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THE FOURIER TRANSFORM ON THE SPHERE

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ABSTRACT

A Fourier transform is set up on the unit sphere by means of the Cayley transform and Vahlen matrices. Basic properties of the transform are investigated, particularly the links to Clifford analysis and Hardy spaces.

Introduction

Vahlen matrices are described in a number of papers of Ahlfors, see for instance [A]. These matrices give a simple way of characterizing Moebius transformations over the one point compactification of euclidean space. This approach was introduced by Vahlen, [V], in 1902, but seems to have been significantly forgotten until its rediscovery in the 1980s. Since that time it has been used by quite a number of authors for various diverse reasons, see for example [Bo, C, EMG, H, Le, Lo, Po, PQ, W], and references therein.

These matrices are ideally suited to describe the conformal covariance of solutions to the Dirac equation in euclidean space, see for instance [R], and, as illustrated in [C, QR], they are also extremely well suited to describe the conformal covariance of the Hilbert space of Clifford algebra valued, square integrable functions defined on an orientable surface lying in R^n. Consequently, they are well suited for describing the conformal covariance of Hardy spaces of monogenic functions associated to square integrable functions over such surfaces. All this is pointed out in some detail in [QR].

In [QR] it is shown that the decomposition theorem obtained in [LMcS] for special classes of monogenic functions defined on sector domains in R^n conformally transforms itself to any domain which is conformally equivalent to this original domain. In particular if we consider the Cayley transformation then we obtain a similar decomposition theorem for special types of monogenic functions defined on the Cayley transformation of the sector domain. In [LMcQ] a one to one correspondence is determined with each function in this decomposition and bounded holomorphic functions lying in special domains in C^{n-1}. To obtain this correspondence the authors use certain projection operators acting on the monogenic extension of the inverse Fourier transform.

A main thrust of this paper is to use Vahlen matrices to construct analogous projection operators over the sphere. It turns out that the south pole of the sphere is a singular point in this theory, so instead of considering the compact sphere one needs to consider the sphere minus the south pole. We call this set the punctured sphere. Having obtained the analogues of these projection operators, we are then in a position to introduce the analogue of the inverse Fourier transform and its monogenic extensions to the inside and exterior of the sphere. Having obtained these extensions it is now a simple matter to transpose many basic ideas in Clifford analysis using these functions to the context described here. In particular we are able to show how the algebra isomorphism described in [LMcQ] transposes itself via the Cayley transformation. Again we obtain a space of bounded holomorphic functions defined on a sector type domain now lying within the complex sphere. However, it is worth pointing out that under conformal transformations the operation of convolution is replaced by a related but not identical operation.

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Preliminaries

In this section we shall develop the necessary basics on Clifford algebras and Clifford analysis that we shall need here. Most all of the results presented here are well known are basic results. However, we do try to clarify some results and give some motivational comments that do not appear elsewhere.

The starting point is to consider a vector $x \in R^n$ and to consider R^n as being embedded in an algebra so that $x^2 = -\|x\|^2$ under the algebra multiplication. If no further relations are assumed then the algebra that we obtain is an example of a Clifford algebra. If we express x as $x_1e_1 + \ldots + x_ne_n$, where $x_1, \ldots x_n \in R$ and $e_1, \ldots, e_n$ is an orthonormal basis for R^n, then it may easily be observed that this algebra is a real, associative, 2^n-dimensional algebra with basis

$$1, e_1, \ldots, e_n, \ldots, e_{j_1} \ldots e_{j_r}, \ldots, e_1 \ldots e_n,$$

where $j_1 < \ldots < j_r$, and $1 \leq r \leq n$. This algebra is an example of a real Clifford algebra, and we shall denote it by A_n. When $n = 1$ we obtain the complex field, when $n = 2$ we obtain the quaternion skew field. For $n > 2$ the algebra is no longer a division algebra for these algebras always contain the zero divisors $(1 \pm e_1e_2e_3)$. However, each non-zero vector $x \in R^n \subset A_n$ has the vector $\frac{-x}{\|x\|^2}$ as a multiplicative inverse. Up to the minus sign this vector corresponds to the Kelvin inverse of the vector x. That the algebra A_n contains Kelvin inversion in its structure is crucial from a function theory viewpoint.

