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HYPERCOMPLEX GEOMETRY

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1. Introduction

A manifold M is said to be *hypercomplex* if there exist three integrable complex structures I_1, I_2, I_3 on M satisfying the quaternion identities: $I_1I_2 = -I_2I_1 = I_3$.

Example 1. Let $\mathbb{H}$ denote the quaternion numbers and consider $(\mathbb{H}\setminus 0)^n = (\mathrm{S}^3 \times \mathbb{R})^n$. Define a hypercomplex structure by

$$I_\lambda(\vec{q}, \vec{x}) = (\vec{q}\lambda, \vec{x})$$

for $(\vec{q}, \vec{x}) \in (S^3)^n \times (\mathbb{R})^n$ and $\lambda \in \{i, j, k\}$. Note that this structure is left invariant. We get compact examples on $(\mathrm{S}^3 \times \mathrm{S}^1)^n$ via $(\mathbb{Z})^n$-quotients of $(\mathrm{S}^3 \times \mathbb{R})^n$.

Thus, the Hopf surface $S^3 \times S^1$ is a simple example of a compact hypercomplex manifold. In the following we shall generalize this example in three directions. The Hopf surface together with the projection $S^3 \times S^1 \to S^3$ is an example of a special Kähler-Weyl 4-manifold M^4 with symmetry, fibering over an Einstein-Weyl 3-space $M^4 \to B^3$. This point of view leads to a construction of hypercomplex 4-manifolds via Abelian monopoles and geodesic congruences on Einstein-Weyl 3-manifolds [6].

We may also think of the Hopf surface as the Lie group $\mathrm{SU}(2) \times S^1$ with a homogeneous hypercomplex structure. Spindel et al. [20] and independently Joyce [12] showed how such homogeneous structures may be constructed on $G \times T^k$ for G a compact Lie group. Using the twistorial description of hypercomplex geometry [16], we may bring complex deformations to bear on these examples and obtain non-homogeneous structures on $G \times T^k$ [17].

The third theme we shall address is the following: to any quaternionic $4n$-manifold M we may associate a hypercomplex $(4n+4)$-space $\mathcal{V}(M)$ [18] generalizing the Swann bundle of a quaternionic Kähler manifold [21]. Joyce [12] showed how to twist this construction with an instanton $P \to M$ to obtain a hypercomplex manifold $\mathcal{V}_P(M)$ fibering over M with fiber the Hopf surface $S^3 \times S^1$. Again such structures may be deformed using twistor theory [16].

2. Kähler-Weyl 4-Manifolds

Consider a hypercomplex 4-manifold M. On M we may define a conformal structure $[g]$: to each non-zero vector X we declare (X, I_1X, I_2X, I_3X) to be orthonormal. Any hypercomplex manifold has a unique torsion-free connection preserving each of the complex structures, the Obata connection D [14]. This connection clearly preserves the conformal structure, so we have a Weyl manifold $(M, [g], D)$ [6]. A Weyl manifold with vanishing trace-free-symmetric part of the Ricci curvature S_0r^D is called *Einstein-Weyl* [5]. In the following we shall see how Einstein-Weyl geometry in 3 and 4 dimensions interacts with hypercomplex geometry.

Let $V_\pm$ be the spin bundles and let L be the bundle coming from the representation $A \mapsto |\det(A)|^{\frac{1}{4}}$. Then the complexified tangent bundle T_cM is equal to $V_+ \otimes V_- \otimes L$ and the curvature

$$R^D = W_+ + W_- + S_0r^D + F_+^D + F_-^D + s^D$$

of the Weyl connection D is contained in

$$L^{-2} \otimes (S^4V_+ \oplus S^4V_- \oplus (S^2V_+ \otimes S^2V_-) \oplus S^2V_+ \oplus S^2V_- \oplus \mathbb{R}).$$

For a hypercomplex manifold, the structure is reduced to $\mathbb{R}_{>0} \times \mathrm{SU}(2)_+$, so the curvature is contained in $L^{-2} \otimes (S^4V \oplus S^2V_+)$. Therefore, half of the Weyl curvature vanishes, $W_- = 0$, the trace-free-symmetric part of the Ricci curvature vanishes, $S_0r^D = 0$, half of the Faraday curvature vanishes, $F_-^D = 0$, and the scalar curvature s^D vanishes. In particular, a hypercomplex manifold is an example of a special selfdual 4-manifold (which is also Einstein-Weyl).

