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**A NOTE ON THE REDUCTION OF SASAKIAN MANIFOLDS**

GUEO GRANTCHAROV AND LIVIU ORNEA

**ABSTRACT.** This is a report on work in process. We show that the contact reduction can be specialized to Sasakian manifolds. We link this Sasakian reduction to Kähler reduction by considering the Kähler cone over a Sasakian manifold. Finally, we present an example of Sasakian manifold obtained by  $SU(2)$  reduction of a standard Sasakian sphere.

1. INTRODUCTION

Reduction technique was naturally extended from symplectic to contact structures by H. Geiges in [6]. Even earlier, Ch. Boyer, K. Galicki and B. Mann defined in [3] a moment map for 3-Sasakian manifolds, thus extending the reduction procedure for nested metric contact structures. Quite surprisingly, a reduction scheme for Sasakian manifolds (contact manifolds endowed with a compatible Riemannian metric satisfying a curvature condition), was still missing.

In this note - presenting work in progress - we fill the gap by defining a Sasakian moment map and constructing the associated reduced space. We then relate Sasakian reduction to Kähler reduction *via* the Kähler cone over a Sasakian manifold.

In a forthcoming paper we shall discuss the compatibility between the Einstein property and the reduction scheme.

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## 2. DEFINITIONS OF SASAKIAN MANIFOLDS

We recall here the notion of a Sasakian manifold, referring to [2] and [4] for details and examples.

**Definition 2.1.** *A Sasakian manifold is a  $(2n+1)$ -dimensional Riemannian manifold  $(N, g)$  endowed with a unitary Killing vector field  $\xi$  such that the curvature tensor of  $g$  satisfies the equation:*

$$(1) \quad R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi$$

where  $\eta$  is the metric dual 1-form of  $\xi$ :  $\eta(X) = g(\xi, X)$ .

Let  $\varphi = \nabla\xi$ , where  $\nabla$  is the Levi-Civita connection of  $g$ . The following formulae are then easily deduced:

$$(2) \quad \varphi\xi = 0, \quad g(\varphi Y, \varphi Z) = g(Y, Z) - \eta(Y)\eta(Z).$$

It can be seen that  $\eta$  is a contact form on  $N$ , whose Reeb field is  $\xi$  (it is also called the characteristic vector field). Moreover, the restriction of  $\varphi$  to the contact distribution  $\eta = 0$  is a complex structure.

The simplest example is the standard sphere  $S^{2n+1} \subset \mathbb{C}^{n+1}$ , with the metric induced by the flat one of  $\mathbb{C}^{n+1}$ . The characteristic Killing vector field is  $\xi_p = -i\vec{p}$ ,  $i$  being the imaginary unit. Other Sasakian structures on the sphere can be obtained by  $D$ -homothetic transformations (cf. [7]). Also, the unit sphere bundle of any space form is Sasakian. A large class of examples is obtained *via* the converse construction of the Boothby-Wang fibration. Moreover, the join of two Sasakian Einstein manifolds is Sasakian Einstein.

The following equivalent definition puts Sasakian geometry in the framework of holonomy groups. Let  $C(N) = N \times \mathbb{R}_+$  be the cone over  $(N, g)$ . Endow it with the warped-product cone metric  $C(g) = r^2g + dr^2$ . Let  $R_0 = r\partial r$  and define on  $C(N)$  the complex structure  $J$  acting like this (with obvious identifications):  $JY = \varphi Y - \eta(Y)R_0$ ,  $JR_0 = \xi$ . We have:

**Theorem 2.1.** [4]  *$(N, g, \xi)$  is Sasakian if and only if the cone over  $N$   $(C(N), C(g), J)$  is Kählerian.*

## 3. THE RESULTS

**Theorem 3.1.** *Let  $(N, g, \xi)$  be a compact  $2n+1$  dimensional Sasakian manifold and  $G$  a compact  $d$ -dimensional Lie group acting on  $N$  by contact isometries. Suppose  $0 \in \mathfrak{g}^*$  is a regular value of the associated moment map  $\mu$ . Then the reduced space  $M = N//G := \mu^{-1}(0)/G$  is a Sasakian manifold of dimension  $2(n-d)+1$ .*

*Proof.* (A sketch.) By [6], the contact moment map  $\mu : N \rightarrow \mathfrak{g}^*$  is defined by

$$\langle \mu(x), \underline{X} \rangle = \eta(X)$$

for any  $\underline{X} \in \mathfrak{g}$  and  $X$  the corresponding field on  $N$ . We know that the reduced space is a contact manifold, *loc. cit.* Hence we only need to check that (1) the Riemannian metric is projected on  $M$  and (2) the field  $\xi$  projects to a unitary Killing field on  $M$  such that the curvature tensor of the projected metric satisfies formula (1).

