Second Meeting on
Quaternionic Structures
in Mathematics and Physics
Roma, 6-10 September 1999

AN INTRODUCTION TO PSEUDOTWISTORS BASIC CONSTRUCTIONS

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R. Penrose has observed in 1976 [11] that the points of the Minkowski space-time can be represented by two-dimensional linear subspaces of a complex four-dimensional vector space on which an hermitian form of signature $(++,--)$ is defined. He called this *flat twistor space*, and the method of investigating deformation of complex structures, yielded from there, the *twistor programme*. This initiated a series of papers and monographs by various authors. In the present research we are dealing with dynamical systems generated by the Hermitian Hurwitz pairs of the signature $(\sigma, s), \sigma + s = 5 + 4\mu, |\sigma + 1 - s| = 2 + 4m; \mu, m = 0, 1, \ldots$ In particular, for (3,2) and its dual (1,4) the role of entropy is indicated as well as the relationship between Hurwitz and Penrose twistors; *Hurwitz twistors* being objects introduced by us. The signatures (1,8) and (7,6) give rise for introducing *pseudotwistors* and *bitwistors*, respectively; for pseudotwistors we can prove [9] a counterpart of the original fundamental Penrose theorem in the local version (on real analytic solutions of the related spinor equations vs. harmonic forms) and in the semi-global version (on holomorphic solutions of those equations vs. Dolbeault cohomology groups). This has to be preceded by basic constructions (which is the core topic of this paper), a study of the related pseudotwistors and spinor equations as well as complex structures on spinors. This will allow us to prove a theorem (which we call the atomization theorem) saying that there exist complex structures on isometric embeddings for the Hermitian Hurwitz pairs concerned so that the embeddings are real parts of holomorphic mappings.

1. Introduction

In 1985 Ławrynowicz and Rembieliński [5] initiated a geometrization of the problem of A. Hurwitz concerned with the composition of quadratic forms and a geometrical study of the related differential operators of the Cauchy-Riemann-, Dirac- and Fueter-types by introducing the so-called Hurwitz pairs, also in the hyperbolic case [17] discussed by Penrose [11]. The results obtained quickly appeared in text-books [12]

Research of the first author partially supported by the State Committee for Scientific Research (KBN) grant PB 2 P03A 010 17 (Section 2 of the paper), and partially by the grants of the University of Łódź no.505/626 and 252 (Sections 1 and 3).

and enabled [7, 8] to formulate and prove *counterparts of two Penrose's fundamental theorems* within the theory of Hurwitz pairs.

Consider the Hurwitz pair consisting of the Hermitian space $\mathbb{C}^4(\kappa) := (\mathbb{C}^4, \kappa)$, equipped with the metric

$$\kappa \equiv I_{2,2} := \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the real space $\mathbb{R}^5(\eta) := (\mathbb{R}^5, \eta)$, equipped with the metric

$$\eta \equiv I_{2,3} := \begin{pmatrix} I_2 & 0 \\ 0 & -I_3 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $(e_1, \ldots, e_4)$ be the canonical basis of $\mathbb{C}^4(\kappa)$. We consider a pair

$$H = (\mathbb{C}^4(\kappa), \mathbb{R}^5(\eta)).$$

If there exists a bilinear mapping $\circ : \mathbb{R}^5(\eta) \times \mathbb{C}^4(\kappa) \to \mathbb{C}^4(\kappa)$ called *multiplication* of elements of $\mathbb{R}^5(\eta)$ by elements of $\mathbb{C}^4(\kappa)$ such that, for $f \in \mathbb{C}^4(\kappa)$ and $\alpha \in \mathbb{R}^5(\eta)$, we have

$$\langle a, a\rangle_\eta \langle\langle f, f\rangle\rangle_\kappa = \langle\langle a \circ f, a \circ f\rangle\rangle_\kappa,$$

where

$$\langle\langle f, g\rangle\rangle_\kappa := f^* \kappa g,\ \ f, g \in \mathbb{C}^n;\ \langle a, b\rangle_\eta := a^T \eta b,\ \ a, b \in \mathbb{R}^p$$

and $*$ denotes the hermitian conjugation, and, moreover, H is *irreducible*, i.e. there exists no subspace V of $\mathbb{C}^4$, $V \neq \{0\}$, $\mathbb{C}^4$, such that $\circ|\mathbb{R}^5(\eta) \times V : \mathbb{R}^5(\eta) \times V \to V$, then H is called an *Hermitian Hurwitz pair* (cf. e.g. [7, 8]). This is of course a particular case of the general definition.

Further, let

$$\epsilon_\alpha \circ e_k = C^1_{\alpha k} e_1 + \cdots + C^n_{\alpha k} e_n,$$

where $(\epsilon_1, \ldots, \epsilon_5)$ is the canonical basis of $\mathbb{R}^5(\eta)$ and let $C_\alpha = (C^j_{\alpha k})$, $\alpha = 1, \ldots, 5$. We define the algebra $\mathcal{A}_{2,3}$ which is generated by $\{C^\#_\alpha C_\beta : \alpha \leq \beta\}$, where $C^\#_\alpha = \kappa C^*_\alpha \kappa^{-1}$. An element $x \in \mathcal{A}_{2,3}$ is called *Hurwitz twistor* [7, 8] whenever x has the form

$$x = \sum_{\alpha<\beta} \xi_{\alpha\beta} C^\#_\alpha C_\beta, \quad \xi_{\alpha\beta} \in \mathbb{C} \tag{1.1}$$