Via the Penrose correspondence, a selfdual conformal 4-manifold M with a conformal Killing vector K corresponds to a 3-dimensional complex twistor space Z with a complex holomorphic vector field K_c [1]. The quotient M/K is an Einstein-Weyl 3-space B with a monopole (w, A) consisting of a section w of L^{-1} and a 1-form A such that $*D^Bw = dA$ [10]. The quotient Z/K_c is the minitwistor space S of B [8].

A conformal 4-manifold $(M, [g])$ with compatible complex structure I has a natural weight-less anti-selfdual 2-form Ω ($\Omega \in L^{-2} \otimes \Lambda_-^2$) and a unique Weyl connection D (i.e. a torsion-free connection preserving the conformal structure) such that $d^D\Omega = 0$ [6]. We called such a structure $(M, [g], I, D)$ a *Kähler-Weyl manifold*.

For a selfdual Kähler-Weyl manifold the twistor space Z contains degree one divisors $\mathcal{D}, \overline{\mathcal{D}}$ corresponding to the complex structures $\pm I$. The line bundle $\mathcal{L}_\tau = [\mathcal{D} - \overline{\mathcal{D}}]$ over Z is clearly trivial on twistor lines. Via the Ward correspondence such a degree zero bundle gives an instanton [1], which in this case is the Ricci form ρ^D. Therefore, the 4-manifold is hypercomplex iff $\mathcal{L}_\tau$ is trivial. When $\mathcal{L}_\tau$ is trivial the meromorphic function defining the divisor $\mathcal{D} - \overline{\mathcal{D}}$ gives a map from Z to $\mathbb{CP}^1$.

If a selfdual Kähler-Weyl manifold has a conformal Killing vector K, preserving the complex structure, then $\mathcal{D}, \overline{\mathcal{D}}$ project to divisors $\mathcal{C}, \overline{\mathcal{C}}$ contained in the minitwistor space S. The space B parameterizes degree two rational curves in S and points in S correspond to oriented geodesics in B. The rational curve in S corresponding to a point x in B intersects $\mathcal{C}, \overline{\mathcal{C}}$ in a pair of points defining a geodesic in B through x with two orientations. In this way we obtain a shear-free geodesic congruence which may be formulated as a a section χ of the bundle $L^{-1} \otimes TB$ satisfying

$$D^B\chi = \tau(id - \chi \otimes \chi) + \kappa * \chi$$

where shear-free means that the conformal structure normal to χ is preserved. The sections τ, κ of L^{-1} are monopoles representing the divergence and twist respectively of the congruence [6].

Conversely, from an Einstein Weyl space $(B^3, [h], D^B)$ with a monopole (w, A) we may construct a selfdual 4-metric

$$g = w^2 h + (dt + A)^2.$$

The twistor space Z is the total space of the monopole line bundle over the minitwistor space S of B. Choose a shear-free geodesic congruence χ. This corresponds to a divisor in S which lifts to a divisor in Z defining a compatible complex structure on the 4-manifold. In fact this conformal 4-space is hypercomplex iff the divergence of χ is proportional to the monopole w used to construct g. This can be seen as follows: the twistor space is the total space of $\mathcal{L}_\tau \xrightarrow{p} S$ and the pull back $p^*\mathcal{L}_\tau$ is trivial over Z, so the Ricci form vanishes. As an example we could take the Einstein-Weyl space given by the round 3-sphere and let χ be a left or right invariant congruence. Since these congruences have vanishing τ any sum w of fundamental solutions to the Laplace equations would give a hypercomplex 4-space. The solution $w = 1$ (in the gauge given by the round sphere) gives the Hopf surface $S^3 \times S^1$.

3. Lie Groups and Hypercomplex Geometry

The hypercomplex structure of the Hopf surface defined in the example in the introduction may be considered as a left invariant structure on the Lie group $S^1 \times \mathrm{SU}(2)$. Consider the Lie group SU(3). The Lie algebra $\mathfrak{g} = \mathfrak{su}(3)$ decomposes as $\mathfrak{g} = \mathfrak{b} \oplus \partial_1 \oplus \mathfrak{f}_1$ where

$$\mathfrak{g} = \begin{pmatrix} \partial_1 & \mathfrak{f}_1 \\ \mathfrak{f}_1 & \mathfrak{b} \end{pmatrix} = \begin{pmatrix} \mathfrak{su}(2) & \mathbb{C}^2 \\ \mathbb{C}^2 & \mathfrak{u}(1) \end{pmatrix} = \mathfrak{su}(3).$$

