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ON SPECIAL 4-PLANAR MAPPINGS OF ALMOST HERMITIAN QUATERNIONIC SPACES

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ABSTRACT. In the paper special 4-planar mappings of almost Hermitian quaternionic spaces are studied. Fundamental equations of these mappings are expressed in linear Cauchy form. Our results improve results of I.N. Kurbatova [9].

4-quasiplanar mappings of an almost quaternionic space have been studied in [5], [9] and [14]. These mappings generalize the geodesic, quasigeodesic and holomorphically projective mappings of Riemannian and Kählerian spaces, see [4], [12], [13], [15], [17], [18], [19]. Similar problems are studied on complex manifolds in [3]. Anti-quaternionic spaces which were studied e.g. in [11], [16] have some properties similar to those of quaternions [1]. This fact can be used in the study of 4-planar mappings.

I.N. Kurbatova studied a special kind of 4-planar mappings (called 4-quasiplanar, see [9]) from a Riemannian space V_n onto another Riemannian space $\bar{V}_n$ where an almost quaternionic structure on V_n is Hermitian and it satisfies additional conditions so that V_n a $\bar{V}_n$ are Apt spaces.

Analyzing the results of [9] (theorems 2 – 6) we noticed that the space $\bar{V}_n$ is implicitly supposed to be Hermitian and this assumption is essential. Hermitian structure of $\bar{V}_n$ is more important than Hermitian structure of V_n and, moreover, it simplifies fundamental equations of 4-planar mappings. In this paper we do not assume V_n to be Hermitian.

1. A well-known definition says that an *almost quaternionic* space is a differentiable manifold M_n with almost complex structures $\overset{1}{F}$ and $\overset{2}{F}$ satisfying

$$\overset{1}{F}{}^h_\alpha \overset{1}{F}{}^\alpha_i = -\delta^h_i; \quad \overset{2}{F}{}^h_\alpha \overset{2}{F}{}^\alpha_i = -\delta^h_i; \quad \overset{1}{F}{}^h_\alpha \overset{2}{F}{}^\alpha_i + \overset{2}{F}{}^h_\alpha \overset{1}{F}{}^\alpha_i = 0, \tag{1}$$

where δ^h_i is the Kronecker symbol, see e.g. [1], [4].

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The tensor $\overset{3}{F}{}_i^h \equiv \overset{1}{F}{}_i^\alpha \overset{2}{F}{}_\alpha^h$ defines an almost complex structure, too. The relations among the tensors $\overset{1}{F}, \overset{2}{F}, \overset{3}{F}$ are the following

$$\overset{1}{F}{}_i^h = \overset{2}{F}{}_i^\alpha \overset{3}{F}{}_\alpha^h = -\overset{3}{F}{}_i^\alpha \overset{2}{F}{}_\alpha^h;\ \overset{2}{F}{}_i^h = \overset{3}{F}{}_i^\alpha \overset{1}{F}{}_\alpha^h = -\overset{1}{F}{}_i^\alpha \overset{3}{F}{}_\alpha^h;\ \overset{3}{F}{}_i^h = \overset{1}{F}{}_i^\alpha \overset{2}{F}{}_\alpha^h = -\overset{2}{F}{}_i^\alpha \overset{1}{F}{}_\alpha^h. \tag{2}$$

Any two of the above three structures $\overset{1}{F}, \overset{2}{F}, \overset{3}{F}$ define the same almost quaternionic structure.

Let $A_n \equiv (M_n, \Gamma, \overset{1}{F}, \overset{2}{F}, \overset{3}{F})$ be an almost quaternionic space with a torsion-free affine connection Γ.

Definition 1 A curve ℓ: $x^h = x^h(t)$ in A_n is called *4-planar* if the tangent vector $\lambda^h = dx^h/dt$ being parallely transported along this curve, remains in the linear 4-dimensional space generated by the tangent vector λ^h and the corresponding vectors $\overset{1}{F}{}_\alpha^h \lambda^\alpha$, $\overset{2}{F}{}_\alpha^h \lambda^\alpha$ and $\overset{3}{F}{}_\alpha^h \lambda^\alpha$.