and $\operatorname{im} x^2 = 0$, where $\operatorname{im} x, x \in \mathcal{A}_{2,3}$ is defined in the following manner: $x \in \mathcal{A}_{2,3}$ can be written uniquely as

$$x = \sum_{k=0}^{4} x_k, \quad x_k = \sum_{\alpha_1<\beta_1<\cdots<\alpha_k<\beta_k} \xi_{\alpha_1\beta_1\ldots\alpha_k\beta_k} C^\#_{\alpha_1} C_{\beta_1} \ldots C^\#_{\alpha_k} C_{\beta_k},$$

with $x_0 = \xi_0 I_4$. We define im $x := x - x_0$ and denote the collection of Hurwitz twistors by $P^1 = \mathcal{J}_H$:

$$\mathcal{J}_H := \{x = \sum_{\alpha<\beta} \xi_{\alpha\beta} C_\alpha^{\#} C_\beta : \text{ im } x^2 = 0\}.$$

2. Dynamical systems generated by the Hermitian Hurwitz pairs of signatures $(3,2)$ and $(1,4)$

Following [1] we are looking in our case for a dynamical system $(X, \tilde{\mu}, T)$ in the sense of ergodic theory. Here X is a measure space, $\tilde{\mu}$ is a measure on the space and T an invertible, measurable map $X \to X$ that preserves $\tilde{\mu}$, i.e., for any measurable set $A \subset X$ we have $\tilde{\mu}[A] = \tilde{\mu} \circ T^{-1}[A]$, and $\tilde{\mu}[X] = 1$. Of course it is natural to take

$$X = \mathbb{C}^4,\ \tilde{\mu} = \langle\langle\ ,\ \rangle\rangle_{I_{2,2}} \quad \text{resp.} \quad X = \mathbb{C}^{16},\ \tilde{\mu} = \langle\langle\ ,\ \rangle\rangle_{I_{8,8}}.$$

If ξ is a finite partitioning of X in measurable sets $C_1, \ldots, C_{N(\xi)}$, i.e.,

$$C_j \subset X, \quad C_j \cap C_k = \emptyset,\ \text{for } j \neq k,\ \text{and } C_1 \cup \cdots \cup C_{N(\xi)} = X,$$

the *entropy* $H_{\tilde{\mu}}(\xi)$ of the partition ξ is the quantity

$$\begin{array}{l} H_{\tilde{\mu}}(\xi) := -\Sigma_{j=1}^{N(\xi)} \tilde{\mu}(C_j) \log_2 \tilde{\mu}(C_j), \\ \text{where } \ \tilde{\mu}(C_j) \log_2 \tilde{\mu}(C_j) = 0 \text{ whenever } \mu(C_j) = 0. \end{array}$$

If $\xi = |C_j|$, $1 \leq j \leq N(\xi)$, and $\tilde{\xi} = \{C_k\}$, $1 \leq k \leq N(\tilde{\xi})$ are two finite partitions of X, we shall denote by $\xi \vee \tilde{\xi}$ the partition of X into $C_j \cap C_k$, where the indices j and k run independently from 1 to $N(\xi)$ and from 1 to $N(\tilde{\xi})$, respectively. For an arbitrary partition $\xi = |C_j|$, $1 \leq j \leq N(\xi)$, we denote by $T^{-1}\xi$ the partition of X into the sets $T^{-1}C_1, \ldots, T^{-1}C_{N(\xi)}$. For all positive integers n we form $\xi^n = T^{-0}\xi \vee \cdots \vee T^{-n+1}\xi$ and consider the limit of $H_{\tilde{\mu}}(\xi^n)/n$ as $n \to \infty$. The limit exists and is called the *entropy* of the partition ξ *for unit time.* We denote it by $b_{\tilde{\mu}}(T|X;\xi)$. For the mapping $T : X \to X$ of the dynamical system $(X, \tilde{\mu}, T)$, the *metric entropy* in the sense of Ja. G. Sinaĭ is the quantity $b_{\tilde{\mu}}(T|X) := \sup_\xi b_{\tilde{\mu}}(T|X;\xi)$, where the upper bound is taken over all finite partitions of X.

In particular, we may take as $b_{\tilde{\mu}}(T|X)$ the entropy in the physical sense (cf., e.g. E. Fermi's book [2]); treating it as a stochastic instant τ [10, 4], analogous to a time instant t, when considering a relativistic particle at $(x, y, z) \in \mathbb{R}^3$ within the spaces $\mathbb{R}^5(I_{1,4})$ and $\mathbb{R}^5(I_{3,2})$ of space-time elements

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 - d\tau^2 \text{ and } - ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 - d\tau^2,$$

where c is a positive constant interpreted as the light velocity. Already in the Newtonian dynamics an additional dimension corresponding to time is needed because of the necessity of introducing an inertical frame and absolute time in connection with the Newtonian laws of dynamics. This means that, in contrast to the Aristotelian physics, the space $\mathbb{E}^3 = \{(x, y, z) \in \mathbb{R}^3, ds^2 = dx^2 + dy^2 + dz^2\}$ together with time $t \in \mathbb{T} = \mathbb{R}$ are no more absolute: there is a projection mapping $\pi : \mathbb{E}^3 \times \mathbb{T} \to \mathbb{T}$ which associates to any element $\tilde{p} \in \mathbb{E}^3 \times \mathbb{R}$ the corresponding instant of time $t = \pi(\tilde{p})$; $\mathbb{T}$ is called the base space. The inverse image of t, $\pi^{-1}(\tilde{p})$ is called a fibre. Each fibre is isomorphic to the Euclidean space $\mathbb{E}^3$, which is therefore called a typical fibre. Such a triple $(\mathbb{E}^3 \times \mathbb{T}, \mathbb{T}, \pi)$ with π being a surjective projection map is called a bundle with the base space $\mathbb{T}$ and bundle space $\mathbb{E}^3 \times \mathbb{T}$. The bundle approach is very convenient and naturally extendable to higher dimensions and curved spaces [12].