To this end, we first describe the metric geometry of the Riemannian submanifold  $\mu^{-1}(0)$ . One proves that the restriction of the Reeb field to  $\mu^{-1}(0)$  is Killing with respect to the induced metric. Moreover, using the Gauss equation we obtain

$$\begin{aligned}
 (3) \quad & g(R^{\mu^{-1}(0)}(X, \xi)Y, Z) - g(R^N(X, \xi)Y, Z) = \\
 & = - \sum_{i=1}^d \|X_i\|^{-2} \{h_i(X, Y)h_i(\xi, Z) - h_i(X, Z)h_i(\xi, Y)\} \\
 & = - \sum_{i=1}^d \|X_i\|^{-2} \{g(X_i, Z)g(\nabla_X^N X_i, \varphi Y) - g(X_i, Y)g(\nabla_X^N X_i, \varphi Z)\}
 \end{aligned}$$

where  $\{\underline{X}_1, \dots, \underline{X}_d\}$  is a basis of  $\mathfrak{g}$  and let  $\{X_1, \dots, X_d\}$  is the corresponding vector fields on  $N$ . (Note that  $\nu_i = \|X_i\|^{-1}\varphi X_{i_p}$  are chosen to be orthonormal in  $p$ ; this is always possible pointwise by appropriate choice of the initial  $\underline{X}_i$ ).

Let now  $\pi : \mu^{-1}(0) \rightarrow M$  and endow  $M$  with the projection  $g^M$  of the metric  $g$  such that  $\pi$  becomes a Riemannian submersion. This is possible because  $G$  acts by isometries. In this setting, the vector fields  $X_i$  span the vertical distribution of the submersion, whilst  $\xi$  is horizontal and projectable (because  $\mathcal{L}_{X_i}\xi = 0$ ). Denote with  $\zeta$  its projection on  $M$ .  $\zeta$  is obviously unitary. To prove that  $\zeta$  is Killing on  $M$ , we just observe that  $\mathcal{L}_\zeta g(Y, Z) = \mathcal{L}_\xi g(Y^h, Z^h)$ , where  $Y^h$  denotes the horizontal lift of  $Y$ . Finally, to compute the values  $R^M(X, \zeta)Y$  of the curvature tensor of  $g^M$ , we use O'Neill formula (cf. [1], (9.28f)) and find

$$R^M(X, \zeta)Y = g(\xi, Y^h)X^h - g(X^h, y^h)\xi = g^M(\zeta, Y)X - g^M(X, Y)\zeta$$

which proves that  $(M, g^M, \zeta)$  is a Sasakian manifold.  $\square$

**Remark 3.1.**  $\mu^{-1}(0)$  is a natural example of a contact CR-submanifold (in the terminology of K. Yano and M. Kon [9], a semi-invariant submanifold in the terminology of A. Bejancu). In general, this means that the tangent space of the submanifold decomposes in three mutually orthogonal distributions:  $\mathbb{R}\xi$ , a distribution  $\mathcal{D}$  on which  $\varphi$  restricts to an endomorphism and a distribution  $\mathcal{D}^\perp$  which is mapped by  $\varphi$  in the normal space of the submanifold. It is known that on a contact CR-submanifold the distribution  $\mathcal{D}^\perp$  is always integrable. Here the integrability of this distribution expresses the fact that it is generated by fundamental vector fields corresponding to a basis of the Lie algebra of the group defining the moment map. In general, the invariant distribution is not integrable. In our case, one can show that its integrability is equivalent with strong restrictions on the geometry of the quotient.

In the following we relate Sasakian reduction to Kähler reduction by using the cone construction. Roughly speaking, we prove that reduction and taking the cone are commuting operations.

Let  $\omega = dr^2 \wedge \eta + r^2 d\eta$  be the Kähler form of the cone  $C(N)$  over a Sasakian manifold  $(N, g, \xi)$ . If  $\rho_t$  are the translations acting on  $C(N)$  by  $(x, r) \mapsto (x, tr)$ , then the vector field  $R_0 = r\partial r$  is the one generated by  $\{\rho_t\}$ . Moreover, the following two relations are useful:

$$(4) \quad \mathcal{L}_{R_0}\omega = \omega, \quad \rho_t^*\omega = t\omega.$$

If a compact Lie group  $G$  acts on  $C(N)$  by holomorphic isometries, commuting with  $\rho_t$ , we obtain a corresponding action of  $G$  on  $N$ . In fact, we can consider  $G \cong G \times \{Id\}$  acting as  $(g, (x, r)) \times (gx, r)$ .

Suppose that a moment map  $\Phi : C(N) \rightarrow \mathfrak{g}$  exists.