A curve is 4-planar if and only if the equations

$$\frac{d\lambda^h}{dt} + \Gamma^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \sum_{s=0}^{3} \underset{s}{\rho}\ \overset{s}{F}{}_\alpha^h \lambda^\alpha$$

hold, where $\overset{0}{F}{}_i^h \equiv \delta_i^h$ is the Kronecker symbol, $\Gamma^h_{\alpha\beta}$ are components of the affine connection on A_n and $\underset{s}{\rho} = \underset{s}{\rho}(t)$ $(s = 0, \ldots, 3)$ denote functions of the parameter t.

Any geodesic curve is a special case of a 4-planar curve where $\rho_1 \equiv \rho_2 \equiv \rho_3 \equiv 0$.

Consider two spaces A_n and $\bar{A}_n$ with the same underlying manifold M_n and the same almost quaternionic structure $(\overset{1}{F}, \overset{2}{F}, \overset{3}{F})$ but with two different torsion-free affine connection Γ and $\overline{\Gamma}$, respectively.

Definition 2 A diffeomorphism f: $A_n \to \bar{A}_n$ is called a *4-planar mapping*, if it maps any geodesic of A_n to a 4-planar curve of $\bar{A}_n$.

Remark. In the following we shall attach to each local map φ around a point $p \in A_n$ the local map $\varphi \circ f^{-1}$ around the point $f(p) \in \bar{A}_n$. This means that any point $x \in A_n$ and the corresponding point $f(x) \in \bar{A}_n$ will have the same local coordinates.

The following theorem holds [5]:

Theorem 1. *A diffeomorphism of A_n onto $\bar{A}_n$ is a 4-planar mapping if and only if in every local coordinate system $x = (x^1, x^2, \ldots, x^n)$ the conditions*

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \sum_{s=0}^{3} \underset{s}{\psi}_{(i}\ \overset{s}{F}{}^h_{j)} \tag{3}$$

hold, where Γ^h_{ij} *and* $\overline{\Gamma}^h_{ij}$ *are components of the affine connections* Γ *and* $\overline{\Gamma}$, *respectively,* $\underset{s}{\psi}{}_i(x)$, $s = 0, \dots, 3$, *are covectors, and* $(i\,j)$ *denotes a symmetrization of indices.*

Using Theorem 1 one can prove the all 4-planar curves of A_n are mapped onto 4-planar curves of $\bar{A}_n$ (I.N. Kurbatova [9] defined *4-quasiplanar mappings preserving almost-quaternionic structure* by the conditions (3)).

Finally, we will consider a special case of $\bar{A}_n$, namely an almost quaternionic Riemannian space $\bar{V}_n \equiv (M_n, \bar{g}, \overset{1}{F}, \overset{2}{F}, \overset{3}{F})$ in which $\overline{\Gamma}$ denote the Levi-Civita connection of $\bar{g}$.

The following theorem holds (see [5]).

Theorem 2. *A diffeomorphism* $f\colon A_n \to \bar{V}_n$ *is a 4-planar mapping if and only if the metric tensor* $\bar{g}_{ij}(x)$ *satisfies the following equations:*

$$\bar{g}_{ij,k} = \sum_{s=0}^{3} \left(\underset{s}{\psi}{}_k\, \bar{g}_{\alpha(i}\, \overset{s}{F}{}^{\alpha}_{j)} + \underset{s}{\psi}{}_{(i}\, \bar{g}_{j)\alpha}\, \overset{s}{F}{}^{\alpha}_{k} \right) \tag{4}$$

where comma denotes the covariant derivative in A_n.

Recall that the covariant derivative of $\bar{g}$ in $\bar{A}_n$ is zero.

