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$Sp(1)^n$-INVARIANT QUATERNIONIC KÄHLER METRIC

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We study $Sp(1)^n$-invariant hyperKähler or quaternionic Kähler manifolds of real dimension $4n$. In the case of $n = 1$, Hitchin classified these kinds of metrics associated with special functions. They are written as

$$g = dt^2 + \sum_{i=1}^{3} f_i(t)\sigma_i^2 \quad on \quad \mathbb{R} \times Sp(1),$$

where σ_1, σ_2, σ_3 are canonical 1-forms associated with $i, j, k \in \mathfrak{sp}(1)$. We obtain a generalization of the Hitchin's result ([2]).

Theorem 0.1. Let $\mathbb{H}$ be the Hamilton's quaternion number field $\mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$. Then $\mathbb{H}^n$ has a natural quaternionic structure I, J, K induced by the action of i, j, k. Since $\mathbb{H}\setminus\{0\}$ is diffeomorphic to $\mathbb{R} \times Sp(1)$ canonically, $(\mathbb{H}\setminus\{0\})^n$ is diffeomorphic to $\mathbb{R}^n \times (Sp(1))^n$. We denote the coordinate of $\mathbb{R}^n$ by $(t_1, t_2, \ldots, t_n)$. Let a Riemannian metric g be written as

$$g = \sum_{i=1}^{n}(dt_i^2 + \sum_{j=1}^{3} f_{ij}(t_1, t_2, \ldots, t_n)\sigma_{ij}^2),$$

where σ_{i1}, σ_{i2}, σ_{i3} are canonical 1-forms associated with $i, j, k \in \mathfrak{sp}(1)$. Then we obtain the following:
(i) If g is hyperKählerian with respect to the quaternionic structure I, J, K, then each $f_{ij}(t_1, t_2, \ldots, t_n)$ depend only on t_i. Hence the Riemannian metric is an n-times product of hyperKähler metric obtained by Hitchin.
(ii) If g is quaternionic Kählerian with respect to the quaternionic structure $\mathbb{R} + \mathbb{R}I + \mathbb{R}J + \mathbb{R}K$, then g is hyperKählerian.

By Hitchin, the coefficient functions f_{ij} satisfy

$$\begin{cases} \frac{d}{dt_i} f_{i1} = 2f_{i2}f_{i3}, \\ \frac{d}{dt_i} f_{i2} = 2f_{i3}f_{i1}, \\ \frac{d}{dt_i} f_{i3} = 2f_{i1}f_{i2}. \end{cases}$$

These equations imply the first integral

$$\begin{cases} f_{i1} - f_{i2} = a_i, \\ f_{i1} - f_{i3} = b_i, \end{cases}$$

where a_i, b_i are constant. Associated to $(a_i \neq 0, b_i \neq 0)$, $(a_i = 0, b_i \neq 0)$ and $(a_i = 0, b_i = 0)$, the metric is the type of Belinski-Gibbons-Page-Pope metric, Eguchi-Hanson metric and conformally flat metric.

One of our backgrounds is a natural metric on a moduli space of self-dual connections on $\mathbb{H}$. It coincides to a framed moduli space of self-dual connections on S^4. The quaternionic Kähler manifold $\mathbb{H}$ has an isometry $Sp(1) \cdot Sp(1)$, that acts on the framed moduli space $\mathcal{M}_k$ on a Hermitian vector bundle V of rank 2 with the second Chern class k.

$$\mathcal{M}_k = \{\nabla : self-dual \quad connection \quad on \quad V, c_2(V) = k\}/gauge \quad group.$$