In the case of the space-time elements in question, the usual variational procedure with respect to the action integral leads to discussion of the Lorentz transformation

$$\begin{aligned} dx =& \rho_1(dx' + v_t^x dt' + v_\tau^x d\tau'), \quad dy = dy', \quad dz = dz', \\ dt =& \rho_2(v_x^t dx' + dt' + v_\tau^t d\tau'), \quad \rho_1, \rho_2, \rho_3 \in \mathbb{R}^+, \\ d\tau =& \rho_3(v_x^\tau dx' + v_t^\tau dt' + d\tau'), \quad v_t^x \text{ etc. } \in \mathbb{R}, \end{aligned}$$

satisfying the condition $A^T \rho A = \rho$, where A is the matrix of the transformation and ρ the metric in question. Hence [10]:

$$\rho_1 = [1 - (\frac{1}{c} v_t^x)^2 \pm (v_\tau^x)^2]^{-1/2}, \quad \rho_2 = [1 - (\frac{1}{c} v_x^t)^2 \mp (c v_\tau^t)^2]^{-1/2},$$
$$\rho_3 = [1 \pm (v_x^\tau)^2 \mp (\frac{1}{c} v_t^\tau)^2]^{-1/2}$$

with various restrictions for v_t^x etc. The corresponding Euler-Lagrange equations, interpreted as the equations of motion, include the stochastic force (cf. [10, 3]):

$$\vec{F} = \pm m v_t^\tau \nabla_{\vec{r}} v_t^\tau, \quad \vec{r} = (x, y, z)$$

(m denoting the mass), being now an intrinsic part of the geometry. More generally, if we consider a one-parameter family of symplectic transformations and the parameter value 0 corresponds to the identity transformation, a Hamiltonian operator is defined as the derivative of the transformations of the family with respect to the parameter (at 0). By differentiating the condition for symplecticity of a transformation, we may find the condition for an operator H to be Hamiltonian: $\omega(H\mathbf{x}, \mathbf{y}) + \omega(\mathbf{x}, H\mathbf{y}) = 0$ for all $\mathbf{x}, \mathbf{y}$ belonging to a symplectic space in question endowed with a skew-scalar product ω; the Hamiltonian is supposed to be related in a standard way with the Lagrangian density appearing in the action integral.

3. Basic constructions for the Hurwitz pairs $(\mathbb{C}^{16}(I_{8,8}), \mathbb{R}^9(I_{\sigma,s}))$, $\sigma + s = 9$

In this section, we recall basic constructions of Hurwitz algebras and give generators of the Hurwitz algebras $\mathcal{H}_{\sigma,s}$, $\sigma + s = 9$. They are counterparts of $\mathcal{A}_{2,3}$. Generally, under a *Hurwitz algebra* we understand a central Clifford algebra whose generators S_α satisfy the condition $S_\alpha^\# = S_\alpha$.

The basic constructions are methods of giving explicit forms for generators of Hurwitz algebras. These constructions involve three different methods. (I) $\mathcal{H}_{\sigma,0} \implies \mathcal{H}_{\sigma+2,0}$, $\sigma \equiv 1 \pmod 2$. Let $S_1, S_2, \ldots S_\sigma$ be generators of $\mathcal{H}_{\sigma,0}$. Then

$$\widetilde{S}_1 = \begin{pmatrix} S_1 & 0 \\ 0 & -S_1 \end{pmatrix}, \ \widetilde{S}_2 = \begin{pmatrix} S_2 & 0 \\ 0 & -S_2 \end{pmatrix}, \dots, \ \widetilde{S}_\sigma = \begin{pmatrix} S_\sigma & 0 \\ 0 & -S_\sigma \end{pmatrix},$$
$$\widetilde{S}_{\sigma+1} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \ \widetilde{S}_{\sigma+2} = \begin{pmatrix} 0 & iI_n \\ -iI_n & 0 \end{pmatrix}, \quad n = 2^{[\frac{1}{2}\sigma - 1]}$$

become generators of $\mathcal{H}_{\sigma+2,0}$.

(II) $\mathcal{H}_{\sigma,0} \implies \mathcal{H}_{\sigma,2}, \quad \sigma \equiv 1 \bmod (2)$. Let $S_1, \dots, S_\sigma$ be generators of $\mathcal{H}_{\sigma,0}$. Then

$$\widetilde{S}_1 = \begin{pmatrix} S_1 & 0 \\ 0 & -S_1 \end{pmatrix}, \dots, \ \widetilde{S}_\sigma = \begin{pmatrix} S_\sigma & 0 \\ 0 & -S_\sigma \end{pmatrix},$$
$$\widetilde{S}_{\sigma+1} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \ \widetilde{S}_{\sigma+2} = \begin{pmatrix} 0 & iI_n \\ iI_n & 0 \end{pmatrix}, \ n = 2^{[\frac{1}{2}\sigma - 1]}$$

become generators of $\mathcal{H}_{\sigma,2}$. (III) $\mathcal{H}_{\sigma,s} \implies \mathcal{H}_{\sigma,s+2}, \quad \sigma + s \equiv 1 \bmod (2), \quad s > 0$. Let $S_1, \dots, S_{\sigma+s}$ be generators of $\mathcal{H}_{\sigma,s}$ of the form

$$S_\alpha = \begin{pmatrix} A_\alpha & iB_\alpha \\ iB_\alpha^* & -D_\alpha \end{pmatrix} \qquad \alpha = 1, 2, \dots, \sigma + s,$$

$$A_\alpha^* = A_\alpha, \qquad D_\alpha^* = D_\alpha,$$

$$A_\alpha, B_\alpha, D_\alpha \in M_n(\mathbb{C}) \qquad n = 2^{[\frac{1}{2}(\sigma+s) - \frac{1}{2}]}.$$

