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SPENCER MANIFOLDS

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Abstract. Almost-complex and hyper-complex manifolds are considered in this paper from the point of view of complex analysis and potential theory. The idea of holomorphic coordinates on an almost-complex manifold $(M, \mathbf{J})$ is suggested by D. Spencer [Sp]. For hypercomplex manifolds we introduce the notion of hyper-holomorphic function and develop some analogous statements. Elliptic equations are developed in a different way than D. Spencer . In general here we describe only the formal aspect of the developed theory.

1. INTRODUCTION.

Differentiable manifolds are described locally by smooth real coordinates. This is typical in differential geometry. Complex-analytic manifolds are equipped locally by complex-analytic coordinates. This give rise to the possibility of applying the theory of holomorphic functions of many complex variables in the local geometry of complex-analytic manifolds. In the case of almost complex manifolds $(M, \mathbf{J})$ one use ordinary real coordinates $(x^1, ..., x^{2n})$. Here we shall consider complex self-conjugate coordinates $(z^1, ..., z^n, \bar{z}^1, ..., \bar{z}^n)$, where $z^k = x^{2k-1} + ix^{2k}$ and $\bar{z}^k = x^{2k-1} - ix^{2k}$. We denote by $\mathbf{J}^*$ the action of $\mathbf{J}$ on differential forms of M, i. e. by definition $(\mathbf{J}^*(\omega)X \overset{def}{=} \omega(\mathbf{J}X)$, where X is a vector field, and ω is a differential form on M. For a fixed index k, we say that z^k is a "holomorphic" coordinate if $\mathbf{J}^*dz^k = idz^k$ and $\mathbf{J*}d\bar{z}^k = -id\bar{z}^k$. For non-holomorphic coordinates z^q we have

$$\mathbf{J}^*dz^q = J^1_q dz^1 + ... + J^n_q dz^n + J^{n+1}_q d\bar{z}^{n+1} + ... + J^{2n}_q d\bar{z}^{2n}$$

In the case z^k is a holomorphic coordinate for each $k = 1, ..., n$, the almost complex structure $\mathbf{J}$ is an integrable one. The interest of the existence of holomorphic coordinates z^k when the index k takes not all values $1, ..., n$ is suggested by Donald Spencer [Sp].

By $\mathbb{H} = \mathbb{H}(\mathbf{1}, i, j, k), ij = k$, we will denote the 4-dimensional quaternionic vector space, i. e. $q \in \mathbb{H}$ means that $q = x^0 + ix^1 + jx^2 + kx^3$, where $x^0, x^1, x^2, x^3 \in \mathbb{R}$. We will use different complex number representation for quaternions q, namely $q = z + \zeta j$, where $z = x^0 + ix^1$and $\zeta = x^2 + ix^3$. So we obtain the right j-complex splitting of $\mathbb{H}$, denoted by $\mathbb{H}^j$, i. e. $\mathbb{H}^j = (\mathbb{R} \oplus i\,\mathbb{R}) \oplus (\mathbb{R} \oplus i\,\mathbb{R})j$. By $\mathbb{R} \oplus i\,\mathbb{R}$ is denoted the tensor product of $\mathbb{R}$ with itself under the basis $(1,0)$ and $(0,i)$. Identifying $\mathbb{R} \oplus i\,\mathbb{R}$ with $\mathbb{C}$ we have that $\mathbb{H}^j$ is isomorphic to $\mathbb{C} \times \mathbb{C}$. Analogously, we will consider the right i-complex splitting of $\mathbb{H}$, namely $\mathbb{H}^i = (\mathbb{R} \oplus j\,\mathbb{R}) \oplus (\mathbb{R} \ominus j\,\mathbb{R})i$, i.e. $q = x^0 + jx^3 + +(x^2 - jx^4)i$. $\mathbb{H}^i$ is isomorphic to $\mathbb{C} \times \mathbb{C}$ too.

By $\mathbb{H}^n$ is denoted the n-dimensional quaternionic vector space (real $4n$-dimensional)

$$\mathbb{H}^n = \{(q^1, ..., q^n) : q^\alpha \in \mathbb{H}, \alpha = 1, ..., n\}$$

According to the above accepted notation we have $q^\alpha = z^\alpha + \zeta^\alpha j$, $\alpha = 1, ..., n$ or

$$\mathbb{H}^n = \mathbb{C}^n + \mathbb{C}^n j\,,\ \ \mathbb{C}^n = \{z^1, ..., z^n : z^\alpha \in \mathbb{C}\}$$

This representation is with respect of the right j-complex splitting $\mathbb{H}^j$. A similar representation of $\mathbb{H}^n$ can be written with respect to the right i-complex splitting $\mathbb{H}^i : \mathbb{H}^i = \mathbb{C}^n + \bar{\mathbb{C}}^n i\,,\ \ \mathbb{C}^n = \mathbb{R}^n \oplus \mathbb{R}^n i\,,\ \bar{\mathbb{C}}^n = \mathbb{R}^n \ominus \mathbb{R}^n i$.

Let $(M, \mathbf{J}, \mathbf{K})$ be a hyper-complex manifold, $\mathbf{JK} + \mathbf{KJ} = 0$, $\dim M_{\mathbb{R}} = 4n$. A pair of complex coordinates (z, ζ) is called *hyper-holomorphic pair* if z is holomorphic with respect to the almost-complex manifold $(M, \mathbf{J})$ and ζ is holomorphic with respect to $(M, \mathbf{K})$.

