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BRANE SOLITONS AND HYPERCOMPLEX STRUCTURES

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ABSTRACT. The investigation of strings and M-theory involves the understanding of various BPS solitons which in a certain approximation can be thought of as solutions of ten- and eleven-dimensional supergravity theories. These solitons have a brane or a intersecting brane interpretation, saturate a bound and are associated with parallel spinors with respect to a connection of the spin bundle of spacetime. A class of intersecting brane configurations is examined and it is shown that the geometry of spacetime is hyper-Kähler with torsion. A relation between these hyper-Kähler geometries with torsion and quaternionic calibrations is also demonstrated.

1. INTRODUCTION

The main achievement in theoretical physics the past few years is the realization that all five string theories [1] are related amongst themselves and that are limits of a another theory which has been called M-theory [2, 3]. The precise nature of M-theory remains a mystery but by now an impressive amount of evidence has been gather which point to its existence. Most of these arise from investigating the low energy approximation of strings and M-theory which are described by ten- and eleven-dimensional supergravities, respectively; see however [4]. The use of supergravity theories in this context is two-fold. First, the conjectured duality symmetries of string theories are discrete subgroups of the continuous duality symmetries of the field equations of supergravity theories [5]. Second, the supergravity theories admit solutions which have the interpretation of extended objects embedded in the spacetime, called branes, which are the BPS solitons of strings and M-theory.. A consequence of their BPS property is that branes are stable under deformations of the various parameters of the theories, like for example coupling constants. Because of this, they can be used to compare the various limits of M-theory and thus establish the relations amongst the various string theories. Extrapolating from the properties of eleven-dimensional and ten-dimensional supergravities, M-theory is thought to have the following essential ingredients:

- A low energy description in terms of the eleven-dimensional supergravity,
- M-2- and M-5-branes, and

- limits that describe all five string theories in ten dimensions.

The solutions of supergravity theories with a brane interpretation have some common properties. The associate spacetime of a p-brane has an asymptotic region isometric to $\mathbb{R}^{(1,p)} \times \mathbb{R}^n$, where, viewing the p-brane as a (p+1)-dimensional submanifold of spacetime, $\mathbb{R}^{(1,p)}$ and $\mathbb{R}^n$ are identified with the worldvolume and transverse directions of the p-brane, respectively; $(p+n)$ is equal either to nine (string theory) or to ten (M-theory). In the above asymptotic region a mass m and a charge q per unit $\mathbb{R}^p \subset \mathbb{R}^{(1,p)}$ volume is defined, i.e m and q are energy and charge densities, respectively. Then using the properties of supergravity theory, a bound can be established [6] as

$$m \geq \alpha\, |q| \ ,$$

where for string theory branes α depends on the string coupling constant λ. The precise dependence of α distinguishes between the various types of branes as follows: $\alpha \sim \lambda^0$ for fundamental strings, $\alpha \sim \lambda^{-1}$ for Dirichlet branes or D-branes for short and $\alpha \sim \lambda^{-2}$ for Neveu-Schwarz 5-branes or NS-5-branes for short. The solutions that are of interest are those that saturate the above bound leading to BPS type configurations. BPS configurations are associated with parallel spinors with respect to a connection which occurs naturally in supergravity theories. The BPS branes of strings and M-theory admit sixteen parallel spinors. Apart from the stability of these BPS solutions that has already been mentioned above, superposition rules have been found that allow to combine two or more such solutions and construct new ones [7]. The solutions that arise from superpositions of BPS branes also admit parallel spinors which typically are less than those of the branes involved in the superposition.

In this paper, the geometry of a class of BPS brane solutions of supergravity theories and that of their superpositions will be described. I shall begin with a description of BPS M-2-brane [8] and M-5-brane [9] solutions of eleven-dimensional supergravity [10]. Then I shall explain the connection between BPS solutions and parallel spinors. I shall continue with the NS-5-brane solution of type II ten-dimensional supergravity theories [11] and show that the geometry of this solution is hyper-Kähler with torsion (HKT). Then I shall explore the various superpositions of NS-5-branes and I shall demonstrate that these superpositions are related to the quaternionic calibrations in $\mathbb{R}^8$ [12]. I shall interpret these superpositions as intersecting NS-5-branes and I shall show that the geometry of these solutions is again HKT. Most of these results have appeared in [13, 14]. Finally, I shall state my conclusions.