Now let us consider the multiplication

$$e_1(x_1e_1 + x_2e_2 + \ldots x_ne_n)e_1 = -x_1e_1 + x_2e_2 + \ldots x_ne_n. \tag{1}$$

So multiplication from the left and right of a general vector x by e_1 gives a reflection of the vector x in the direction of e_1. It is at this point that our insistance on the appearance of a minus sign in the formula $x^2 = -\|x\|^2$ when setting up the algebra A_n is crucial. If instead we had set up the algebra so that $x^2 = \|x\|^2$ then equation (1) would be changed to give reflections in the direction of the vectors $e_2 \ldots e_n$. When n is odd this would give a rotation, and we would not be able to obtain a reflection in the direction of e_1, or any other direction, by this method.

Returning to the algebra A_n we similarly have that for each $y \in S^{n-1}$, the unit sphere in R^n, the product yxy gives a reflection in the direction of the vector y. By induction it follows that for $y_1, \ldots, y_p \in S^{n-1}$ the product $y_1 \ldots y_pxy_p \ldots y_1$ gives an orthogonal transformation over R^n. As the vectors $y_1, \ldots y_p$ are arbitrary points on S^{n-1} and each orthogonal transformation can be expressed as a finite product of reflections it follows that the group

$$Pin(n) = \{a \in A_n : a = y_1 \ldots y_p, y_j \in S^{n-1}, 1 \leq j \leq p \in N\},$$

where N denotes the set of natural numbers, is a covering group of the orthogonal group, $O(n)$. In fact, [Po, AtBS], $Pin(n)$ is a double covering of $O(n)$. Other properties of A_n, and $Pin(n)$ can be found in [Po, AtBS], and elsewhere.

We shall need the folowing two anti-automorphisms on A_n:

$$\sim: A_n \to A_n : e_{j_1} \ldots e_{j_r} \to e_{j_r} \ldots e_{j_1}$$

and

$$- : A_n \to A_n : e_{j1} \ldots e_{j_r} \to (-1)^r e_{j_r} \ldots e_{j_1}$$

.

We shall denote $\sim (A)$ by $\tilde{A}$ for each $A \in A_n$, and $-(A)$ by $\bar{A}$. For $A, B \in A_n$ the real part of $\bar{A}\, B$ gives the euclidean inner product over R^{2^n} of A with B, and corresponds to the usual complex congugation in the case $n = 1$. Here we have used the basis we have introduced for A_n as an orthonormal basis for the space A_n.

We now proceed to introduce Vahlen matrices:

Definition 1 *A 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with A_n valued coefficents, is called a Vahlen matrix if (i) a, b, c and d are all products of vectors (ii) $\tilde{a}\, c, \tilde{c}\, d, \tilde{d}\, b, \tilde{b}\, a \in R^n$ (iii) $a\, \tilde{d} - b\, \tilde{c} \in R \backslash \{0\}$.*

It is a relatively simple matter using the pin group and the anti- automorphism $\sim$ to see that conditions (ii) and (iii) in the previous definition can be interchanged with other equivalent conditions, see for instance [R].

The set $V(n)$ of Vahlen matrices form a group under matrix multiplication. Moreover, [A], each Moebius transformation over the one point compactification $R^n \cup \{\infty\} \approx S^n$ can be expressed as $\phi(x) = y = (ax+b)(cx+d)^{-1}$, where a, b, c and d are coefficents of a Vahlen matrix. Also, [A], each Vahlen matrix gives rise to a Moebius transformation. To motivate oneself to see that Vahlen matrices can be used to describe Moebius transformations one can simply ask if a Moebius transformation can be expressed in the form $(ax+b)(cx+d)^{-1}$ using Clifford numbers. Conditions (i)-(iii) from definition 1 soon become apparant.

We shall also need the Dirac operator $D = \sum_{i=1}^n e_i \frac{\partial}{\partial x_i}$. Note that $D^2 = -\triangle_n$, where $\triangle_n$ is the Laplacian in R^n.