2. Holomorphic coordinates

2.0.1. *Almost-holomorphic functions.* By definition a function $f : U \to \mathbf{C}$, where U is an open subset of M, is called *almost holomorphic* or almost complex if $\bar{\partial} f = 0$. The above definition can be reformulated in the following equivalent form:

$$f \textit{ is almost holomorphic iff } \mathbf{J}^* df = i df$$

Respectively, f is *almost-antiholomorphic iff* $\mathbf{J}^* df = -i df$. For the proof of the equivalence it is enough to take in view that the exterior derivative d is decomposed as $d = \partial + \bar{\partial}$ over the space of smooth functions on M. Another form of this definition is obtained taking the real and imaginary parts of f, i.e. $f = u + iv$. In view of $df = du + idv$ we receive $\mathbf{J}^* du + i\mathbf{J}^* dv = idu - dv$. This means that $\mathbf{J}^* du = -dv$ and $\mathbf{J}^* dv = du$. As the obtained two equations are not independent, we can state the following Cauchy- Riemann type form of the definition

$f = u + iv$ is almost-holomorphic iff $\mathbf{J}^* dv = du$ or equivalently $\mathbf{J}^* du = -dv$.

Respectively: *$f = u + iv$ is almost-anti holomorphic iff $\mathbf{J}^* dv = -du$ or equivalently $\mathbf{J}^* du = dv$*

Remark: For an almost complex manifold $(M, \mathbf{J})$ with non-integrable $\mathbf{J}$, the decomposition $d = \partial + \bar{\partial}$ is not valid over differential (p, q)-forms on $(M, \mathbf{J})$.

The following proposition is well-known:

Proposition 1. The almost complex structure $\mathbf{J}$ of the almost complex manifold $(M, \mathbf{J})$, $dim_{\mathbf{R}} M = 2n$, is an integrable almost complex structure if and only if for every point $p \in M$, there is a neighborhood U of p and almost holomorphic functions

$f_j : U \to \mathbb{C}$, $j = 1, ..., n$, which differentials at p, i.e. $d_p f_j$, $j = 1, ..., n$, are $\mathbb{C}$-linear independent.

Remark: Taking $(U; f_1, ..., f_n)$ as local coordinate system (as f_j are functionally independent on a neighborhood of p), we obtain a local complex-analytic coordinate system $(U; z_1, ..., z_n)$, where $z^k = f_k$.

3. Spencer coordinates

We say that a local Spencer coordinate system of type m is defined on an almost complex manifold $(M, \mathbf{J})$ if the following conditions hold:

1.) There exist an open subset U of M and m different functionally independent almost holomorphic functions $f_j : U \to \mathbb{C}$, $j = 1, ..., m$, such that

2.) The sequence $(f_1, ..., f_m)$ is a maximal sequence of functionally independent on U almost-holomorphic functions. 3.) The sequence

$$(U, w^1, ..., w^m, z^{m+1}, ..., z^n, \bar{w}^{n+1}, ..., \bar{w}^{n+m}, \bar{z}^{n+m+1}, ..., \bar{z}^{2n})$$

where $w^j = f_j$, $j = 1, ..., m$, determines a local self-conjugate system on $(M, \mathbf{J})$.

An almost complex manifold which is equipped with an atlas of local Spencer coordinate systems is by definition an almost-complex manifold of Spencer type m. It is to remark that the notion of Spencer type is correctly defined in the category of almost complex manifolds. This follows by the fact that each composition of almost-holomorphic mappings and each inverse of almost-holomorphic diffeomorphism are almost-holomorphic too.

Lemma 1: The matrix representation of $\mathbf{J^*}$ in each local Spencer coordinate system

$$(U, w^1, ..., w^m, z^{m+1}, ..., z^n, \bar{w}^{n+1}, ..., \bar{w}^{n+m}, \bar{z}^{n+m+1}, ..., \bar{z}^{2n})$$

where $w^j = f_j$, $j = 1, ..., m$, are functionally independent almost holomorphic functions, seems as follows

$$\begin{pmatrix} iE_m & * & 0 & * \\ 0 & * & 0 & * \\ 0 & * & -iE_m & * \\ 0 & * & 0 & * \end{pmatrix}$$

E_m being the unit $m \times m$ matrix.

Proof. It is enough to take in view that:

$$(dw^1, ..., dw^m, dz^{m+1}, ..., dz^n, d\bar{w}^{n+1}, ..., d\bar{w}^{n+m}, d\bar{z}^{n+m+1}, ..., d\bar{z}^{2n})$$

is basis of the cotangent space and

$$\mathbf{J}^* dw^j = \mathbf{J}^* df_j = i df_j = i dw^{\,j}, \; j = 1, ..., m \;\blacksquare$$

Consequences: The first m equations of the system $J^* df = i df$ are just the conditions $\partial f / \partial \bar{z}_j = 0$, $j = 1, ..., m$

We shall consider the mapping from U to $\mathbb{C}^m$ defined by $f_1, ..., f_m$. This mapping is a smooth submersion as it can be considered as a composition of the diffeomorphism defined by Spencer coordinates of U in $\mathbb{C}^n \times \bar{\mathbb{C}}^n$ and the projection of $\mathbb{C}^n \times \bar{\mathbb{C}}^n$ on $\mathbb{C}^m$, $m < n$. This mapping will be denoted by f_U, and the image of U by f_U will be denoted U_m^c. It is an open subset of $\mathbb{C}^m$, which will be called a naturally associated m-dimensional open set to the considered local Spencer coordinate system.