The proof follows from the fact that formulas (3) and (4) are equivalent in our special case.

2. Now we shall prove the following two lemmas.

Consider the spaces A_n, $\bar{A}_n$ and let "," or "|" before an index denote a covariant derivative w.r. to the corresponding local variable on A_n and $\bar{V}_n$, respectively.

Lemma 1. *Let a 4-planar mapping* $A_n \to \bar{A}_n$ *be given and let* $\underset{s}{\psi}{}_i$ *denote the corresponding covectors from (3). Then*

$$\overset{s}{F}{}^{\alpha}_{i,\alpha} = \overset{s}{F}{}^{\alpha}_{i|\alpha} \qquad s = 1, 2, 3. \tag{5}$$

holds if and only if the covectors $\underset{s}{\psi}{}_i$ *are expressed by formulas*

$$\underset{s}{\psi}{}_i = -\frac{n}{n-4} \psi_\alpha\, \overset{s}{F}{}^{\alpha}_{i}, \qquad s = 1, 2, 3, \quad \psi_i \equiv \underset{0}{\psi}{}_i. \tag{6}$$

The proof of the above Lemma 1 is a consequence of (5) and fundamental equations of 4-planar mappings (3). We use also algebraic properties of quaternionic structures (1) and (2).

A manifold with an affine connection Γ and an almost complex structure F is said to be an *Apt space* (see [2], [4], [9], or *nearly Kählerian space* or *Tachibana space* [4], [6],

[7], [8], [10], [20]) if its structure F satisfies $F^{\alpha}_{i,\alpha} = 0$. A space $A_n = (M_n, \Gamma, \overset{1}{F}, \overset{2}{F}, \overset{3}{F})$ to be an *almost quaternionic Apt space* if

$$\overset{s}{F}{}^{\alpha}_{i,\alpha} = 0, \qquad s = 1, 2, 3.$$

Lemma 1 implies that an Apt spaces A_n is 4-planarly mapped on an Apt space $\bar{A}_n$ iff (6) holds. Evidently Kählerian spaces are Apt spaces and also quaternionic Kählerian spaces are Apt spaces.

Contracting (3) with respect to h and j we got the lemma

Lemma 2. *If for a 4-planar mapping $A_n \to \bar{A}_n$ the formulae (6) hold and the spaces A_n and $\bar{A}_n$ are equiaffine, then the vector ψ_i is a gradient, i.e. there exists a function ψ such that $\psi_i = \psi_{,i}$.*

3. Now we shall show that if a 4-planar mapping from A_n onto a Riemannian space $\bar{V}_n$ is given, then the formulae (3) and (4) are both equivalent to the following formula:

$$\bar{g}^{ij}{}_{,k} = -\sum_{s=0}^{3} \left(\underset{s}{\psi}{}_k \, \bar{g}^{\alpha(i} \overset{s}{F}{}^{j)}_{\alpha} + \underset{s}{\psi}{}_{\alpha} \, \bar{g}^{\alpha(i} \overset{s}{F}{}^{j)}_{k} \right) \tag{7}$$

where $\bar{g}^{ij}$ is the inverse matrix of metric tensor $\bar{g}_{ij}$. In fact, (7) is a consequence of the identity $\bar{g}^{ij}{}_{,k} = -\bar{g}_{\alpha\beta,k} \bar{g}^{\alpha i} \bar{g}^{\beta j}$.