Think of $\mathfrak{b} \oplus \partial_1$ as $\mathbb{H}$ and think of ∂_1 as the imaginary quaternions acting on $\mathfrak{f}_1$ via the adjoint representation. Applying left translations we obtain in this way a hypercomplex structure on SU(3). Now, let G be a compact semi-simple Lie group. The Lie algebra $\mathfrak{g}$ decomposes as follows

$$\mathfrak{g} = \mathfrak{b} \oplus_{j=1}^n \partial_j \oplus_{j=1}^n \mathfrak{f}_j,$$

where $\mathfrak{b}$ is Abelian, ∂_j is isomorphic to $\mathfrak{su}(2)$ and $[\partial_j, \mathfrak{f}_j] \subset \mathfrak{f}_j$. The rank r of G is equal to $n + \dim \mathfrak{b}$ and if we add $2n - r$ Abelian factors we can think of $(2n-r)\mathfrak{u}(1) \oplus \mathfrak{b} \oplus_{j=1}^n \partial_j$ as $\mathbb{H}^n$. Since $[\partial_j, \mathfrak{f}_j] \subset \mathfrak{f}_j$ we can proceed as with SU(3) above to get a left invariant hypercomplex structure on $T^{2n-r} \times G$ [12]. In this way we get homogeneous hypercomplex structures on for example

$$\mathrm{SU}(2\ell+1), T^1 \times \mathrm{SU}(2\ell), T^\ell \times \mathrm{SO}(2\ell+1), T^\ell \times \mathrm{Sp}(\ell), T^{2\ell} \times \mathrm{SO}(4\ell),$$
$$T^{2\ell-1} \times \mathrm{SO}(4\ell+2), T^2 \times E_6, T^7 \times E_7, T^8 \times E_8, T^4 \times F_4 \text{ and } T^2 \times G_2.$$

The issue is now how to get more than these homogeneous examples. For a general hypercomplex manifold (M^{4n}, I_1, I_2, I_3) we note that we have a 2-sphere of complex structures $I_{\mathbf{v}} = v_1 I_1 + v_2 I_2 + v_3 I_3$ for $\mathbf{v} = (v_1, v_2, v_3) \in S^2$. The *twistor space* of M is the space $W = M \times S^2$ of these compatible complex structures [15, 16]. This space is a complex manifold of dimension $2n+1$: the complex structure $\mathcal{I}$ at $(x, \mathbf{v}) \in M \times S^2$ is standard along the 2-sphere and it is equal to $I_{\mathbf{v}}(x)$ along $T_x M$. The integrability of $\mathcal{I}$ is a consequence of M being hypercomplex. The holomorphic projection $W \xrightarrow{p} S^2 = \mathbb{CP}^1$ has fiber $p^{-1}(z)$ which is M together with the complex structure determined by the point $z \in \mathbb{CP}^1$. The non-holomorphic projection $W \xrightarrow{\pi} M$ has as fibers, rational curves of normal bundle $\mathcal{O}(1) \otimes \mathbb{C}^{2n}$.

The idea is to deform the hypercomplex structure on M by deforming the map $W \xrightarrow{p} \mathbb{CP}^1$ [17]. Consider the sheaf $\mathcal{D}$ defined by the exact sequence

$$0 \to \mathcal{D} \to \Theta_W \xrightarrow{dp} p^*\Theta_{\mathbb{CP}^1} \to 0.$$

where Θ is the tangent sheaf. The deformations of the map p (and therefore the deformations of the hypercomplex geometry on M) is measured by the cohomology groups of the sheaf $\mathcal{D}$ [9]: $H^0(W, \mathcal{D})$ is the space of hypercomplex symmetries, $H^1(W, \mathcal{D})$ is the parameter space of deformations and $H^2(W, \mathcal{D})$ is the obstruction space.