Lemma 2: Each almost holomorphic function h, defined on a local Spencer coordinate system U is represented as a superposition of a holomorphic function H defined on U_m^c and the almost holomorphic functions $f_1, ..., f_m$ defined on U, i.e.

$$h = H \circ (f_1, ..., f_m) = H(f_1, ..., f_m)$$

Proof: As $w_j = f_j,\ j = 1, ..., m$, is a system of smooth functionally independent on U functions, we have $h = H(w^1, ..., w^m)$ with $H \in \mathcal{C}^\infty(U)$. But

$$\bar{\partial} H = (\bar{\partial} H/\partial \bar{w}^1)d\bar{w}^1 + ... + (\partial H/\partial \bar{w}^m)d\bar{w}^m$$

and in view of $\bar{\partial} H = \bar{\partial} h = 0$, we get that the above written (0,1)-form is a zero-form, or $\partial H/\partial \bar{w}_j = 0$, $j = 1, ...m$. ■

Lemma 3: Let $(w^1, ..., w^m)$ and $(v^1, ..., v^m)$ be two systems of holomorphic coordinates on U_m^c defined by two different systems of almost holomorphic on U systems $(f_1, ..., f_m)$ and $(h_1, ..., h_m)$. Then there exists a bijective holomorphic transition mapping between the mentioned two coordinate systems.

Proof. According to *Lemma 2* we have $v_j = H_j(w^1, ..., w^m)$, $j = 1, ..., n$,where H_j are holomorphic functions of $(w_1, ..., w_m)$. The system $H = (H_1, ..., H_m)$ defines the mentioned transition mapping as the differentials dH_j are $\mathbb{C}$-linear independent. ■

Recapitulating we obtain the following

Proposition 2: On each paracompact almost complex manifold $(M, \mathbf{J})$ of constant Spencer type m there exists a locally finite covering U_j by self-conjugated Spencer's coordinate system $(Uj, z_j^1, ..., z_j^m, ...)$ such that in every intersection $U_j \cap U_k$ the holomorphic coordinates $z_j^1, ..., z_j^m$ change holomorphically in the other holomorphic coordinates $z_k^1, ..., z_k^m$.

4. Local submersions and local foliations

As it was remarked above the mapping $f_U : U \to \mathbb{C}^m$, defined by the almost holomorphic functions $(f_1, ..., f_m)$ is a local submersion. According to the introduced notations

$$f_U(U) = U_m^c \subset \mathbb{C}^m$$

The leaves of this submersion are defined as the stalks of the mapping f_U. Each leaf is a smooth $(2n - 2m)$-dimensional submanifold of U on which all functions f_j have constant value. Transversal leaves are defined as univalent inverse images of U_m^c, i.e. as sections of U over U_m^c.

We shall consider the set of all open subsets $U_m^c \subset \mathbb{C}^m$, corresponding to different mappings f_U, U open subset of M. This set together with the transition mappings described in *Lemma 3* defines a pseudo-group of holomorphic transition mappings between open subsets of $\mathbb{C}^m$ denoted as follows

$$\Gamma\{U^c_m, V^c_m, ...; H : U^c_m \to V^c_m, ...\}$$

We shall denote by $\mathbf{C}^m/\Gamma$ the set of equivalent points of $\mathbb{C}^m$ with respect to the natural equivalence defined by the holomorphic transition mappings. With this in mind we consider the family $\{f_U : U \to M\}$ and will define a glued mapping

$$f : M \to \mathbb{C}^m/\Gamma$$

as follows: if $p \in M$ we take an open subset U such that $p \in U$ and we set

$$f(p) = \{\text{the equivalence class of the point } \ f_U(p).\}$$

Under the assumption that $\mathbb{C}^m/\Gamma$ is equipped with the standard complex structure $\mathbf{i}$ defined by holomorphic coordinates $(w^1, ..., w^m)$ we can formulate the following

Lemma 4. The glued mapping $f : M \to \mathbb{C}^m/\Gamma$ is an almost holomorphic mapping between $(M, \mathbf{J})$ and $(\mathbb{C}^m/\Gamma, \mathbf{i})$.

Proof. As the glued mapping f coincides locally with some f_U we have:
$\mathbf{J}^* df_U = \mathbf{J}^* d(f_1, ..., f_m) = \mathbf{J}^*(df_1, ..., df_m) = (\mathbf{J}^* df_1, ..., \mathbf{J}^* df_m) = \mathbf{i}(df_1, ..., df_m) = id\ f_U$. So each f_U is an almost holomorphic mapping ■

Lemma 5. The sheaf of almost holomorphic functions on M is the inverse image of the sheaf of holomorphic functions on $\mathbb{C}^m/\Gamma$.

Proof. The mentioned sheaf on M is defined by the presheaf $\{U, \mathcal{O}_M(U)\}$ where U varies in the set of all open subsets of M and $\mathcal{O}_M(U)$ is defined as follows:
$\mathcal{O}_M(U) = \{h \circ f_U \mid h \in \mathcal{O}_{\mathbb{C}^m/\Gamma} f_U(U))\}$.

4.1. **Hypercomplex manifolds and hyperholomorphic functions.** Let M be a 4n-dimensional ($\mathcal{C}^\infty$) smooth manifold. A hypercomplex structure on M is defined by a pair of two almost complex structures $\mathbf{J}$ and $\mathbf{K}$ such that $\mathbf{JK} + \mathbf{KJ} = 0$. It is easy to see that the composition $\mathbf{JK}$ is an almost-complex structure too. Moreover, for each triple of real numbers b, c, d, such that $b^2 + c^2 + d^2 = 1$, the linear combination $b\mathbf{J} + c\mathbf{K} + d(\mathbf{JK})$ is an almost-complex structure on M. So there is a family of almost complex structures on M parametrized by the points of sphere Σ^2. (See for instance [AM], [ABM]).