The tangent space of $\mathcal{M}_k$ is represented as the first cohomology of the following elliptic complex:

$$0 \longrightarrow End(V) \xrightarrow{\nabla} End(V) \otimes T^*\mathbb{R}^4 \xrightarrow{pr_- \circ d^\nabla} End(V) \otimes \wedge_- \longrightarrow 0$$

where $\bigwedge^2 T^*\mathbb{R}^4$ is decomposed into the self-dual part $\bigwedge_+$ and the anti-self-dual part $\bigwedge_-$, $pr_- : \bigwedge^2 T^*\mathbb{R}^4 \longrightarrow \bigwedge_-$ is the natural projection. The tangent space of the moduli space is represented as a subset of $End(V)$-valued 1-forms. The L_2-metric of $End(V)$-valued 1-forms induces a Riemannian metric on the moduli space $\mathcal{M}_k$

$$\langle \alpha, \beta \rangle = \int_{\mathbb{R}^4} tr(\alpha \wedge \beta).$$

Furthermore the quaternionic structure I, J, K induces a hyperKählerian structure with respect to the Riemannian metric. It is known that the dimension of $\mathcal{M}_k$ is $8k$. These are represented as elements of

$$M_{k,k+1}(\mathbb{H}) = \{(A, B) | A \in M_{k,1}(\mathbb{H}), \quad B \in M_{k,k}(\mathbb{H})\}$$

by the A.D.H.M. construction. We denote

$$M^0_{k,k+1}(\mathbb{H}) = \{(A, B) | (A, B) \in M_{k,k+1}(\mathbb{H}), \quad tr(B) = 0\}.$$

It corresponds to a hyperKähler submanifold in $\mathcal{M}_k$, whose dimension is equal to $8k - 4$. We denote it by $\mathcal{M}^0_k$. The conformal group $(Sp(1) \times Sp(1))/\mathbb{Z}_2 \times \mathbb{R}^+ \times \mathbb{H}$ on $\mathbb{H}$ and the gauge group $Sp(1)/\mathbb{Z}_2$ at the infinity act on $\mathcal{M}^0_k$

i. $(q, p) \in (Sp(1) \times Sp(1))/\mathbb{Z}_2, \quad x \mapsto qxp^{-1} \quad (A, B) \mapsto (Ap, qBp)$,
ii. $\lambda \in \mathbb{R}^+, \quad x \mapsto \frac{1}{\lambda}x \quad (A, B) \mapsto (\lambda A, \lambda B)$,
iii. $a \in \mathbb{H}, \quad x \mapsto x - a \quad (A, B) \mapsto (A, B + aid)$,
iv. $r \in Sp(1)/\mathbb{Z}_2, \quad (A, B) \mapsto (rA, B)$.

We denote vector fields generated from the action i, ii by $V_1(\lambda)$, $V_2(a)$. Then the norms of $V_1(\lambda)$, $V_2(a)$ are constant on each orbit.

Proposition .

$$\|V_1(\lambda)\|^2 = \lambda^2 C_1$$

$$\|V_2(a)\|^2 = \sum_{i,j=0}^{3} C_{2ij} a_i a_j,$$

where C_1, C_2 are constant, $a = a_0 + ia_1 + ja_2 + ka_3$.

The $Sp(1) \times \mathbb{R}^+$ acts on $\mathcal{M}_k^0$. The reduced space $\mathbb{P}(\mathcal{M}_k^0)$ is known to be quaternionic Kählerian ([1]). These are not smooth manifolds, they have singularities. Now in the case $k = 2$, $\mathcal{M}_2^0$ and $\mathbb{P}(\mathcal{M}_2^0)$ are examples that are hyperKähler or quaternionic Kähler space of dimension $4n$ with $Sp(1)^n$-symmetry. In fact $\mathcal{M}_2^0$ is a hyperKähler space of dimension 3×4 with $(Sp(1) \times Sp(1))/\mathbb{Z}_2 \times Sp(1)/\mathbb{Z}_2$-symmetry and $\mathbb{P}(\mathcal{M}_2^0)$ is a quaternionic Kähler space of dimension 2×4 with $Sp(1)/\mathbb{Z}_2 \times Sp(1)/\mathbb{Z}_2$-symmetry.

References

[1] C. P. Boyer and B. M. Mann, *The hyperKähler geometry of the ADHM construction and quaternionic geometric invariant theory*, Proceedings of symposia in pure Math., 54(1993)Part 2, 45-83.

[2] N. J. Hitchin, *Twistor spaces, Einstein metrics and isomonodromic deformations*, J. Differential Geometry, 42(1995)no.1, 30-111.