2. Eleven-Dimensional Supergravity

I shall not attempt to give a full description of eleven-dimensional supergravity. This can be found in the original paper of Cremmer, Julia and Scherk who constructed the theory [10]. Here I shall only emphasize some aspects of the geometric structure of the theory. In field theoretic terms, the theory describes the dynamics of the graviton g, a three-form gauge potential A and a gravitino ψ. The latter is a spinor-valued one-form which does not enter in the analysis below and so it will be neglected in what follows. Geometrically, let $(N; g, F, \nabla)$ be an eleven-dimensional spin manifold N of signature $(-,+,\ldots,+)$ equipped with a metric g, a closed four-form F, locally $F = dA$, and a connection ∇. In the physics literature ∇ is called superconnection and

$$\nabla : \qquad C^\infty(S) \to \Omega^1(N) \otimes C^\infty(S) \ ,$$

where S is the spin bundle over N and rank $(S) = 32$. This connection can be written as

$$\nabla = D + T(F) \ ,$$

D is the connection of S induced from the Levi-Civita connection of the metric g and

$$T_M(F)dx^M = -\frac{1}{144} F_{NPQR}\big(\Gamma_M{}^{NPQR} - 8\delta^N{}_M \Gamma^{PQR}\big)dx^M \ ,$$

where $\{\Gamma^M; M = 0, \ldots, 10\}$ is a basis in the Clifford algebra $\mathrm{Clif}(1,10)$ and $\Gamma^{M_1 M_2 \cdots M_n} = \Gamma^{[M_1}\Gamma^{M_2}\cdots\Gamma^{M_n]}$. The dynamics of the theory is described[1] by the action

$$S = \int d^{11}x \sqrt{|\det g|}\big(R(g) - 2|F|^2\big) - \frac{4}{3} A \wedge F \wedge F \ ,$$

where $R(g)$ is the Ricci scalar of the metric g and the norm of F is taken with respect to g. The above action consists from the Einstein-Hilbert term, the standard kinetic term for F and a Chern-Simons term, respectively. The equations of the fields g and A can be derived by varying the above action.

There are two classes of solutions to the field equations depending on whether or not $F = 0$. If $F = 0$, then the field equations imply that the Ricci tensor of g vanishes. Therefore a large class of solutions is $\mathbb{R}^{(1,10-n)} \times M^n$, where M^n is a manifold of appropriate special holonomy, i.e $SU(k)$, $k = 2,3,4$ $(n = 2k)$; $Sp(2)$ $(n = 8)$; G_2 $(n = 7)$; $\mathrm{Spin}(7)$ $(n = 8)$. Such solutions admit parallel spinors and have found application in the various compactifications of M-theory [15]. The other class of solutions is that for which $F \neq 0$. For such solutions to have a brane interpretation, it is required that they have an asymptotic region which is isometric to either $\mathbb{R}^{(1,2)} \times \mathbb{R}^8$ or $\mathbb{R}^{(1,5)} \times \mathbb{R}^5$. The former asymptotic behaviour is that of M-2-brane while the latter is that of M-5-brane. Then after imposing appropriate decaying conditions on the fields as they approach these asymptotic regions, the charges per unit volume of the M-2- and M-5-branes can be defined as follows:

$$q_2 = \int_{S^7} ({}^*F - A \wedge F) \ ,$$

[1] The conventions for forms are as follows: $\omega = \frac{1}{p!}\omega_{a_1,\ldots,a_p} dx^{a_1} \wedge \cdots \wedge dx^{a_p}$, $|\omega|^2 = \frac{1}{p!}\omega_{a_1,\ldots,a_p}\omega^{a_1,\ldots,a_p}$ and ${}^{**}\omega = -(-1)^{p(n-p)}\omega$, where * is the Hodge star operation and $n = 11$ in the present case.

where $S^7 \subset \mathbb{R}^8$, and

$$q_5 = \int_{S^4} F \ ,$$

where $S^4 \subset \mathbb{R}^5$, respectively. Then adapting the positive mass theorem of general relativity to this case, the bounds can be established,

$$m_2 \geq \alpha_2 |q_2|$$

or

$$m_5 \geq \alpha_5 |q_5| \ ,$$

where m_2 and m_5 are the M-2-brane and M-5-brane masses per unit volume, respectively. The manifolds that saturate those bounds admit sixteen parallel spinors with respect to the connection ∇.