Then the generators of $\mathcal{H}_{\sigma,s+2}$ are given by

$$\widetilde{S}_\alpha = \begin{pmatrix} A_\alpha & 0 & 0 & iB_\alpha \\ 0 & D_\alpha & iB_\alpha^* & 0 \\ 0 & iB_\alpha & -A_\alpha & 0 \\ iB_\alpha^* & 0 & 0 & -D_\alpha \end{pmatrix}, \quad \alpha = 1, 2, \dots, \sigma + s,$$

$$\widetilde{S}_{\sigma+s+1} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad \widetilde{S}_{\sigma+s+2} = \begin{pmatrix} 0 & iI_{\frac{1}{2}n} \otimes \sigma_3 \\ iI_{\frac{1}{2}n} \otimes \sigma_3 & 0 \end{pmatrix},$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Applying the above construction methods, we can find generators of the Hurwitz algebras of $(\mathbb{C}^4(I_{2.2}), \mathbb{R}^5(I_{\sigma,s}))$, $\sigma + s = 5$, and $(\mathbb{C}^{16}(I_{8,8}), \mathbb{R}^9(I_{\sigma,s}))$, $\sigma + s = 9$. At first we notice that

$$\begin{array}{ccc} (\mathbb{C}^n(I_{\frac{1}{2}n,\frac{1}{2}n}), & & \mathbb{R}^{\sigma+s}(I_{\sigma,s})) \\ \swarrow & & \searrow \\ \mathcal{H}_{\sigma-1,s} & & \mathcal{H}_{s-1,\sigma} \end{array}$$

which implies that a Hurwitz pair gives rise to two Hurwitz algebras $\mathcal{H}_{\sigma-1,s}$ and $\mathcal{H}_{s-1,\sigma}$. Explicitly, in the case of $\sigma+s=5$,

for $\sigma=1$	and $s=4$	we get	$\mathcal{H}_{0,4},\ \mathcal{H}_{3,1}$
2	3		$\mathcal{H}_{1,3},\ \mathcal{H}_{2,2}$
3	2		$\mathcal{H}_{2,2},\ \mathcal{H}_{1,3}$
4	1		$\mathcal{H}_{3,1},\ \mathcal{H}_{0,4}$

In the case of $\sigma+s=9$,

for $\sigma=1$	and $s=8$	we get	$\mathcal{H}_{0,8},\ \mathcal{H}_{7,1}$
2	7		$\mathcal{H}_{1,7},\ \mathcal{H}_{6,2}$
3	6		$\mathcal{H}_{2,6},\ \mathcal{H}_{5,3}$
4	5		$\mathcal{H}_{3,5},\ \mathcal{H}_{4,4}$
5	4		$\mathcal{H}_{4,4},\ \mathcal{H}_{3,5}$
6	3		$\mathcal{H}_{5,3},\ \mathcal{H}_{2,6}$
7	2		$\mathcal{H}_{6,2},\ \mathcal{H}_{1,7}$
8	1		$\mathcal{H}_{7,1},\ \mathcal{H}_{0,8}$

We give the generators of Hurwitz algebras involved explicitly as well as the type of the basic construction. We take the Pauli matrices as follows:

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

(IV) $\mathcal{H}_{\sigma,s}$ with $\sigma+s=3$ and $\sigma+s=5$.

For $\mathcal{H}_{3,0}$ the generators can be chosen as $S_\alpha=\sigma_\alpha$, $\alpha=1,2,3$. Subsequently, we take

$$\text{for } \mathcal{H}_{3,2}\quad \begin{array}{l} S_1=\begin{pmatrix}\sigma_1&0\\0&-\sigma_1\end{pmatrix},\quad S_2=\begin{pmatrix}\sigma_2&0\\0&-\sigma_2\end{pmatrix},\quad S_3=\begin{pmatrix}\sigma_3&0\\0&-\sigma_3\end{pmatrix},\\ \qquad S_4=\begin{pmatrix}0&I_2\\-I_2&0\end{pmatrix},\quad S_5=\begin{pmatrix}0&iI_2\\iI_2&0\end{pmatrix}\end{array}$$

with the type of basing construction $\mathcal{H}_{3,0}\overset{\text{(II)}}{\to}\Longrightarrow\mathcal{H}_{3,2}$;

for $\mathcal{H}_{1,2}$ $\quad S_1=i\sigma_1,\ S_2=i\sigma_2,\ S_3=\sigma_3$;

$$\text{for } \mathcal{H}_{1,4}\quad \begin{array}{l} S_1=\begin{pmatrix}0&i\sigma_1\\i\sigma_1&0\end{pmatrix},\quad S_2=\begin{pmatrix}0&i\sigma_2\\i\sigma_2&0\end{pmatrix},\quad S_3=\begin{pmatrix}I_2&0\\0&-I_2\end{pmatrix},\\ S_4=\begin{pmatrix}0&I_2\\-I_2&0\end{pmatrix},\quad S_5=\begin{pmatrix}0&i\sigma_3\\i\sigma_3&0\end{pmatrix}\end{array}$$