We shall consider almost-holomorphic functions on hypercomplex manifolds. The definition remains the same as in the above considered case, for instance on $(M, \mathbf{J}, \mathbf{K})$ we have $\mathbf{J}$-almost- holomorphic function which are complex-valued function f on $(M, \mathbf{J})$ such that $\mathbf{J}^* df = idf$ using the right-side j-complex splitting of $\mathbb{H}$. Respectively $\mathbf{K}$-almost- holomorphic functions g on $(M, \mathbf{J}, \mathbf{K})$ are the almost-holomorphic with respect to$(M, \mathbf{K})$ such that $\mathbf{K}^* dg = jdg$ using an i-complex splitting of $\mathbb{H}$. Let$(M, \mathbf{J}, \mathbf{K})$ be a hypercomplex manifolds and $\mathbb{H}$ be 4-dimensional quaternionic vector space. According to Sommese [So] the right-side multiplication by i and j are given respectively by the matrices S and T, called standard quaternionic structures.

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

In the paper of Sommese the matrix T *is* denoted by K.

As we have $S^2 = -\mathbf{1}, T^2 = -\mathbf{1}, (ST)^2 = -\mathbf{1}$, and $ST + TS = 0$, we can consider $(\mathbb{H}, S, T)$ as a special hypercomplex manifold. (See [So]). A function F defined on an open subset $U \subset M$ with valued in $\mathbb{H}$ is called $\mathbf{J}$-hyper-holomorphic function on U if $dF \circ \mathbf{J} = S \circ dF$, or $\mathbf{J}^* dF = S dF$. Using the right-side j-complex splitting $\mathbb{H}^j$ we take the compositions of F with the projections of $\mathbb{H}$ on the first and the second components of $\mathbb{H}^j$. So F is represented by a pair of complex valued functions denoted respectively by f and φ. If we set $F = u + i\,v + j\,\zeta + k\,\eta$, where u, v, ζ, η are real-valued functions on U, we can write $\varphi = u + iv +$ ($\zeta + i\eta)j$, with $f = u + iv$, $\varphi = \zeta + i\eta$. Complexifying the matrix S, i.e. setting

$$S = \begin{bmatrix} \mathbf{i} & \mathbf{0} \\ \mathbf{0} & -\mathbf{i} \end{bmatrix}, \qquad i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and taking $dF = df + d\varphi j$, we calculate that

$$J^* df + J^* d\varphi j = i\, df - i\, d\varphi j.$$

Having in mind the splitting $\mathbb{H}^j$, we get $J^* df = i\, df$ and $J^* d\varphi = -i\, d\varphi$, which means that f is $\mathbf{J}$-almost-holomorphic function on U and φ is $\mathbf{J}$-almost-antiholomorphic.

For the definition of $\mathbf{K}$-hyper-holomorphic function on U we shall use the other complex splitting of $\mathbb{H}$, namely $\mathbb{H}^i$. A function $G : M \to \mathbb{H}^i$, i.e. $G = g + \psi i$, $g = u' + j\zeta'$, $\psi = v' - j\eta'$, will be called $\mathbf{K}$-hyper-holomorphic function on U if $dG \circ K = T \circ dG$ or $K^* dG = T dG$. Taking a (2×2)-representation of the matrix T, i.e.

$$T = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix}, \qquad \mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

after a short calculation we get

$$K^* dg + i\, K^* d\psi = d\psi - i\, dg$$

It follows that $K^* dg = d\psi$ and $K^* d\psi = -dg$. This result is in terms of $\mathbb{H}^i$.

Now we will translate the obtained result in terms of $\mathbb{H}^j$. From $K^*(du' + d\zeta' j) = \mathrm{d}v' - \mathrm{d}\eta' j$ we get

$$K^* du' = dv' \quad \text{and} \quad K^* d\zeta' = -d\eta'.$$

Analogously, from $K^*(dv' - d\eta' j) = -(du' + d\zeta' j)$ we get

$$K^* dv' = -du' \quad \text{and} \quad K^* d\eta' = -d\zeta'.$$

But the system $K^* du' = dv'$, $K^* dv' = -du'$ is just the Cauchy-Riemann system, which says that the function $u' + iv'$ is $\mathbf{J}$-almost-antiholomorphic, i. e. $J^* d(u' + iv') = -id(u' + iv')$. The function $\zeta' + i\eta'$ is $\mathbf{J}$-almost-holomorphic.

4.2. **Hyper-Spencer coordinates.** Hyper-holomorphic coordinates on a hyper-complex manifold $(M, \mathbf{J}, \mathbf{K})$ can be introduced by functionally independent quaternionic-valued functions $f_\alpha + \varphi_\alpha j$, $\alpha = 1, ..., m$, $m = (1/2) dim M_{\mathbb{R}}$, or by the complex-valued function $(f_\alpha, \varphi_\alpha)$. We are interested of the possibility to have $m < (1/2)\dim M_{\mathbb{R}}$. More precisely, a $\mathbf{J}$-hyper- Spencer coordinate system is defined locally on M as a maximal system of m functionally independent $\mathbf{J}$-hyper-holomorphic functions. A hypercomplex manifold equipped with an atlas of local $\mathbf{J}$-hyper-Spencer coordinate systems is called a hypercomplex manifold of Spencer type m.