To be specific, the BPS M-2-brane solution [8] is

$$\begin{aligned} ds^2 &= h^{\frac{1}{3}}\big(h^{-1} ds^2(\mathbb{R}^{(1,2)}) + ds^2(\mathbb{R}^8)\big) \\ {}^*F &= \mp \tfrac{1}{2} \star dh \ , \end{aligned}$$

where

$$h = 1 + \frac{q_2}{|y|^6}$$

is a harmonic function on $\mathbb{R}^8$, $y \in \mathbb{R}^8$, the Hodge star operation on F is with respect to the metric g on N and the Hodge star operation on dh is with respect to the flat metric on $\mathbb{R}^8$. The M-2-brane is located at $y = 0$. There are two asymptotic regions. One is as $|y| \to \infty$, where the spacetime N becomes isometric to $\mathbb{R}^{(1,2)} \times \mathbb{R}^8$ as expected. The other is as $|y| \to 0$ in which case N becomes isometric to $AdS_4 \times S^7$; AdS_4 is a Minkowski signature analogue of the standard hyperbolic four-manifold. It turns out though that the BPS membrane solution develops a singularity behind the $AdS_4 \times S^7$ region.

The BPS M-5-brane solution [9] is

$$\begin{aligned} ds^2 &= h^{\frac{2}{3}}\big(h^{-1} ds^2(\mathbb{R}^{(1,2)}) + ds^2(\mathbb{R}^5)\big) \\ F &= \mp \tfrac{1}{2} \star dh \ , \end{aligned}$$

where

$$h = 1 + \frac{q_5}{|y|^3}$$

is a harmonic function on $\mathbb{R}^5$, $y \in \mathbb{R}^5$ and the Hodge star operation on dh is with respect to the flat metric on $\mathbb{R}^5$. The M-5-brane is located at $y = 0$. There are also two asymptotic regions in this case. One is as $|y| \to \infty$ where the spacetime N becomes isometric to $\mathbb{R}^{(1,5)} \times \mathbb{R}^5$ as expected. The other is as $|y| \to 0$ in which case N becomes isometric to $AdS_7 \times S^4$; AdS_7 is a Minkowski signature analogue of the standard hyperbolic seven-manifold. The BPS M-5-brane solution is not singular.

Despite much progress in constructing solutions of eleven-dimensional supergravity which admit parallel spinors, there is no systematic way to tackle the problem of constructing solutions of the theory with any number of parallel spinors. Of course this is related to understanding the properties of the connection ∇. To give an example of the subtleties involved, let us consider the case of solutions with thirty two parallel spinors which is the maximal number possible. A straightforward example of a such spacetime is the Minkowski $N = \mathbb{R}^{(1,10)}$ space with vanishing F. However, this is not all. There are two more cases, $N = AdS_4 \times S^7$ with F the volume form of AdS_4 and $N = AdS_7 \times S^4$ with F the volume form of S^4.

A related and still unresolved problem is to construct *localized* solutions of eleven-dimensional supergravity theory with the interpretation of intersecting M-2- and M-5-branes. In particular consistency of the M-theory picture suggests that there should be a solution that has the interpretation of a M-2-brane ending orthogonally on a M-5-brane associated with eight parallel spinors. No such solution has been found so far; For a recent discussion of this see [16]. Other BPS solutions involving M-5-branes and M-2-branes have been found though, like a solution which has the interpretation of a M-2-brane 'passing through' a M-5-brane [17].

3. Type IIA Strings

A geometric insight into the properties of the connection ∇ of eleven-dimensional supergravity can be given after reducing M-theory to type IIA string theory. It turns out that in a sector of the supergravity theory associated with type IIA strings, IIA supergravity, the connection ∇ of the spin bundle is induced from certain connections with torsion of the tangent bundle of spacetime.

For this consider the eleven-dimensional spacetime $N = S^1 \times M$ with

$$\begin{aligned} ds^2(N) &= e^{\frac{4}{3}\phi} d\theta^2 + e^{-\frac{2}{3}\phi} ds^2(M) \\ F &= d\theta \wedge H \ , \end{aligned} \tag{3.1}$$

where θ is the angle which parameterizes the circle of radius r. It is assumed that the vector field $X = \frac{\partial}{\partial\theta}$ is an isometry of N which leaves in addition F invariant. In field theoretic terms, the metric γ in $ds^2(M)$ describes the graviton in ten dimensions, the closed three-form H is the $NS \otimes NS$ form field strength and ϕ is the dilaton; γ, H and ϕ are the so called common sector fields of string theory. The IIA supergravity has additional fields which can also be derived from eleven dimensions but the above sector will suffice for our purpose. The type IIA string coupling constant is related to the radius of the circle S^1 as $\lambda = r^{\frac{3}{2}}$ [3]. So for small radius, the string coupling constant is small and M-theory reduces to IIA strings.