In what follows we shall asumme a quaternionic structure on $\bar{V}_n$ which is Hermitian, i.e. we have

$$\bar{g}_{i\alpha} \overset{s}{F}{}^{\alpha}_{j} + \bar{g}_{j\alpha} \overset{s}{F}{}^{\alpha}_{i} = 0, \qquad s = 1, 2, 3. \tag{8}$$

(8) is equivalent with

$$\bar{g}^{i\alpha} \overset{s}{F}{}^{j}_{\alpha} + \bar{g}^{j\alpha} \overset{s}{F}{}^{i}_{\alpha} = 0, \qquad s = 1, 2, 3, \tag{9}$$

or with

$$\bar{g}^{\alpha\beta} \overset{s}{F}{}^{i}_{\alpha} \overset{s}{F}{}^{j}_{\alpha} = \bar{g}^{ij}, \qquad s = 1, 2, 3. \tag{10}$$

Using (9) the equations of 4-planar mappings are simplified to

$$\bar{g}^{ij}{}_{,k} = -2\psi_k \, \bar{g}^{ij} - \sum_{s=0}^{3} \underset{s}{\psi}{}_{\alpha} \, \bar{g}^{\alpha(i} \overset{s}{F}{}^{j)}_{k} \tag{11}$$

Suppose now that the covector ψ_i is a gradient, i.e. $\psi_i \equiv \underset{0}{\psi}{}_i \equiv \psi_{,i}$ where ψ is a function. We define the tensor

$$a^{ij} \equiv e^{2\psi} \bar{g}^{ij}.$$

Then (11) can we rewritten in the form

$$a^{ij}{}_{,k} = \sum_{s=0}^{3} \underset{s}{\lambda}{}^{(i} \overset{s}{F}{}^{j)}_{k}, \tag{12}$$

where

$$\underset{s}{\lambda}{}^{i} \equiv -\underset{s}{\psi}{}_{\alpha}\bar{g}^{\alpha i}\,. \tag{13}$$

By the definition of the tensor a^{ij} (10) is equivalent with

$$a^{\alpha\beta}\,\overset{s}{F}{}^i_\alpha\,\overset{s}{F}{}^j_\alpha = a^{ij}, \qquad s=1,2,3. \tag{14}$$

Due to the fact that $\bar{V}_n$ is Hermitian and using (13) we see that the formula (6) is equivalent with

$$\underset{s}{\lambda}{}^i = \frac{n}{n-4}\lambda^\alpha\,\overset{s}{F}{}^i_\alpha\,, \qquad s=1,2,3, \qquad \lambda^i \equiv \underset{0}{\lambda}{}^i. \tag{15}$$

Now we come back to the affine case. Let a space A_n be given as before and let the system of equations (12), (14) and (15) has a solution for a regular matrix function a^{ij} and a vector function λ^i. Then one can prove that the inverse matrix $||\tilde{g}_{ij}|| = ||a^{ij}||^{-1}$ defines a Riemannian metric $\tilde{g}$ on M_n and the covector $\lambda^\alpha\tilde{g}_{\alpha i}$ is a gradient gradψ. By the conformal change $\bar{g}_{ij} = e^{2\psi}\,\tilde{g}_{ij}$ we obtain a new metric $\bar{g}$ for which A_n becomes a Hermitian almost quaternionic space $\bar{V}_n$. Moreover, there exists a 4-planar mapping $A_n \to \bar{V}_n$.

This results coincides with the result by N.S. Sinyukov for geodetic mappings and the results by V.V. Domashev and J. Mikeš for holomorphically projective mappings of Kählerian spaces etc., see [12], [13], [18], [19]. Now we can conclude the above results with

Theorem 3. *Under the condition (5) an equiaffine space A_n admits a 4-planar mapping on a Hermitian quaternionic space $\bar{V}_n$ if and only if there exists a regular tensor a^{ij} on A_n satisfying (12), (14), and (15).*

The result analogous to Theorem 3 was proved by I.N. Kurbatova [9] under the assumption that A_n is Hermitian and from the proof it is evident that also $\bar{V}_n$ is supposed the be Hermitian.

4. Analysing the equation of I.N. Kurbatova [9] analogous to (12) we can modify this equation as a system of linear differential equations of Cauchy type. In what follows we give more simple modification which uses also conditions (14).