Having in mind the interconnection between $\mathbf{J}$-hyper-holomorphic functions and $\mathbf{J}$-almost-holomorphic ones we derive the analogues of the *Lemmas 1,2* and *3* of the previous paragraphs. Let us remark that in view that f_α are $\mathbf{J}$-almost- holomorphic, and φ_α are $\mathbf{J}$-almost-antiholomorphic, the corresponding matrix representations of J^* is as follows (according to *Lemma 1*)

$$\begin{bmatrix} iE_m & * & 0 & * \\ 0 & * & 0 & * \\ 0 & * & -iE_m & * \\ 0 & * & 0 & * \end{bmatrix} \times \begin{bmatrix} -iE_m & * & 0 & * \\ 0 & * & 0 & * \\ 0 & * & iE_m & * \\ 0 & * & 0 & * \end{bmatrix}$$

Analogously, $\mathbf{K}$-hyper-Spencer coordinates can be introduced with the help of $\mathbf{K}$-hyper holomorphic mappings. The *Proposition 2* remains valid for $\mathbf{J}$-holomorphic transition functions and $\mathbf{K}$-holomorphic transition functions. When the transition transformations are simultaneously $\mathbf{J}$- and $\mathbf{K}$-holomorphic it follows that they are affine.

Full coordinate systems defined by $m = (1/2)\dim M_{\mathbb{R}}$ functions which are both $\mathbf{J}$ and $\mathbf{K}$ hyper-holomorphic lead to quaternionic manifolds.

5. Elliptic Equations

5.1. **Potential structures on almost-complex manifolds.** Let $(M, \mathbf{J})$ be an almost complex manifold. We shall consider the following globally defined on M Pfaffian form: $\omega = J^* du$, where $u = u(p), p \in M$, is a real-valued smooth (at least of class $\mathcal{C}^2$) function. In the case the 1-form ω is closed, we will say that ω defines a *potential structure* on the almost complex manifold $(M, \mathbf{J})$. On each local real coordinate system $(U, x = (x^k)), x^k \in \mathbb{R}, k = 1, ..., 2n$, we have a matrix representation of $\mathbf{J}$, i.e. $J = \| J^k_j(x) \|$, where $J^k_j(x)$ are smooth real functions on U. By J_j is denoted the j-row of the mentioned matrix and ∇u is the gradient of u. It is easy to see

$$Jdu = \sum_{q=1}^{2n} (J_q \cdot \nabla u) dx^q$$

where

$$J_q \cdot \nabla u = \sum_{p=1}^{2n} J^p_q \frac{\partial u}{\partial x^p},$$

For each potential structure on $(M, \mathbf{J})$ the following two statements hold.

Consequence 1. On every simply connected domain $\Omega \subset M$ it holds that

$$\int_\gamma J^* du = 0$$

for each closed curve γ in Ω.

Consequence 2. The following system

$$\frac{\partial(J_q \bullet \nabla u)}{\partial x^s} = \frac{\partial(J_s \bullet \nabla u)}{\partial x^q},$$

s, $q = 1, \cdots, 2n$, is satisfied locally.

5.2. **Almost pluri-harmonic functions.** By $(M, \mathbf{J}, \omega)$ is denoted an almost-complex manifold $(M, \mathbf{J})$ equipped with potential structure ω. Then the 1-form $\omega = J^* du$ is close, and we have $dJ^* du = 0$. In this case we will say that the function u is an almost-pluriharmonic function. The interconnection between almost-pluriharmonic functions and almost-holomorphic ones (with respect to $\mathbf{J}$) is like to this one between pluriharmonic functions and holomorphic ones. This follows directly form the Cauchy-Riemann equations $J^* du = -dv$, $J^* dv = du$. Clearly the real part u and the imaginary part v of the almost-holomorphic function $f = u + iv$ are almost-pluriharmonic functions.

5.3. **Elliptic equations on almost-complex manifolds.** We denote by $\triangle_{\mathbf{J}}$ the following differential operator of second order (in terms of coordinates)

$$\Delta_{\mathbf{J}} = \sum_{s,p=1}^{2n} A_{sp} \frac{\partial^2}{\partial x^s \partial x^p} + \sum_{p=1}^{2n} B_p \frac{\partial}{\partial x^p}$$

where

$$A_{sp} = \sum_{q=1}^{n} (J_q^s J_q^p + \delta_q^s \delta_q^p),$$

and

$$B_p = \sum_{s,q=1}^{2n} J_q^s \left(\frac{\partial J_q^p}{\partial x^s} - \frac{\partial J_s^p}{\partial x^q} \right),$$

δ_q^s, δ_q^p are the Kronecker symbols. Setting $A_J = \|A_{sp}\|$, we obtain

$$A_J = JJ^* + E_{2n}$$

where J^* is the transpose of J and E_{2n} is the unity $2n \times 2n$ matrix.

We emphasize here that now we work with real coordinates, but not with complex self-conjugate ones. However this corresponds to the Spencer type 0. In the other extreme case of Spencer type n we have complex-analytic (holomorphic) coordinates.

This is the case of complex analytic manifold with the standard almost-complex structure denoted by $\mathbf{S}^0$ (it is different from $\mathbf{S}$ in the previous paragraph).

$$-\mathbf{S}^0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \times ... \times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (n \text{ times})$$

As $S^0(S^0)^* = E_{2n}$ we get $A_{S^0} = 2E_{2n}$ and $\Delta_{\mathbf{S}^0} = 2\Delta$, where Δ is the Laplace operator in $2n$ real variables.