The dynamics of the common sector fields in ten dimensions can be described by the action

$$S = \int d^{10}x \, e^{-2\phi} \sqrt{|\det \gamma|} \left(R(\gamma) - 2|H|^2 + 4|d\phi|^2 \right) ,$$

where the norms are taken with respect to the metric γ on M.

As it has been mentioned, a simplification occurs in the description of the connection ∇ using $N = S^1 \times M$ and 3.1 as above. For this, first observe that the spin bundle S decomposes as $S = S^+ \otimes S^-$, where S^+ and S^- are spin bundles over M

with $\operatorname{rank} S^+ = \operatorname{rank} S^- = 16$. This is due to the decomposition of the spinor representation of $\mathrm{spin}(1,10)$ into the sum of the two irreducible spinor representations of $\mathrm{spin}(1,9)$. Next it turns out that the connection ∇ decomposes into two connections one on S^+ and one on S^- which are induced by the connections

$$\nabla^\pm = D \pm H$$

of the tangent bundle, respectively, where D is the Levi-Civita connection of the metric γ. The connections $\nabla^\pm$ are metric connections with torsion the closed three-forms $\pm H$, respectively. There are two more conditions that arise from reducing the connection ∇ of eleven-dimensional supergravity to IIA supergravity which involve the dilaton ϕ. However, these two conditions do not give rise to additional restrictions on the parallel spinors of $\nabla^\pm$ connections in the examples that we shall investigate below. So we shall not consider them further.

The above simplification in the structure of the connection ∇ has some profound consequences. One of them is that the existence of parallel spinors with respect to the connection ∇ of S depends on the holonomy of the connections $\nabla^\pm$ of the tangent bundle of M.

The NS-5-brane solution of IIA supergravity [11] is

$$\begin{aligned} ds^2(M) &= ds^2\big(\mathbb{R}^{(1,5)}\big) + h ds^2(\mathbb{Q}) \\ H &= \mp\tfrac{1}{2} \star dh \\ e^{2\phi} &= h \ , \end{aligned}$$

where the Hodge star operation on dh is with respect to the flat metric on $\mathbb{R}^4$ and h is a harmonic function on $\mathbb{R}^4 = \mathbb{Q}$, $\mathbb{Q}$ is the quaternionic line,

$$h = 1 + \frac{1}{|q|^2} \ ,$$

$q \in \mathbb{Q}$. The NS-5-brane is located at $q = 0$. The spacetime M is diffeomorphic to $\mathbb{R}^{(1,5)} \times \big(\mathbb{Q} - \{0\}\big)$ and it has two asymptotic regions, $\mathbb{R}^{(1,5)} \times \mathbb{R}^4$ as $|q| \to \infty$ and $\mathbb{R}^{(1,5)} \times \mathbb{R} \times S^3$ as $|q| \to 0$. In what follows we shall choose $H = -\frac{1}{2} \star dh$.

The non-trivial part of the metric of M is that on $\mathbb{Q} - \{0\}$. To investigate the geometry on $\big(\mathbb{Q} - \{0\}\big)$, we introduce two hypercomplex structures $\mathbf{I} = \{I_1, I_2, I_3\}$ and $\mathbf{J} = \{J_1, J_2, J_3\}$ as follows:

$$\begin{aligned} I_1(dq) &= i\, dq \ , \qquad I_2(dq) = j\, dq \ , \qquad I_3(dq) = k\, dq \ , \\ J_1(dq) &= -dq\, i \ , \qquad J_2(dq) = -dq\, j \ , \qquad J_3(dq) = -dq\, k \ , \end{aligned}$$

where i, j, k are the imaginary unit quaternions. Observe that the two hypercomplex structures commute, $[\mathbf{I}, \mathbf{J}] = 0$. The metric on $\mathbb{Q} - \{0\}$ is hermitian with respect to both hypercomplex structures. In addition, the hypercomplex structure $\mathbf{I}$ is compatible with the ∇^- connection ($\nabla^- \mathbf{I} = 0$) and the hypercomplex structure $\mathbf{J}$ is compatible with the ∇^+ connection ($\nabla^+ \mathbf{J} = 0$); Note that the torsion H has support on $\mathbb{Q} - \{0\}$. Therefore the holonomy of $\nabla^\pm$ is in $SU(2)$. In fact the holonomy of $\nabla^\pm$ is $SU(2)$ and so the NS-5-brane admits sixteen parallel spinors. This fact follows from representation theory. As it will be demonstrated shortly, the geometry of the NS-5-brane can be summarized by saying that it admits two commuting strong HKT structures.