with the type of basing construction $\mathcal{H}_{1,2}\overset{\text{(III)}}{\to}\Longrightarrow\mathcal{H}_{1,4}$; (V) $\mathcal{H}_{\sigma,s}$ with $\sigma+s=7$. Let

$$\mathrm{diag}\,(A,B) := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \text{ etc. and } \mathrm{diag}^*(A,B) := \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} \text{ etc.,}$$

where A, B, etc. denote square matrices. For $\mathcal{H}_{1,6}$ the generators can be chosen as

$$S_\alpha = \mathrm{diag}^*(i\sigma_\alpha, i\sigma_\alpha, i\sigma_\alpha, i\sigma_\alpha),\ \alpha = 1,2,3,$$

$$\begin{array}{l} S_4 = \ \mathrm{diag}^*(-I_2, I_2, -I_2, I_2), \\ S_7 = \mathrm{diag}(\sigma_3, -\sigma_3, -\sigma_3, \sigma_3), \end{array} \quad S_5 = \begin{pmatrix} 0 & I_4 \\ -I_4 & 0 \end{pmatrix}, \quad S_6 = \begin{pmatrix} 0 & 0 & iI_2 & 0 \\ 0 & 0 & 0 & -iI_2 \\ iI_2 & 0 & 0 & 0 \\ 0 & -iI_2 & 0 & 0 \end{pmatrix}$$

with the type of construction $\mathcal{H}_{1,2} \overset{(\mathrm{III})}{\to} \Longrightarrow \mathcal{H}_{1,4} \overset{(\mathrm{III})}{\to} \Longrightarrow \mathcal{H}_{1,6} \overset{(\mathrm{III})}{\to} \Longrightarrow \mathcal{H}_{1,8}$;

$$\text{for } \mathcal{H}_{5,2} \quad \begin{array}{ll} S_\alpha = \mathrm{diag}(\sigma_\alpha, -\sigma_\alpha, -\sigma_\alpha, \sigma_\alpha), \quad \alpha = 1,2,3, & \\ S_4 = \begin{pmatrix} 0 & I_2 & 0 & 0 \\ I_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_2 \\ 0 & 0 & -I_2 & 0 \end{pmatrix}, & S_5 = \begin{pmatrix} 0 & iI_2 & 0 & 0 \\ -iI_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -iI_2 \\ 0 & 0 & iI_2 & 0 \end{pmatrix}, \\ S_6 = \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix}, & S_7 = \begin{pmatrix} 0 & -iI_4 \\ -iI_4 & 0 \end{pmatrix} \end{array}$$

with the type of construction $\mathcal{H}_{3,0} \overset{(\mathrm{I})}{\to} \Longrightarrow \mathcal{H}_{5,0} \overset{(\mathrm{II})}{\to} \Longrightarrow \mathcal{H}_{5,2}$;

$$\text{for } \mathcal{H}_{3,4} \quad \begin{array}{l} S_\alpha = \mathrm{diag}(\sigma_\alpha, \sigma_\alpha, -\sigma_\alpha, -\sigma_\alpha), \quad \alpha = 1,2,3, \\ S_4 = \mathrm{diag}^*(-I_2, I_2, -I_2, I_2), \quad S_6 \text{ resp. } S_7 \text{ as } S_5 \text{ resp. } S_6 \text{ for } \mathcal{H}_{1,6}, \\ S_5 = \mathrm{diag}^*(iI_2, iI_2, iI_2, iI_2) \end{array}$$

with the type of construction $\mathcal{H}_{3,0} \overset{(\mathrm{II})}{\to} \Longrightarrow \mathcal{H}_{3,2} \overset{(\mathrm{II})}{\to} \Longrightarrow \mathcal{H}_{3,4}$;

$$\text{for } \mathcal{H}_{7,0} \quad \begin{array}{l} S_\alpha = \mathrm{diag}(\sigma_\alpha, -\sigma_\alpha, \sigma_\alpha, -\sigma_\alpha), \quad \alpha = 1,2,3, \\ S_4,\ S_5 \text{ as for } \mathcal{H}_{1,6}, \quad S_6 = \begin{pmatrix} 0 & iI_4 \\ iI_4 & 0 \end{pmatrix}, \quad S_7 = \begin{pmatrix} 0 & I_4 \\ -I_4 & 0 \end{pmatrix} \end{array}$$

with the type of construction $\mathcal{H}_{3,0} \overset{(\mathrm{I})}{\to} \Longrightarrow \mathcal{H}_{5,0} \overset{(\mathrm{I})}{\to} \Longrightarrow \mathcal{H}_{7,0}$.

4. Pseudotwistors related to Hermitian Hurwitz pairs

In this section we define the pseudotwistors for the Hermitian Hurwitz pairs

$$(\mathbb{C}^{16}(I_{8,8}), \quad \mathbb{R}^9(I_{\sigma,s})),$$

$\sigma + s = 9$, and discuss the duality of them and the Penrose diagrams. These are the counterparts of $(\mathbb{C}^4(I_{2,2},\ \mathbb{R}^5(I_{2,3}))$ in [7, 8].