Proposition 3:$\triangle_{\mathbf{J}}$ is an elliptic differential operator.

Proof: It is sufficient to consider the following inequality

$$\sum_{s=1}^{2n}\sum_{p=1}^{2n} A_{sp}\xi_s\xi_p = \sum_{q=1}^{2n}\left(\sum_{s=1}^{2n} J_q^s\xi_s\right)^2 + \sum_{q=1}^{2n}\left(\sum_{s=1}^{2n} \delta_q^s\xi_s\right)^2 \geq \sum_{q=1}^{2n}\xi_q^2. \qquad \blacksquare$$

Considering the PDE

$$\triangle_{\mathbf{J}} u = 0,$$

we can state the following

Theorem :*Each almost pluriharmonic function u satisfies locally the equation* $\triangle_{\mathbf{J}} u = 0$

Proof: Let u be almost pluriharmonic, i.e. $dJdu = 0$, or the 1-form J^*du is closed. According to the previous paragraph u satisfies locally the following system of PDEs

$$\frac{\partial(J_q \bullet \nabla u)}{\partial x^s} = \frac{\partial(J_s \bullet \nabla u)}{\partial x^q},$$

s, $q = 1, \cdots, 2n$. Now replacing

$$J_q \bullet \nabla u = \sum_{p=1}^{2n} J_k^p \frac{\partial u}{\partial x^p} \quad \text{and} \quad J_s \bullet \nabla u = \sum_{p=1}^{2n} J_s^p \frac{\partial u}{\partial x^p}$$

in (4) we obtain the system

$$\sum_{p=1}^{2n}\left(\frac{\partial\left(J_k^p \frac{\partial u}{\partial x^p}\right)}{\partial x^s} - \frac{\partial\left(J_s^p \frac{\partial u}{\partial x^p}\right)}{\partial x^k}\right) = 0,$$

k, $s = 1, \cdots, 2n$. Multiplying each of the above written equations by J_q^s and summing with respect to s we obtain

$$\sum_{p=1}^{2n}\sum_{s=1}^{2n}\left(J_k^p J_q^s \frac{\partial^2 u}{\partial x^s \partial x^p} - J_q^s J_s^p \frac{\partial^2 u}{\partial x^k \partial x^p}\right) = \sum_{p=1}^{2n}\sum_{s=1}^{2n} J_q^s \left(\frac{\partial J_s^p}{\partial x^k} - \frac{\partial J_k^p}{\partial x^s}\right)\frac{\partial u}{\partial x^p}.$$

As we have

$$\sum_{s=1}^{2n} J_q^s J_s^p = -\delta_q^p$$

and

$$\frac{\partial^2 u}{\partial x^k \partial x^p} = \sum_{s=1}^{2n} \delta_k^s \frac{\partial^2 u}{\partial x^s \partial x^p},$$

we obtain

$$\sum_{p=1}^{2n} \sum_{s=1}^{2n} \left(J_k^p J_q^s + \delta_q^p \delta_k^s \right) \frac{\partial^2 u}{\partial x^s \partial x^p} = \sum_{p=1}^{2n} \sum_{s=1}^{2n} J_q^s \left(\frac{\partial J_s^p}{\partial x^s} - \frac{\partial J_k^p}{\partial x^s} \right) \frac{\partial u}{\partial x^p}.$$

Now taking $q = k$ and summing with respect to k we get exactly

$$\triangle_{\mathbf{J}} u = 0. \quad \blacksquare$$

In the case $\mathbf{J} = \mathbf{S}^0$ the above written equation is just the classical Cauchy-Riemann system.

Consequences:

1. Each almost pluriharmonic function and respectively every almost holomorphic function of class $\mathcal{C}^2$ on a smooth manifold are of class $\mathcal{C}^\infty$ too.

2. For connected smooth manifolds the maximum principle holds.

3. In the case of real analytic manifold M,equipped with real-analytic structure $\mathbf{J}$, each $\mathbf{J}$-pluriharmonic and each $\mathbf{J}$-almost-holomorphic function is real analytic.

4. In the case of connected real analytic manifold M with real-analytic structure $\mathbf{J}$ the principle of unicity of the analytic continuation holds.

Remark: This theorem is inspired from the paper [BKW]. The first announcement is in [DM]

5.4. **The equation $dJ^* d\,\mathrm{u} = 0$ in terms of vector fields - commutators and anti-commutators.** Applying the well known formula

$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$, ω is 1-form, X, Y are vector fields

to the 1-form $\omega = \mathbf{J}du$ we present the equation (2) in terms of expressions of vector fields, namely

$$[X, Y]_{\mathbf{J}}(u) = \mathbf{J}[X, Y](u)$$

where $[X, Y]_{\mathbf{J}} \overset{def}{=} X \circ \mathbf{J}Y - Y \circ \mathbf{J}X$. It is to remark that $[X, Y]_{\mathbf{J}}$ is not a vector field. For instance:

$$\begin{aligned} [X, Y]_{\mathbf{J}}(fh) = \; & [X, Y]_{\mathbf{J}}(f)h + f\; [X, Y]_{\mathbf{J}}(h) + X(f)(\mathbf{J}Y)(h) - \\ & (\mathbf{J}X)(f)Y(h) + X(h)(\mathbf{J}Y)(f) - (\mathbf{J}X)(h)Y(f) \end{aligned}$$