4. Hyper-Kähler Manifolds with Torsion

Let $(M, g, \mathbf{J})$ be a Riemannian hyper-complex manifold with metric g and hypercomplex structure $\mathbf{J}$; $\dim M = 4k$. The manifold $(M, g, \mathbf{J})$ admits a HKT structure [18] if

- The metric g is hermitian with respect to all three complex structures.
- There is a compatible connection ∇ with both the metric g and the hypercomplex structure $\mathbf{J}$ which has torsion a *three form* H.

There are two types of HKT structures on manifolds, the *strong* and the *weak*, depending on whether or not the torsion three-form is a closed, respectively.

Torsion has appeared in the physics literature since the early attempts to incorporate it in a relativistic theory of gravity. In supersymmetry, metric connections with torsion a closed three-form have appeared in the context of IIA and IIB supergravities but the relation to HKT geometry was not established. Connections with torsion a *closed* three-form appeared next in the investigation of two-dimensional supersymmetric sigma models [19, 20]. For a class of models, the sigma model manifold satisfies conditions which can be organized in one or two copies of what it is now called *strong* HKT structure. The general case of connections with torsion *any* three-form were found in the investigation of one-dimensional supersymmetric sigma models [21, 22]. For a class of models, the sigma model manifold satisfies conditions which can be organized as one or two copies of what it is now called *weak* HKT structure. The definition of the HKT structure as a new structure on manifolds was given in [18]. In the same paper, the strong and weak HKT structures were introduced, a formulation of a HKT structure in terms of conditions on Kähler forms was given, and a twistor construction for the HKT manifolds was proposed. The latter two properties of HKT manifolds are similar to those of hyper-Kähler manifolds [23]. There is also a generalization of the Quanternionic Kähler structure on manifolds to include torsion. The Quaternionic Kähler manifolds with torsion (QKT) have been introduced in [24] and further investigated in [25]. QKT manifolds admit a twistor construction similar to that of Quaternionic Kähler ones [26].

A straightforward consequence of the definition of HKT manifolds is that the holonomy of the connection ∇ is in $Sp(k)$. Some other developments related to these connections with three-form torsion are the vanishing theorems of [27, 28] for certain cohomology groups. Many examples of HKT manifolds have been constructed. These include a class of group manifolds with strong HKT structures in [29]. Specifically, the Hopf surface $S^1 \times S^3$ admits two strong HKT structures. Homogeneous weak HKT manifolds have been constructed in [30] using the hypercomplex structures of [31]. Inhomogeneous weak HKT structures have been given on $S^1 \times S^{4k-1}$ in [32].

In physics, the NS-5-brane solution constructed in the previous section clearly admits two strong HKT structures each associated with the hypercomplex structures on $\mathbb{Q} - \{0\}$ defined by left and right quaternionic multiplication, respectively. Other examples are certain (strong and weak) HKT structures that appear on the moduli spaces of five-dimensional black holes [22, 33, 34].

The HKT structure has many properties some of them found in [18] and more have been derived in [32]. One of them is the following: Let M be hypercomplex manifold with respect to $\mathbf{J}$ and equipped with a three-form H. M admits a HKT structure if

$$d\omega_{\mathbf{J}} - 2i_{\mathbf{J}}H = 0 \ , \tag{4.1}$$

where $\omega_{\mathbf{J}}$ are the three Kähler forms associated with the hyper-complex structure and $i_{\mathbf{J}}$ are the inner derivations with respect to the three complex structures. This equation will be used later to construct new HKT manifolds. In fact if two of the above conditions are satisfied, they imply the third. Observe also that the torsion of an HKT manifold can be specified from the metric and the complex structures [18]. So in what follows, we shall not give the expression for the torsion.

5. Quaternionic Calibrations

Calibrations have been introduced by Harvey and Lawson [35] to construct a large class of minimal submanifolds. Here, I shall use calibrations to find a new class of solutions of IIA supergravity that has the interpretation of intersecting branes. This new class of solutions admits a strong HKT structure.

A calibration of degree k is a k-form ω such that for every k-plane η in $\mathbb{R}^n$

$$\omega(\vec{\eta}) \leq 1 \ ,$$

where $\vec{\eta}$ is the co-volume form of η.