Let $(\mathbb{C}^{16}(I_{8,8}),\ \mathbb{R}^9(I_{\sigma,s})),\ \sigma + s = 9$ be one of the Hurwitz pairs and let C_α, $\alpha = 1, 2, \ldots, 9$, be the corresponding Hurwitz matrices. Then we have the Hurwitz algebras:

$$\mathcal{A} = \bigoplus\nolimits_{k=0}^{4} \mathcal{A}_{2k},$$
$$\mathcal{A}_{2k} = \{\sum\nolimits_{1 \leqq \alpha_1 < \beta_1 < \cdots < \alpha_k < \beta_k \leqq 9} \xi_{\alpha_1 < \beta_1 \ldots \alpha_k \beta_k} C^{\#}_{\alpha_1} C_{\beta_1} \ldots C^{\#}_{\alpha_k} C_{\beta_k}\}.$$

Definition. An element $\xi \in \mathcal{A}$ is called a *pseudotwistor* of $\mathcal{A}$, if $\operatorname{im} \xi^2 = 0$. If $\xi \in \mathcal{A}_{2k}$, it is said to be of *degree* k. Here we denote the non-scalar part of ξ^2 is denoted by $\operatorname{im} \xi^2$.

Example 1. Scalar elements are pseudotwistors.

Example 2. Monomials

$$C^{\#}_{\alpha_1} C_{\beta_1} \ldots C^{\#}_{\alpha_k} C_{\beta_k}, \quad \alpha_1 < \beta_1 < \alpha_2 < \cdots < \beta_k,$$

are pseudotwistors. Hence we see that any element of $\mathcal{A}$ can be written as a linear combination of pseudotwistors.

Example 3. There are elements of $\mathcal{A}$ which are not pseudotwistors, e.g, $\xi = C^{\#}_1 C_2 + C^{\#}_3 C_4$. We note that $\operatorname{im} \xi^2 = 2C^{\#}_1 C_2 C^{\#}_3 C_4$.

Now we introduce an analogue of Wick's theorem in the fermionic algebras to Hurwitz algebras. We describe it with the use of a simple example: For

$$\xi_1 = C^{\#}_1 C_3 + C^{\#}_1 C_5, \quad \xi_2 = C^{\#}_2 C_5,$$

we have

$$\xi_1 \xi_2 = -C^{\#}_1 C_2 C^{\#}_3 C_5 + \eta_{55} C^{\#}_1 C_5.$$

Because of contractions, we have a term of lower degrees. We denote this term by

$$C^{\#}_1 C_2 = \eta_{55} C^{\#}_1 \underline{C_5 C^{\#}_2 C_5},$$

where underlining indicates the way of contraction. The first term is called *the normal* product of ξ_1 and ξ_2 and is denoted by

$$: \xi_1 \xi_2 := -C^{\#}_1 C_2 C^{\#}_3 C_5.$$

Hence we can write

$$\xi_1 \xi_2 =: \xi_1 \xi_2 : + : \overline{\xi_1 \xi_2} :$$

where $\overline{}$ denotes contraction. In general, for $\xi_1, \xi_2 \in \mathcal{A}$, we have

$$\xi_1 \xi_2 =: \xi_1 \xi_2 : + : \overline{\xi_1 \xi_2} : + : \overline{\xi_1 \xi_2}^{(2)} : + \ldots,$$

where $\overline{\xi_1 \xi_2}^{(2)} := \overline{\overline{\xi_1 \xi_2}}$ etc., i.e. $\overline{}^{(\ell)}$ denotes the ℓ-time contraction.

Example 4. For $\xi_1 = C^{\#}_1 C_3 C^{\#}_5 C_7, \quad \xi_2 = C^{\#}_4 C_8 + C^{\#}_5 C_8 + C^{\#}_1 C_3$, we get

$$\xi_1 \xi_2 =: \xi_1 \xi_2 : + : \overline{\xi_1 \xi_2} : + : \overline{\overline{\xi_1 \xi_2}} :,$$
$$: \xi_1 \xi_2 := C^{\#}_1 C_3 C^{\#}_5 C_7 \cdot C^{\#}_4 C_8,$$

$$: \overline{\xi_1 \xi}_2 := -C_1^{\#} C_3 C_7^{\#} C_8,$$

$$: \overline{\overline{\xi_1 \xi}}_2 := C_5^{\#} C_7.$$

We have the following

Lemma 4.1. An element $\xi \in \mathcal{A}$ is a pseudotwistor if and only if

$$: \xi\ \xi := 0, : \overline{\xi\ \xi} := 0, \ldots, \overline{\xi\ \xi}^{(k-1)} := 0.$$

We are going to prove

Theorem 4.2. The pseudotwistor space of degree k

$$\mathcal{J}^{(k)} = \{\xi \in \mathcal{A}^{(2k)} : \text{ im } \xi^2 = 0\}, \quad k = 1, 2, 3, 4,$$

has a decomposition

$$\mathcal{J}^{(k)} = \mathcal{J}_-^{(k)} + \mathcal{J}_+^{(k)}$$

and introduces a flag structure, i.e.,
(i) $\mathcal{J}_-^{(k)} \subset G(2k-1, 8)$ and $\mathcal{J}_+^{(k)} \subset G(2k, 8)$ and
(ii) $\xi = \xi_- + \xi_+$ implies $\xi_- \subset \xi_+$.

Proof. The proof is divided into five steps.

