 > 0$ and having choosen φ such that:

$$\varphi''\left(\frac{1}{2}M^2\right) < 0,$$

we can conclude that the second variation at the equilibrium of interest is negative definite thus e_1 is nonlinear stable. ∎

Similar arguments lead us to:

THEOREM 5.2 *The equilibrium state e_2 is:* (**i**) *unstable, if $c_1 < c_2$;* (**ii**) *nonlinear stable, if $c_1 > c_2$.*

THEOREM 5.3 *The equilibrium state e_3 is always nonlinear stable.*

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West University of Timişoara, B-dul B. Pârvan no. 4, 1900, Timişoara, ROMÂNIA