The contact set G_ω of a calibration is the subset of $\mathrm{Gr}(k, \mathbb{R}^n)$ of k-planes that saturate the above bound. Usually G_ω is a homogeneous space. There are many examples of calibrations, like Kähler and Special Lagrangian, which have been extensively investigated both in mathematics and physics. In the present case, the relevant class of calibrations are the quaternionic calibrations that have been described by Daroc, Harvey and Morgan in [12]. For this, we identify $\mathbb{R}^8 = \mathbb{Q}^2$ and introduce the hypercomplex structures $\mathbb{I} = \{I_1, I_2, I_3\}$ and $\mathbb{J} = \{J_1, J_2, J_3\}$ on $\mathbb{Q}^2$ induced by left and right quaternionic multiplication, respectively, i.e.

$$\begin{aligned} I_1(du) &= i\, du \ , \qquad I_2(du) = j\, du \ , \qquad I_3(du) = k\, du \ , \\ J_1(du) &= -du\, i \ , \qquad J_2(du) = -du\, j \ , \qquad J_3(du) = -du\, k \ , \end{aligned}$$

where i, j, k are the imaginary unit quaternions. In addition, we define

$$\phi_{\mathbb{J}} = \frac{1}{3}\sum_{r=1}^{3} \omega_{J_r} \wedge \omega_{J_r} \ ,$$

where ω_{J_r} is the Kähler form of the J_r complex structure with respect to the standard metric on $\mathbb{Q}^2$, and similarly $\phi_{\mathbb{I}}$ for $\mathbb{I}$. There are several calibration forms that can be constructed using the Kähler forms above. The calibration forms of quaternionic calibrations are found by averaging the calibration form $\phi_{\mathbb{J}}$ with various calibration forms constructed using the $\mathbb{I}$ hypercomplex structure. These calibration forms and their contact sets are summarized in the following table:

Quaternionic Calibrations in $\mathbb{R}^8$

Calibration Form ω	Contact Set G_ω
$\frac{1}{2}(\phi_{\mathbb{I}} + \phi_{\mathbb{J}})$	S^1
$\frac{1}{5}\omega_{I_1} \wedge \omega_{I_1} + \frac{3}{5}\phi_{\mathbb{J}}$	S^2
$\frac{1}{4}\Omega + \frac{3}{4}\phi_{\mathbb{J}}$	S^3
$\phi_{\mathbb{J}}$	S^4

In the second case above, instead of I_1 any other of the complex structures of $\mathbb{I}$ can be used. In the third case, the form Ω is the $Spin(7)$ invariant form associated with $\mathbb{I}$. In the last case, the contact set is the grassmannian of quaternionic lines in $\mathbb{Q}^2$, $Gr(1;\mathbb{Q}^2) = S^4$. The contact sets of the quaternionic calibrations are computed by observing that the groups that leave invariant the above forms act transitively on the calibrated planes. For more details see [12].

6. HKT Geometries and Branes

The strategy of constructing HKT geometries in eight dimensions is to superpose a HKT geometry on $\mathbb{Q} - \{0\}$ along four-planes in $\mathbb{R}^8$. The four-planes which will be used are in the contact sets of quaternionic calibrations that has been described in the previous section. The construction [13, 14] involves the following steps:

(i) Consider the HKT metric

$$d\sigma^2 = \frac{1}{|q|^2} dq d\bar{q} \tag{6.1}$$

on $\mathbb{Q} - \{0\}$, where $q \in \mathbb{Q}$. The torsion of this HKT geometry is that of the NS-5-brane that we have described.

(ii) Introduce the maps

$$\tau : \quad \begin{aligned} &\mathbb{Q}^2 \to \mathbb{Q} \\ &u \to \tau(u) = p_1 u^1 + p_2 u^2 - a \ , \end{aligned}$$

where p_1, p_2, a are quaternions.

(iii) Define the metric

$$ds^2 = \sum_\tau r^2(\tau) \tau^* d\sigma^2 + ds^2(\mathbb{Q}^2) \ ,$$

where the sum is over a finite number of maps τ, $r(\tau) \in \mathbb{R}$ and $ds^2(\mathbb{Q}^2)$ is the standard flat metric on $\mathbb{Q}^2$.