As above notations, let  $\{\underline{X}_1, \dots, \underline{X}_d\}$  be a basis of  $\mathfrak{g}$  and let  $\{X_1, \dots, X_d\}$  be the corresponding vector fields on  $C(N)$ . We see that  $X_i$  are independent on  $r$ , hence can be considered as vector fields on  $N$ . Furthermore, the commutation of  $G$  with  $\rho_t$  implies

$$(5) \quad \Phi(\rho_t(p)) = t\Phi(p).$$

Now embed  $N$  in the cone as  $N \times \{1\}$  and let  $\mu := \Phi|_{N \times \{1\}}$ . We can prove that this is precisely the moment map of the action of  $G$  on  $N$ .

Let  $P = \Phi^{-1}(0)/G$  be the reduced Kähler manifold. The key remark is that because of (5),  $\Phi^{-1}(0)$  is the cone  $N' \times \mathbb{R}_+$  over  $N' = \{x \in N ; (x, 1) \in \Phi^{-1}(0)\}$ . Moreover, since the actions of  $G$  and  $\rho_t$  commute, one has an induced action of  $G$  on  $N'$ . Then

$$\Phi^{-1}(0)/G \cong (N' \times \mathbb{R}_+)/G \cong N'/G \times \mathbb{R}_+$$

The manifold  $N'//G \times \mathbb{R}_+$  is Kähler, as reduction of a Kähler manifold, but we still have to check that this Kähler structure is a cone one. For the more general, symplectic case, this was done in [5]. Let  $g_0$  be the reduced Kähler metric and  $g'$  be the Sasakian reduced metric on  $N'//G$ . It is easily seen that the lift of  $g_0$  to  $\Phi^{-1}(0)$  coincides with the lift of the cone metric  $r^2 g' + dr^2$  on horizontal fields. This implies that the cone metric coincides with  $g_0$ .

Summing up we have proved:

**Theorem 3.2.** *Let  $(N, g, \xi)$  be a Sasakian manifold and let  $(C(N), C(g), J)$  be the Kähler cone over it. Let a compact Lie group  $G$  act by holomorphic isometries on  $C(N)$  and commuting with the action of the 1-parameter group generated by the field  $R_0$ . If a moment map with regular value 0 exists for this action, then a moment map with regular value 0 exists also for the induced action of  $G$  on  $N$ . Moreover, the reduced space  $C(N)//G$  is the Kähler cone over the reduced Sasakian manifold  $N//G$ .*

The advantage of defining the Sasakian reduction *via* Kähler reduction, as done in [3] for 3-Sasakian manifolds, is the avoiding of curvature computations.

4. EXAMPLES:  $SU(2)$  ACTIONS ON SASAKIAN SPHERES

**Example 4.1.** In a future note we are planning to consider with details  $S^1$  actions on Sasakian manifolds but now we concentrate to the actions of  $SU(2)$  with homogeneous reduced spaces. Consider the standard Sasakian structure on  $S^{4n-1} \subset \mathbb{C}^{2n}$  given by the "round metric" and vector field  $\zeta$  generated by the left action of  $S^1 = e^{it}$ . Then the right action of the unit quaternions on  $S^{4n-1} \subset \mathbb{H}^n$  by:

$$(q, (q_0, \dots, q_{n-1})) \mapsto (q_0q, \dots, q_{n-1}q).$$

satisfies the conditions of Theorem 3.1. The associated moment map is the same as the 3-Sasakian moment map of the  $S^1$  action given in [3]:

$$\mu(q) = \sum q_a i \bar{q}_a$$

The reason is that in both cases the coordinates  $(\mu_1, \mu_2, \mu_3)$  of  $\mu$  are given by a scalar product of the vector fields generated by the left actions of  $i, j$  and  $k$  with  $\zeta$ . So using the result from [3] we have:

$$\mu^{-1}(0) \cong SU(n+1)/SU(n-1)$$

. The reduced space is diffeomorphic to the homogeneous space  $SU(n+1)/SU(2) \times SU(n-1)$  which is a  $S^1$  bundle over  $SU(n+1)/S(U(2) \times U(n-1))$ , a Hermitian symmetric space. Note also that the latter space is a quaternionic Kähler manifold and is the base for the 3-Sasakian fibration with  $S^3$  fiber, obtained as a reduced space after the 3-Sasakian reduction mentioned above. One can also check that the reduced metric is the homogeneous Einstein metric arising from the Wang and Ziller's construction, [8].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT, STORRS, CT 06268, U.S.A.,  
ON LEAVE FROM: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOFIA, 1124 SOFIA, BUL-  
GARIA.

*E-mail address:* `geogran@indigo.math.uconn.edu`

UNIVERSITY OF BUCHAREST, FACULTY OF MATHEMATICS, 14 ACADEMIEI STR.  
70109 BUCHAREST, ROMANIA

*E-mail address:* `lornea@imar.ro`