We consider covariant derivatives of (14) in A_n, i.e.

$$a^{\alpha\beta}_{\ ,k}\,\overset{r}{F}{}^i_\alpha\,\overset{r}{F}{}^j_\beta + a^{\alpha\beta}\,\overset{r}{F}{}^i_{\alpha,k}\,\overset{r}{F}{}^j_\beta + a^{\alpha\beta}\,\overset{r}{F}{}^i_\alpha\,\overset{r}{F}{}^j_{\beta,k} = a^{ij}_{\ ,k}\,, \qquad r=1,2,3.$$

Putting (12) into the above equation we get

$$\sum_{s=0}^{3}\left(\underset{s}{\lambda}{}^{(i}\,\overset{s}{F}{}^{j)}_k - \underset{s}{\lambda}{}^{\alpha}\,\overset{r}{F}{}^{(i}_\alpha\,\overset{r}{F}{}^{j)}_\beta\,\overset{s}{F}{}^\beta_k\right) = a^{\alpha\beta}\,\overset{r}{F}{}^{(i}_{\alpha,k}\,\overset{r}{F}{}^{j)}_\beta\,. \tag{16}$$

For $r = 1$, using (1), (2) and (15) we have

$$\lambda^{(i}\delta_k^{j)} - \lambda^\alpha \overset{1}{F}{}_\alpha^{(i} \overset{1}{F}{}_k^{j)} = \frac{n-4}{4}\, a^{\alpha\beta} \overset{1}{F}{}_{\alpha,k}^{(i} \overset{1}{F}{}_\beta^{j)} \tag{17}$$

and contracting (17) with respect to j and k we have the following expression of the vector λ^i:

$$\lambda^i = \frac{n-4}{n(2n+1)}\, a^{\alpha\beta} \left(\overset{1}{F}{}_{\alpha,\gamma}^{i} \overset{1}{F}{}_\beta^{\gamma} + \overset{1}{F}{}_\alpha^{i} \overset{1}{F}{}_{\beta,\gamma}^{\gamma} \right) . \tag{18}$$

It implies that λ^i can be expressed as a linear functions in a^{ij}. It implies

Theorem 4. *Under the condition (5) an equiaffine space A_n admits a 4-planar mapping onto a Hermitian almost quaternionic space $\bar{V}_n$ if and only if the following system of differential equations of Cauchy type is solvable with respect to the unknown functions a^{ij}:*

$$a^{ij}{}_{,k} = \sum_{s=0}^{3} \underset{s}{\lambda}{}^{(i} \overset{s}{F}{}_k^{j)} , \tag{19}$$

where

$$\underset{s}{\lambda}{}^{i} = \frac{n}{n-4} \lambda^\alpha \overset{s}{F}{}_\alpha^{i} , \quad s = 1,2,3, \quad \lambda^i = \frac{n-4}{n(2n+1)}\, a^{\alpha\beta} \left(\overset{1}{F}{}_{\alpha,\gamma}^{i} \overset{1}{F}{}_\beta^{\gamma} + \overset{1}{F}{}_\alpha^{i} \overset{1}{F}{}_{\beta,\gamma}^{\gamma} \right)$$

and the matrix (a^{ij}) should satisfying addition $|a^{ij}| \neq 0$ and the algebraic condition

$$a^{\alpha\beta} \overset{s}{F}{}_\alpha^{i} \overset{s}{F}{}_\alpha^{j} = a^{ij}, \qquad s = 1,2,3.$$

The system (19) does not have more than one solution for the initial Cauchy conditions $a^{ij}(x_o) = \underset{o}{a}{}^{ij}$ under the conditions (20). Therefore the general solution of (19) does not depend on more than $N_o = (n/2)^2$ parameters. The question of existence of a solution of (19) leads to the studium of integrability conditions, which are linear equations w.r. to the unknowns $a^{ij}(x)$ with coefficients from the space A_n.

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