The manifold $K = \mathbb{Q}^2 - \cup_\tau\{\tau^{-1}(0)\}$ admits a HKT structure associated with the hypercomplex structure $\mathbf{J}$ induced by the hypercomplex structure $\mathbb{J}$ on $\mathbb{Q}^2$. To show this, the torsion of the HKT structure on K is given by pulling back the torsion of the NS-5-brane with respect to the maps τ and then summing up over the various maps τ. The key observation is that the differential $d\tau$ commutes with the hypercomplex structures on $\mathbb{Q} - \{0\}$ and K defined by *right* quaternionic multiplication. Using this, the condition 4.1 for K can be written as

$$\sum_\tau \tau^*\big(d\omega_{\mathbf{J}} - 2i_{\mathbf{J}}H\big) = 0 \ ,$$

where the expression inside the brackets is that of the condition 4.1 for the HKT structure (i) on $\mathbb{Q} - \{0\}$ and so vanishes identically. Thus K admits a HKT structure with respect to the ∇^+ connection and $\mathbf{J}$ hypercomplex structure.

Observe that $d\tau$ for generic parameters $\{p_1, p_2\}$ does not commute with the hypercomplex structure induced by left quaternionic multiplication on $\mathbb{Q} - \{0\}$ and K. So K does not admit a HKT structure with respect to this hypercomplex structure.

To make connection with the calibrations of the previous sections as promised, we consider a HKT geometry constructed using several maps τ with generic parameters $(p_1, p_2; a)$. Such a HKT geometry is independent from the parameterization of the maps τ and depends only on the arrangements of quaternionic lines $\tau^{-1}(0)$ in $\mathbb{Q}^2$. To see this, observe that two maps τ and τ' give the same HKT structure if their parameters are related as

$$(p_1', p_2'; a') = (up_1, up_2; ua)$$

for some $u \in \mathbb{Q}$, $u \neq 0$. So the inequivalent HKT structures associated with each map τ are parameterized by the bundle space of the canonical quaternionic line bundle over the Grassmannian $\mathrm{Gr}(1; \mathbb{Q}^2)$. In turn the quaternionic lines defined by $\mathrm{Ker}\, d\tau$ are in $\mathrm{Gr}(1; \mathbb{Q}^2)$ which is precisely the contact set of last calibration in the table given in the previous section. Observe that the calibration form and the HKT connection ∇^+ are compatible with the same hypercomplex structure $\mathbf{J}$.

The HKT geometries that we are considering are complete provided that the subspaces $\tau^{-1}(0)$ are in general position. Near the intersection of two such subspaces, the HKT metric is isometric to that of $(\mathbb{Q} - \{0\}) \times (\mathbb{Q} - \{0\})$, where the metric on $\mathbb{Q} - \{0\}$ is given as in 6.1.

Finally, the above HKT geometries can be used to construct new solutions of IIA supergravity as

$$\begin{aligned} ds^2(M) &= ds^2(\mathbb{R}^{(1,1)}) + ds^2(K) \\ e^{2\phi} &= (\det\gamma)^{\frac{1}{4}} \ . \end{aligned}$$

The IIA supergravity three-form field strength H is given is terms of the torsion of the HKT manifold K. The brane interpretation of such solution is that of NS-5-branes intersecting on a string. The NS-5-brane associated with the map τ is located at $\tau^{-1}(0)$.

7. Special Cases

As we have seen for a generic choice of maps τ the HKT geometries in eight dimensions found in the previous section were associated with quaternionic lines in $\mathbb{Q}^2$ given by $\mathrm{Ker}\, d\tau$. This establishes a correspondence between HKT geometries and quaternionic calibrations with calibration form $\phi_{\mathbb{J}}$. This correspondence can be extended to the rest of the quaternionic calibrations. For this, instead of considering generic maps τ with parameters $(p_1, p_2; a)$ to construct the HKT geometries, we restrict them in an appropriate way. There are four cases to consider, including the HKT geometry on K that has been mentioned above, as illustrated in the following table:

<u>HKT Geometries in Eight Dimensions</u>

$\bar{p}_1 p_2$	Ker $d\tau$
$\mathbb{R}$	S^1
$\mathbb{C}$	S^2
Im$\mathbb{Q}$	S^3
$\mathbb{Q}$	S^4

In the first column we denote the restriction on the parameters of the map and in the second column the set that ker $d\tau$ lies as we vary the map τ in the same class. Comparing the above table with that which contains the contact sets of quaternionic calibrations in section five, we observe that ker $d\tau$ lies in a contact set in all four cases.