Some properties of $[X, Y]_{\mathbf{J}}$

Considering the natural splitting

$$\mathbb{C}TM = T^{1,0}M \oplus T^{0,1}M$$

we can take the restriction of $[X, Y]_{\mathbf{J}}$ on $T^{1,0}M$. This means that

$$\mathbf{J}X = iX \quad \text{and} \quad \mathbf{J}Y = iY$$

where $X, Y \in T^{1,0}M$. So we have

$$[X, Y]_{\mathbf{J}} = X \circ (iY) - Y \circ (iX) = i[X, Y]$$

Analogously

$$[X, Y]_{\mathbf{J}} = (-i)[X, Y] \quad \text{on } T^{0,1}M$$

Now we take X$\in T^{1,0}M$ and $Y \in T^{0,1}M$

$$[X, Y]_{\mathbf{J}} = X \circ (iY) - Y \circ (-iX) = i(X \circ Y + Y \circ X) = i\{X, Y\}$$

Here $\{X, Y\}$ denotes the anticommutator of X and Y. Analogously, if $X \in T^{0,1}M$ and $Y \in T^{1,0}M$:

$$[X, Y]_{\mathbf{J}} = -i\{X, Y\}$$

5.5. **Potential structures on hypercomplex manifolds.** On a hypercomplex manifold $(M, \mathbf{J}, \mathbf{K})$ we can consider two separate potential structures, namely

$$\omega_1 = \mathbf{J}^* du \quad \text{and} \quad \omega_2 = \mathbf{K}^* d\zeta$$

or the sum

$$\omega = \mathbf{J}^* du + \mathbf{K}^* d\zeta$$

The corresponding almost-pluriharmonic functions u, v, ζ, η satisfy the equations:

$$d\mathbf{J}^* du = d\mathbf{J}^* dv = 0 \;\; \text{and} \;\; d\mathbf{J}^* d\zeta = d\mathbf{J}^* d\eta = 0$$

We have also the natural defined elliptic operators $\Delta_{\mathbf{J}}$ and $\Delta_{\mathbf{K}}$. According to the proved theorem:

$$d\mathbf{J}^* du = d\mathbf{J}^* dv = 0 \quad \Rightarrow \quad \Delta_{\mathbf{J}} u = \Delta_{\mathbf{J}} v = 0$$

and

$$d\mathbf{K}^* d\zeta = d\mathbf{K}^* d\eta = 0 \quad \Rightarrow \quad \Delta_{\mathbf{K}} \zeta = \Delta_{\mathbf{K}} \eta = 0$$

For the sum $\omega = \mathbf{J}^* du + \mathbf{K}^* d\zeta$ a pair of functions (u, ζ) appears, namely the solutions of the following second order equation:

$$d\mathbf{J}^* du + d\mathbf{K}^* d\zeta = 0$$

In terms of vector fields the above written equations seem as follows

$$[X, Y]_{\mathbf{J}} u = \mathbf{J}[X, Y](u) \quad \text{and} \quad [X, Y]_{\mathbf{K}} u = \mathbf{K}[X, Y](u)$$

6. Generation of almost-complex structures

6.1. **Remarks on the local equation of almost-holomorphic functions.** Let $(M, \mathbf{J})$ be an almost-complex manifold, $dimM = 2n$. Having in mind the question of the local integration of the equation $\mathbf{J}^* df = idf$, we shall examine how "far away" a non- integrable almost complex structure $\mathbf{J}$ is from the classical complex structure related with the standard almost-complex structure $\mathbf{S}$.

Let p be a point of M. Taking an open neighborhood U of the point p, small enough, we can accept that U is a neighborhood of the origin in $\mathbb{R}^{2n}$ (p to be the origin). Now we shall replace $\mathbf{J}$ by it matrix representation J on U and J^* will denote the transposed matrix. We will use general real coordinates $x = (x^1, ..., x^{2n}) \in \mathbb{R}^{2n}$. Let G denotes a non-degenerate $(2n \times 2n)$ matrix, such that $G^{-1}J^*(0)G = S^*$, where S^* is the transposed matrix of S,

$$S = \begin{bmatrix} 0 & -E_n \\ E_n & 0 \end{bmatrix}, \ E_n \text{ being the unit } n \times n \text{ matrix.}$$

For $x \in U$ we set:

$$G^{-1}J(x)G = \begin{bmatrix} A(x) & B(x) + E_n \\ C(x) - E_n & D(x) \end{bmatrix}$$

$A(x), B(x), C(x), D(x)$ are $n \times n$ matrices.

Clearly we have for $x = 0$:

$$\begin{bmatrix} A(x) & B(x) + E_n \\ C(x) - E_n & D(x) \end{bmatrix} = S^* \text{ and } A(0) = B(0) = C(0) = D(0) = 0$$

Moreover, we have $(G^{-1}J(x)G)^2 = -E_{2n}$, which implies the following identities:

$A^2(x) + (B(x) + E_n)(C(x) - E_n) = -E_n$
$A(x)(B(x) + E_n) + (B(x) + E_n)D(x) = 0_n$
$(C(x) - E_n)A(x) + D(x)(C(x) - E_n) = 0_n$
$(C(x) - E_n)(B(x) + E_n) + D^2(x) = -E_{2n}$

From the last system it follows that locally is valid:

$A(x) = -(C(x) - E_n)^{-1}D(x)(C(x) - E_n)$
$B(x) + E_n = -(C(x) - E_n)^{-1}(D^2(x) + E_n)$

Indeed, as

$$\det(C(0) - E_n) = (-1)^n \neq 0$$

the inverse matrix $(C(x) - E_n)^{-1}$exists in some neighborhood of the origin $\mathbf{0} \in \mathbb{R}^n$.