Definition 2 *Suppose that U is a domain in R^n and $f, g : U \to A_n$ are a pair of differentiable functions satisfying the equations $Df = 0$ and $gD = 0$, where $gD = \sum_{i=1}^n \frac{\partial g}{\partial x_i} e_i$. Then f is called a left monogenic function and g is called a right monogenic function.*

It is easy to see that f is left monogenic if and only if $\tilde{f}$ is right monogenic. An example of a function which is both left and right monogenic is the Clifford-Cauchy kernel $G(x) = \frac{1}{\omega_n} \frac{-x}{\|x\|^n}$, where ω_n is the surface area of the unit sphere in R^n. Other examples include the conjugate harmonic functions used in [StW]. The study, and application of monogenic functions is refered to as Clifford analysis. This analysis contains a natural generalization of many aspects of one variable complex analysis, see for instance [BDS]. The following two theorems are basic to this analysis.

Theorem 1 *Suppose that $f, g : U \to A_n$ are respectively left and right monogenic functions, and S is a compact, piecewise C^1 or Lipschitz, and orientable surface lying in U. Then $\int_S g(x)n(x)f(x)d\sigma(x) = 0$, where $n(x)$ is the outward pointing normal vector to S at x and σ is the Lebesgue measure on S.*

Theorem 2 *Suppose that f, S and U are as in theorem 1, and that S bounds a subdomain V of U. Then for each $y \in U$ we have*

$$f(y) = \int_S G(x-y)n(x)f(x)d\sigma(x)$$

.

Theorem 1 is a generalization of Cauchy's theorem while theorem 2 is a generalization of the Cauchy integral formula. The earliest known reference for theorem 2 is [Di], where it is established for the case $n = 3$. Theorem 1 is a straightforward consequence of Stokes' theorem, while theorem 2 follows from Stokes' theorem, the homogeneity of the kernel G and Kelvin inversion, so in essence they are deduced the same way as in the complex plane.

It is natural to ask what type of diffeomorphisms preserve solutions to the Dirac equation. It is a consequence of Stokes' theorem and theorem 1 that such a diffeomorphism would have to preserve the normality of the vector $n(x)$ for each $x \in S$ and each surface S of the type introduced in theorem 1. Consequently, the diffeomorphism would have to be conformal. A theorem due to Liouville [L] tells us that in dimensions 3 and above the only such transformations are Moebius transformations. In [R] we establish:

Theorem 3 *Suppose that $f : U \to A_n$ is a left monogenic function in the variable $y = \phi(x) = (ax + b)(cx + d)^{-1}$, then the function $J(\phi, x)f(\phi(x))$ is left monogenic in the variable x, where $J(\phi, x) = \widetilde{(cx + d)}\|cx + d\|^{-n}$.*

This result is in complete analogy to the same result over Minkowski space, see [JV]. Although most every result in Clifford analysis carries over from the setting described here to the setting of the Clifford algebra generated from the identity $x^2 = \|x\|^2$, with virtually no change to the method of proof, it is worth noting that the algebra A_n describes a slightly larger conformal group than the one obtained in the other setting. In [QR] and [C] it is noted that $\|cx + d\|^{-2n+2} = \pm J(\phi, x) \stackrel{\sim}{J} (cx + d)$. Now $\|cx + d\|^{-2n+2}$ is the Jacobian, over a surface, for the Moebius transformation $\phi(x) = (ax + b)(cx + d)^{-1}$. As a consequence of these observations we may deduce:

Theorem 4 *Suppose that S_1 and S_2 are two surfaces and that $\phi(S_1) = S_2$, where $\phi(x) = (ax+b)(cx+d)^{-1}$. If $\alpha(y) \in L^2(S_1, A_n)$, where $L^2(S, A_n)$ is the A_n valued space of square integrable functions defined on the surface S, then $J(\phi, x)\alpha(\phi(x)) \in L^2(S_2, A_n)$.*

In the previous theorem we are assuming that each surface is an orientable, locally Lipschitz continuous manifold of dimension $n-1$, locally embedded but possibly globally immersed in $R^n \cup \{\infty\}$. It should be noted that in much of what follows we make use of the fact that $L^2(S, A_n) = L^2(S\backslash E, A_n)$, where E is a subset of S with measure zero.

Theorem 4 establishes an isometry between the Hilbert module $L^2(S_1, A_n)$ and the Hilbert module $L^2(S_2, A_n)$.