The holonomy groups of the connections $\nabla^{\pm}$ in each of the above cases are given in the following table:

$\bar{p}_1 p_2$:	$\mathbb{R}$	$\mathbb{C}$	Im$\mathbb{Q}$	$\mathbb{Q}$
∇^-:	$Sp(2)$	$SU(4)$	$Spin(7)$	$SO(8)$
∇^+:	$Sp(2)$	$Sp(2)$	$Sp(2)$	$Sp(2)$

The holonomy of ∇^+ is in $Sp(2)$ in all cases because it is compatible with the $\mathbf{J}$ hypercomplex structure. Now if $\bar{p}_1 p_2$ is real for all τ involved in the construction of HKT geometry, then $d\tau$ commutes with the hypercomplex structures of K and $\mathbb{Q} - \{0\}$ induced by *left* quaternionic multiplication. This leads to another HKT structure on M compatible with the ∇^- connection. So the holonomy of ∇^- is in $Sp(2)$ as well. This HKT geometry was found in [36] and a special case in [37]. If $\bar{p}_1 p_2$ is complex, say, with respect to the I_1 complex structure, then $d\tau$ commutes with the complex structures of K and $\mathbb{Q} - \{0\}$ induced by left quaternionic multiplication with the quaternionic unit i. This makes the ∇^- connection compatible with the I_1 complex structure which implies that the holonomy of ∇^- is in $U(4)$. In fact it turns out that the holonomy of ∇^- is in $SU(4)$. Observe that the Kähler form of I_1 appears in the construction of the calibration form in this case. A similar analysis can be done for the remaining case.

Some of the above HKT geometries can be related to toric hyper-Kähler geometries [23, 36]. In particular, the HKT geometries associated with maps τ such that $\bar{p}_1 p_2 \in \mathbb{R}$ are T-dual (mirror symmetry) to toric hyper-Kähler manifolds [22]. In this case mirror symmetry transforms manifolds of one class, HKT manifolds, to manifolds of another class, hyper-Kähler manifolds. This is because the T-duality above is performed as many times as the number of tri-holomorphic vector fields of the toric hyper-Kähler manifold which is *less* than the middle dimension of the manifold. This is unlike the case of mirror symmetry for Calabi-Yau spaces where T-duality is performed in as many directions as the middle dimension of the manifold [38]. The HKT geometries with $\bar{p}_1 p_2 \in \mathbb{R}$ also appear in the context moduli spaces of a class of black holes in five dimensions [22].

It is of interest to ask the question whether it is possible to construct supergravity solutions which have the interpretation of intersecting branes using other calibrations from those employed above. To be more specific instead of the quaternionic calibrations, one may also use Kähler or special Lagrangian calibrations to do the superposition. Unfortunately, in both these cases, a superposition similar to that employed for quaternionic calibrations does not lead to solutions of supergravity field equations. This may due to the fact that the resulting geometries depend on the particular parameterization of the maps τ. On the other hand string perturbation theory considerations seem to suggest that superpositions of the kind employed above lead to BPS brane configurations [39, 40, 41]. However, it is not known how to construct in a systematic way the corresponding supergravity solution from a BPS brane configuration of string theory.

8. Concluding Remarks

The understanding of the non-perturbative properties of string theory requires the investigation of various solitons. In the low energy approximation these solitons have an interpretation as branes or as intersecting branes and are solutions of various supergravity theories. A class of such solutions was presented and their construction was related to quaternionic calibrations.

The problem of finding the intersecting brane solutions of supergravity theories has not been tackled in complete generality. Although many examples of such solutions are known, there does not seem to be a systematic way to find a solution for each BPS brane configuration of string theory. The resolution of this will require a better understanding of the supercovariant derivative of supergravity theories. The method of calibrations that I have presented led to the construction of a large class of these solutions but it has limitations some of which has already mentioned. However, the solutions that we are seeking for which the form field strengths do not vanish ($F \neq 0$) are in the same universality class as hyper-Kähler, Calabi-Yau and other special holonomy manifolds as far as the holonomy of the supercovariant connections of the supergravity theories is concerned. So it may be that powerful methods of algebraic and differential geometry that have been developed to construct examples of the latter may be extended to find examples of the former.

After the end of the conference, the moduli space of a class of five-dimensional black holes was determined and it was found to be a weak HKT manifold [33]. This result was further generalized in [34] to a larger class of four- and five-dimensional black holes. The moduli spaces of all five-dimensional black holes that admit at least four parallel spinors are HKT manifolds.

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