S t e p A. We are going to choose a special basis of the algebra $\mathcal{A}$ in question. We take

$$x_2 = C_1^{\#} C_2, \quad x_3 = C_1^{\#} C_3, \ \ldots, \ x_9 = C_1^{\#} C_9.$$

Then we arrive at the following basis $\{x_\alpha x_\beta : \ 2 \leqq \alpha \leqq \beta \leqq 9\}$ of $\mathcal{A}$:

$$x_2 x_3 = -\eta_{1,1} C_2^{\#} C_3, \quad x_2 x_4 = -\eta_{1,1} C_2^{\#} C_4, \quad \ldots, \quad x_8 x_9 = -\eta_{1,1} C_8^{\#} C_9,$$

where η is the metric of $\mathbb{R}^9(I_{\sigma,s})$. An element $x_k \in \mathcal{A}_{2k}, \ k = 1, 2, 3, 4,$ can be written as

$$x_1 = \sum_{\beta_1=2}^{9} \xi_{1\beta_1} x_{\beta_1} + \sum_{2 \leqq \alpha_1 < \beta_1} \xi_{\alpha_1 \beta_1} x_{\alpha_1} x_{\beta_1},$$

$$x_2 = \sum_{2 \leqq \beta_1 < \alpha_1 < \beta_2} \xi_{1\beta_1 \alpha_2 \beta_2} x_{\beta_1} x_{\alpha_2} x_{\beta_2} + \sum_{2 \leqq \alpha_1 < \beta_1 < \alpha_2 < \beta_2} \xi_{\alpha_1 \beta_1 \alpha_2 \beta_2} x_{\alpha_1} x_{\beta_1} x_{\alpha_2} x_{\beta_2},$$

$$\begin{aligned} x_3 = & \sum_{2 \leqq \beta_1 < \alpha_2 < \cdots < \beta_3} \xi_{1\beta_1 \alpha_2 \beta_2 \alpha_3 \beta_3} x_{\beta_1} x_{\alpha_2} x_{\beta_2} x_{\alpha_3} x_{\beta_4} \\ & + \sum_{2 \leqq \alpha_1 < \beta_1 < \cdots < \alpha_3 < \beta_3} \xi_{\alpha_1 \beta_1 \ldots \alpha_3 \beta_3} x_{\alpha_1} x_{\beta_1} x_{\alpha_2} x_{\beta_2} x_{\alpha_3} x_{\beta_3}, \end{aligned}$$

$$x_4 = \ \xi_{12345678} x_2 x_3 \ldots x_8 + \xi_{23\ldots9} x_2 x_3 \ldots x_9.$$

Let $x_k = x_1^{(k)} + x_2^{(k)}$, $k = 1, 2, 3, 4$. Since the degree k is indicated, we denote it simply by $x = x_1 + x_2$.

S t e p B. From the above formulae we calculate directly:

$$x^2 =: x_1 x_2 : + x_2 x_2 :, \quad k = 1,$$

where

$$: x_1 x_2 := \sum_{\beta_1, \alpha_1', \beta_1' \geq 2} \xi_{1\beta_1} \xi_{\alpha_1'\beta_1'} x_{\beta_1} x_{\alpha_1'} x_{\beta_1'} \quad (\text{as.} = \text{antisymmetric}),$$

$$: x_1 x_2 := \sum_{\alpha_1, \beta_1, \alpha_1', \beta_1' \geq 2} \xi_{\alpha_1\beta_1} \xi_{\alpha_1'\beta_1'} x_{\alpha_1} x_{\beta_1} x_{\alpha_1'} x_{\beta_1'} \quad (\text{as.});$$

$$\text{im } x^2 =: x_1\overline{x_1} : + : x_1\overline{\overline{x_2}} : + : x_2\overline{\overline{x_2}} : + : x_1 x_2 : + : x_2 x_2 :,$$

where

$$: x_1\overline{x_1} := \sum_{\alpha_2, \beta_1, \beta_2, \alpha_2', \beta_1', \beta_2' \geq 2} \underline{\xi_{1\beta_1\alpha_2\beta_2}\ \xi_{1\beta_1'\alpha_2'\beta_2'}} x_{\alpha_2} x_{\beta_2} x_{\alpha_2'} x_{\beta_2'} \quad (\text{as.}),$$

$$: x_1\overline{\overline{x_2}} := \sum_{\alpha_j, \alpha_j'; \beta_j, \beta_j' \geq 2} \underline{\underline{\xi_{1\beta_1\alpha_2\beta_2}}\ \xi_{\alpha_1'\beta_1'\alpha_2'\beta_2'}}\ x_{\beta_2} x_{\alpha_2'} x_{\beta_2'} \quad (\text{as.}),$$

$$: x_2\overline{\overline{x_2}} := \sum_{\alpha_j, \alpha_j'; \beta_j, \beta_j' \geq 2} \underline{\underline{\xi_{\alpha_1\beta_1\alpha_2\beta_2}}\xi_{\alpha_1'\beta_1'\alpha_2'\beta_2'}} x_{\alpha_2} x_{\beta_2} x_{\alpha_2'} x_{\beta_2'} \quad (\text{as.}),$$

$$: x_2 x_2 := \sum_{\alpha_j, \alpha_j'; \beta_j, \beta_j' \geq 2} \xi_{\alpha_1\beta_1\alpha_2\beta_2} \xi_{\alpha_1'\beta_1'\alpha_2'\beta_2'} x_{\alpha_1} x_{\beta_1} x_{\alpha_2} x_{\beta_2} x_{\alpha_1'} x_{\beta_1'} x_{\alpha_2'} x_{\beta_2'} \quad (\text{as.}),$$