For $M = T^k \times G$ the twistor space W is a homogeneous complex manifold and one may expect that $H^j(W, \mathcal{D})$ is computable via Bott-Borel-Weil-Hirzebruch theory for representations and cohomology. Consider the natural map Φ from W to G/U where U is a maximal torus in G. The spaces $Z = G/U$ is a complex manifold and is called the Borel flag [2, 7]. The cohomology of the Borel flag has indeed been studied using representation theory and this will help us getting information about the cohomology on W: let X be M with a complex structure $X = p^{-1}(z)$. The restriction of Φ to X has fiber E which is a product of elliptic curves. We may compute $H^j(X, \mathcal{O}_X)$, say, using a Leray spectral sequence

$$E_2^{p,q} = H^p(Z, R^q\Phi_*\mathcal{O}_X), E_\infty^{p,q} = H^{p+q}(X, \mathcal{O}_X).$$

We find $R^q\Phi_*\mathcal{O}_X = \mathcal{O}_Z \otimes H^q(E, \mathcal{O}_E)$ and since $H^p(Z, \mathcal{O}_Z)$ vanishes for $p \geq 1$ [3], the spectral sequence is easy to handle and we get

$$H^q(X, \mathcal{O}_X) = E_\infty^{0,q} = E_2^{0,q} = H^q(E, \mathcal{O}_E) \cong \Lambda^q \mathbb{C}^n.$$

In much the same way we can compute the cohomology $H^j(W,\mathcal{O}_W), H^j(W,\Phi^*\Theta_Z)$ etc. via vanishing results of Bott [4]. Then using the sequences

$$0 \to \mathcal{O}_W \to p^*\Theta_{\mathbb{CP}^1} \to \mathcal{O}_{X_1\cup X_2} \to 0$$

$$0 \to \mathcal{D} \to \Theta_W \xrightarrow{dp} p^*\Theta_{\mathbb{CP}^1} \to 0,$$

we are able to find $H^j(W,\mathcal{D})$.

It turns out that the obstruction space $H^2(W,\mathcal{D})$ is non-trivial. Therefore we study the possible obstructions using Kuranishi theory [13]. However, we can prove that for the U-invariant part of $H^1(W,\mathcal{D})$ the obstruction vanishes and we obtain (see [17] for a more precise formulation of the theorem):

Theorem 1. *Suppose G is a compact semi-simple Lie group of rank r and containing n factors of $\mathfrak{sp}(1)$. Then the local moduli at a generic deformation of left-invariant hypercomplex structures on $T^{2n-r}\times G$ is a smooth manifold of dimension $n(n+r)$. The identity component of the group of hypercomplex symmetries of a generic deformation is the Abelian group T^{2n}.*

In the introduction we defined one hypercomplex structure on $(S^3\times S^1)^n$. Inspired by the theory of Abelian varieties, we shall now construct a family of hypercomplex structures on $(S^3\times S^1)^n$ and use the theorem above to secure completeness. Let $(q_1,\dots,q_n;x_1,\dots,x_n) = (\mathbf{q};\mathbf{x})$ be coordinates for $(\mathrm{S}^3)^n\times\mathbb{R}^n$. Here the q_j are unit quaternions. Choose a hypercomplex structure on $\mathbb{H}^n$ by right multiplication of unit quaternions. Then we define a hypercomplex structure on $(\mathrm{S}^3\times\mathbb{R})^n$ through the embedding into $\mathbb{H}^n$.

For $1\le j\le n$, define an action generated by

$$\gamma_j(\mathbf{q};\mathbf{x}) = (e^{2\pi i\theta_{1j}}q_1,\dots,e^{2\pi i\theta_{nj}}q_n;\mathbf{x}+\mathbf{v}_j).$$

The action of γ_j is represented by the column vectors $\mathbf{v}_j$ and $\Theta_j = (\theta_{1j},\dots,\theta_{nj})^T$, where θ_{ij} are in $\mathbb{R}/\mathbb{Z}$.

Assume that the vectors $\{\mathbf{v}_1,\dots,\mathbf{v}_n\}$ are linearly independent. Let $\Gamma\cong\mathbb{Z}^n$ be the group generated by $\{\gamma_1,\dots,\gamma_n\}$. We call

$$(\Theta|V) = (\Theta_1,\dots,\Theta_n|\mathbf{v}_1,\dots\mathbf{v}_n)$$

the *period matrix* of the manifold $(\mathrm{S}^3\times\mathbb{R})^n/\Gamma$. Thus the groups Γ are parameterized by the space $\mathbb{R}/\mathbb{Z})^{n^2}\times\mathrm{GL}(n,\mathbb{R})$. However, different period matrices may generate the same group. In fact, the period matrices $(\Theta|V)$ and $(\hat\Theta|\hat V)$ generate the same group if and only if there is a matrix $M=(m_{ij})$ in $\mathrm{GL}(n,\mathbb{Z})$ such that