Now lets consider the equation $(J^* - iE_{2n})df = 0$. It follows that

$$(G^{-1}J^*G - iE_{2n})df = 0$$

and also

$$\begin{bmatrix} A(x) - iE_n & B(x) + E_n \\ C(x) - E_n & D(x) - iE_n \end{bmatrix} df = 0$$

*Proposition:*The following block matrix identity is valid:

$\left[\begin{array}{cc} A(x)-iE_n & B(x)+E_n \end{array}\right] = (A(x)-iE_n)(C(x)-E_n)^{-1}\left[\begin{array}{cc} C(x)-E_n & D(x)-iE_n \end{array}\right]$

Proof: Let consider the right side of the identity:
$(A(x)-iE_n)(C(x)-E_n)^{-1}\left[\begin{array}{cc} C(x)-E_n & D(x)-iE_n \end{array}\right] =$
$=\left[\begin{array}{cc} A(x)-iE_n & (A(x)-iE_n)(C(x)-E_n)^{-1}(D(x)-iE_n) \end{array}\right]$
But:
$(A(x)-iE_n)(C(x)-E_n)^{-1}(D(x)-iE_n) = B(x)+E_n$, as $A(x) = -(C(x)-E_n)^{-1}D(x)(C(x)-E_n)$.
The last equality becomes:
$(-(C(x)-E_n)^{-1}D(x)(C(x)-E_n)-iE_n)(C(x)-E_n)^{-1}(D(x)-iE_n) =$
$=(C(x)-E_n)^{-1}(-D(x)-iE_n)(C(x)-E_n)(C(x)-E_n)^{-1}(D(x)-iE_n) =$
$=-(C(x)-E_n)^{-1}(D(x)+iE_n)(D(x)-iE_n) =$
$=-(C(x)-E_n)^{-1}(D^2(x)+iE_n) = B(x)+E_n.$ ■

Corollary: The first n equations of the considered system

$$(J^* - iE_{2n})df = 0$$

follow from the last n ones. So we obtain that locally this system is equivalent to the next one:

$$\left[\begin{array}{cc} C(x)-E_n & D(x)-iE_n \end{array}\right] df = 0$$

or:

$$\left[\begin{array}{cc} E_n & (C(x)-E_n)^{-1}(D(x)-iE_n) \end{array}\right] df = 0$$

Setting $P(x) \overset{def}{=} (C(x)-E_n)^{-1}D(x)$ and $Q(x) \overset{def}{=} (C(x)-E_n)^{-1}$, we receive the following block matrix form of the considered equation of almost holomorphic functions:

$$\left[\begin{array}{cc} E_n & P(x)+iQ(x) \end{array}\right] df = 0.$$

6.2. **Local reconstruction of J by the matrices P and Q.** We will use the following equalities:
$C - E_n = Q^{-1}$; $D = Q^{-1}P$; $A = -QQ^{-1}PQ = -PQ^{-1}$;
$B + E_n = -Q((Q^{-1}P)^2 + E_n) = -PQ^{-1}P - E_n$.
The matrix J can be reconstructed as follows:

$$J = \left[\begin{array}{cc} -PQ^{-1} & -PQ^{-1}P - Q \\ Q^{-1} & Q^{-1}P \end{array}\right] \qquad (*)$$

The mentioned reconstruction (*) can be considered as a generation of the matrix representation of J on the open set U by the pair of matrices (P, Q). Denoting by $\mathcal{M}(U, n)$ the algebra of all $(n \times n)$-matrices equipped with the topology of coordinate convergence, we can consider the Cartesian product $\mathcal{M}(U, n) \times \mathcal{M}(U, n)$ with the product topology as a continuous family which generates the set $\mathcal{J}(U, 2n)$ of all $(2n \times 2n)$-matrices J, which verify the matrix equation

$$J^2 + E_{2n} = 0,$$

as a kind of moduli space (locally). More precisely, the following proposition holds

Proposition 4: For each $J \in \mathcal{J}(U, 2n)$ there is a pair $(P, Q) \in \mathcal{M}(U, n) \times \mathcal{M}(U, n)$ such that J is generated by (P, Q) in the sense of the rule (*). Conversely, each pair (P, Q) defines a J according to the rule (*). Each sequence (P_n, Q_n) of elements of $\mathcal{M}(U, n) \times \mathcal{M}(U, n)$ determines a sequence of elements of $\mathcal{J}(U, 2n)$, and the limit of the second sequence corresponds by the rule (*) to the limit of the first sequence.

The proof is clear.

6.3. **Global reconstruction of J.** The problem of global reconstruction of almost complex structures on a smooth manifold by an appropriate algebraic objects is much more difficult. It seems that an approach can be developed on real-analytic almost complex manifold (M, J) having local matrix representation for J with real-analytic coefficients. Now we shall consider the sheaf of germs of almost complex structures, denoted by $\mathcal{J}(M)$, and the sheaf of germs of pairs of matrices (P, Q). Supposing that each J can be considered as a global section of the sheaf $\mathcal{J}(M)$, we can develop the rule (*) for germs of $\mathcal{J}(M)$ and germs of pairs (P, Q) at each point $p \in M$. The set of global sections of $\mathcal{J}(M)$ must be generated by the sections of the sheaf of germs of pairs (P, Q).

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