$$: x_1 x_2 := \sum_{\alpha_j, \alpha_j'; \beta_j, \beta_j' \geq 2} \xi_{1\beta_1\alpha_2\beta_2} \xi_{\alpha_1'\beta_1'\alpha_2'\beta_2'} x_{\beta_1} x_{\alpha_2} x_{\beta_2} x_{\alpha_1'} x_{\beta_1'} x_{\alpha_2'} x_{\beta_2'} \quad (\text{as.});$$

$$\text{im } x^2 =: x_1\overline{x_1}^{(3)} : + : x_1\overline{x_2}^{(4)} : + : x_2\overline{x_2}^{(4)} :,$$

where

$$: x_1\overline{x_1}^{(3)} := \sum_{\alpha_j, \alpha_j'; \beta_j, \beta_j' \geq 2} \xi_{1\beta_1\underline{\alpha_2\beta_2\alpha_3\beta_3}} \xi_{\underline{1\beta_1'\alpha_2'}\beta_2'\alpha_3'\beta_3'} x_{\beta_1} x_{\alpha_2} x_{\beta_1'} x_{\alpha_2'} \quad (\text{as.}),$$

$$: x_1\overline{x_2}^{(4)} := \sum_{\alpha_j, \alpha_j'; \beta_j, \beta_j' \geq 2} \xi_{1\beta_1\underline{\alpha_2\beta_2\alpha_3\beta_3}} \xi_{\underline{\alpha_1'\beta_1'\alpha_2'\beta_2'}\alpha_3\beta_3'} x_{\beta_1} x_{\alpha_1'} x_{\beta_1'} \quad (\text{as.}),$$

$$: x_2\overline{x_2}^{(4)} := \sum_{\alpha_j, \alpha_j'; \beta_j, \beta_j' \geq 2} \xi_{\alpha_1\beta_1\underline{\alpha_2\beta_2\alpha_3\beta_3}} \xi_{\underline{\alpha_1'\beta_1'\alpha_2'\beta_2'}\alpha_3'\beta_3'} x_{\alpha_1} x_{\beta_1} x_{\alpha_1'} x_{\beta_1'} \quad (\text{as.});$$

$$\text{im } x^2 = 0, \qquad k = 4,$$

where

$$x = \xi_{12345678} x_2 x_3 \ldots x_8 + \xi_{23\ldots 9} x_2 x_3 \ldots x_9.$$

S t e p C. We observe that $\mathcal{J}^{(1)}$ determines a flag structure

$$M_1 = \{(L_1, L_2) : \ L_1 \subset L_2 \subset \mathbb{C}^8, \dim\ L_1 = 1, \dim\ L_2 = 2\}.$$

Indeed, let us write down the system of equations. We fix the indices $\beta_1, \alpha_1^{'}, \beta_1^{'}$ and write down : $x_1 x_2 := 0$. If we take $(2,3,4)$, for example, we obtain

$$\xi_{12}\xi_{34} - \xi_{13}\xi_{24} + \xi_{14}\xi_{23} = 0.$$

The second equation : $x_1 x_2 := 0$ for $(\alpha_1, \beta_1, \alpha_1^{'}, \beta_1^{'}) = (2,3,4,5)$ implies

$$\xi_{23}\xi_{45} - \xi_{24}\xi_{35} + \xi_{25}\xi_{34} = 0.$$

Hence we see that $(x_1)_{234} \subset (x_2)_{2345}$. From the same discussion in the general case, we conclude that

$$\xi_{-} \subset \xi_{+},$$

where $\xi_{-} = \sum_{\beta' \geq 2} \xi_{1\beta'} x_{\beta'}$, $\xi_{+} = \sum_{2 \leq \alpha_1' \beta'} \xi_{\alpha' \beta'} x_{\alpha'} x_{\beta'}$, which proves the assertion of Step C.

S t e p D. $\mathcal{J}^{(2)}$ determines a flag-structure

$$M_2 = \{(L_3, L_4) : \ L_3 \subset L_4 \subset \mathbb{C}^8, \dim \ L_3 = 3, \dim \ L_4 = 4\}.$$

Indeed, let

$$L_k(\alpha_1, \ldots, \beta_l) = L_k \cap \{\alpha_1 = \cdots = \beta_l = 0\}.$$

Then : $x_1 x_1 := 0$ implies that $L_3(\beta_1)$ determines a 2-dimensional subspace in $\mathbb{C}^8$ for a fixed β_1. Hence we have a family of 2-dimensional subspaces. By this we infer that : $x_1 x_1 := 0$ determines a 3-dimensional subspace L_3 of $\mathbb{C}^8$. Hence : $x_2 x_2 := 0$ is the system of Plücker relations for 4-dimensional subspaces, so we have $L_4 \subset \mathbb{C}^8$. From : $x_1 x_2^{(2)} := 0$, we conclude that

$$L_3(\beta_1, \alpha_2) \subset L_4(\alpha_1^{'}, \beta_1^{'}), \qquad \beta_1 = \alpha_1^{'}, \quad \alpha_2 = \beta_1^{'},$$

and hence $L_3 \subset L_4$. By this we obtain the desired correspondence.

S t e p E. A similar observation leads to the conclusion that $\mathcal{J}^{(3)}$ determines a flag-structure

$$M_3 = \{(L_5, L_6) : \ L_5 \subset L_6 \subset \mathbb{C}^8, \ \dim \ L_5 = 5, \ \dim \ L_6 = 6\}.$$

Conclusions of Steps A-E suffice to complete the proof of the theorem.

Remark. We have not used in full the conditions to be pseudotwistors. In fact, we shall see more subtle structures of pseudotwistors, which will lead us to the "quaternary analysis" for the Hermitian Hurwitz pairs [9].

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