$$(\hat\Theta|\hat V) = (\Theta M|VM).$$

The quotient space $(\mathrm{S}^3\times\mathbb{R})^n/\Gamma$ is a hypercomplex manifold because the actions of Γ commute with the right multiplications of the quaternions on $(q_1,\dots,q_n)$. The quotient space is clearly diffeomorphic to $(\mathrm{S}^3\times\mathrm{S}^1)^n$. Using the fact that symmetries lifts to holomorphic maps of the twistor space (which is built out of a complex projective space), it is seen that hypercomplex manifolds $(\mathrm{S}^3 \times \mathbb{R})^n/\Gamma$ and $(\mathrm{S}^3 \times \mathbb{R})^n/\Gamma'$ are equivalent if and only if there exist period matrices $(\Theta|V)$ and $(\Theta'|V')$ for Γ and Γ' respectively such that $V = V'$, and $\Theta_j = \pm\Theta'_j$. Thus we obtain

Theorem 2. *The quotient space* $((\mathbb{R}/\mathbb{Z})^{n^2} \times \mathrm{GL}(n, \mathbb{R}))/(\mathbb{Z}_2^n \times \mathrm{GL}(n, \mathbb{Z}))$ *is a complete moduli space for hypercomplex structures on the product manifold* $(\mathrm{S}^3 \times \mathrm{S}^1)^n$.

The constructions above are currently being modified to work for the case of nilpotent automorphisms and for combinations of the semi-simple and the nilpotent situation in joint work with Grantcharov and Poon.

4. The Swann Bundle

Now we turn to the third theme where $S^3 \times S^1$ appears as the fiber of a bundle. The definition of a hypercomplex manifold is equivalent to requiring that the holonomy group lies in $\mathrm{GL}(n, \mathbb{H})$. More generally for a quaternionic manifold M the frame bundle has a torsion free connection with holonomy in

$$\mathrm{GL}(n, \mathbb{H})\,\mathrm{GL}(1, \mathbb{H}) = (\mathbb{R}_{>0} \times \mathrm{SL}(n, \mathbb{H}) \times \mathrm{Sp}(1))/\{\pm 1\}.$$

This group acts on $\mathbb{H}/\mathbb{Z}_2$ by

$$\rho(\lambda, A, q)(\eta) = \lambda^{\frac{n}{n+1}} \eta q^{-1}.$$

The associated bundle is denoted by $\mathcal{U}(M)$ and was studied by Swann for M a quaternionic Kähler manifold [21]. For M quaternionic $\mathcal{U}(M)$ is hypercomplex [18]. The group $\mathbb{H}^*$ acts from the left on $\mathcal{U}(M)$ and the center $\mathbb{Z}$ preserves the hypercomplex structure. The quotient $\mathcal{U}(M)/\mathbb{Z}$ is denoted $\mathcal{V}(M)$ and is a compact hypercomplex manifold which we call the *Swann bundle* [18], [19]. Now, let P be an S^1-instanton on M. Then Joyce [12] introduces the twisted bundle $\mathcal{V}_P(M) = P \times_{S^1} \mathcal{V}(M)$ which again provides us with an example of a compact hypercomplex manifold. The fiber from $\mathcal{V}_P(M)$ to M is $S^3 \times S^1$.

Example 2. Let M be the complex projective plane and let $P \to M$ be the instanton given by the Hopf fibration $S^5 \to \mathbb{CP}^2$. Then in this case the hypercomplex manifold $\mathcal{V}_P(M)$ is equal to $\mathrm{SU}(3)/\mathbb{Z}_2$.

We may now apply complex deformation theory to these twisted Swann bundles. The twistor space W of $\mathcal{V}_P(M)$ fibers over the twistor space Z of M and via the Leray spectral sequence we are able to compute the cohomology $H^j(W, \mathcal{D})$ in terms of the cohomology on Z [16].

Example 3. Let M be the connected sum $2\mathbb{CP}^2$ equipped with a Poon conformal structure c_λ, $\lambda \in (0, 1)$. Then the deformation theory gives a 4-parameter space of T^3-symmetric hypercomplex structures on the 8-manifold $\mathcal{V}(2\mathbb{CP}^2)$. Furthermore, we can integrate and find these hypercomplex manifolds locally as a (Joyce-) hypercomplex quotient [11] of $\mathbb{H}^4$ with a T^2 action. The space is realized as a subspace of $\mathbb{C}^6 \times \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ given by simple equations [16].

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