In the next section we shall be working with the Fourier transform over $R^{n-1} = \{x_2e_2+\ldots+x_ne_n : x_2, \ldots, x_n \in R\}$. It is thus necessary to introduce the complexification, $A_n(C)$, of the algebra A_n. The anti-automorphisms $-$ and $\sim$ carry over linearly from the real algebra A_n to the complex algebra $A_n(C)$. We shall also need the real anti-automorphism

$$\dagger : A_n(C) \to A_n(C) : \dagger(Z) = \sum_{j_1,\ldots,j_r} z^*_{j_1\ldots j_r}\, \overline{e_{j_1} \ldots e_{j_r}},$$

where $Z = z_0 + \ldots + z_{1,\ldots,n}e_1 \ldots e_n \in A_n(C)$, summation is taken over the basis we introduced earlier for A_n, and $*$ denotes the usual complex conjugation. In future we will denote $\dagger(Z)$ by $Z^\dagger$.

Definition 3 *For a surface S a function $f : S \to A_n(C)$ is said to be square integrable if*

$$\| \int_S f(x) f^\dagger(x) d\sigma(x) \| < \infty.$$

For $f, g \in L^2(S, A_n(C))$ we may introduce the $A_n(C)$ valued quadratic form

$$< f, g >_S = \int_S f(x) g^\dagger(x) d\sigma(x).$$

This is a well defined inner product on $L^2(S, A_n(C))$, and the identity component of this quadratic form gives the usual Hermitean inner product, $<,>_{S,H}$, on $L^2(S, A_n(C))$. As $\bar{J} = J^\dagger$ then it is straightforward to deduce:

Proposition 1 *If $S_1 = \phi(S_2)$ for some Moebius transformation $\phi(x) = (ax+b)(cx+d)^{-1}$ then:*
(1) $\alpha(y) \in L^2(S_1, A_n(C))$ if and only if $J(\phi, x)\alpha(\phi(x)) \in L^2(S_2, A_n(C))$.
(2) $< f(y), g(y) >_{S_1,H} = < J(\phi, x) f(\phi(x)), J(\phi, x) g(\phi(x)) >_{S_2,H}$.

The Fourier Transform

In the previous section we illustrated that there is an isometry between $L^2(S_1, A_n(C))$ and $L^2(S_2, A_n(C))$ provided that the surfaces, S_1 and S_2 are conformally equivalent. Also, the Fourier transform provides an isometry between $L^2(R^{n-1}, A_n(C))$ and itself. We shall be interested here in looking at the Fourier transformation combined with the Cayley transformation, $\phi(x) = (e_1 x + 1)(x + e_1)^{-1}$. This transform takes R^{n-1} to the sphere, S^{n-1} and upper half space, $R^{n,+} = \{x_1 e_1 + \ldots + x_n e_n \in R^n : x_1 > 0\}$ to the unit disc, $D = \{x \in R^n : \|x\| < 1\}$.

For $f \in L^2(R^{n-1}, A_n(C))$ we may consider the Fourier transform

$$c(n) \int_{R^{n-1}} e^{i<y,v>} f(x) dx^{n-1},$$

where $c(n)$ is a normalization constant which depends on dimension, and y, $v \in R^{n-1}$.

Under the Cayley transformation this integral becomes

$$c(n) \int_{S^{n-1}} e^{i<\psi(x),\psi(u)>} \bar{J}(\psi, x) J(\psi, x) f(\psi(x)) d\sigma(x),$$

where ψ is the inverse of the Cayley transformation. Consequently, we may define the Fourier transform over the sphere to be

$$F_{S^{n-1}}(\alpha)(u) = \int_{S^{n-1}} E_{S^{n-1}}(x, u) \alpha(x) d\sigma(x), \tag{2}$$

where

$$E_{S^{n-1}} = c(n) J(\psi, u) e^{-i<\psi(x),\psi(u)>} \bar{J}(\psi, x)$$

and $\alpha(x) \in L^2(S^{n-1}, A_n(C))$. Consequently we obtain:

Theorem 5 *The transformation*

$$F_{S^{n-1}} : L^2(S^{n-1}, A_n(C)) \to L^2(S^{n-1}, A_n(C))$$

is an isometry.

We shall call the transform $F_{S^{n-1}}$ the spherical Fourier transform. It is worth noting that unlike the classical Fourier transform over R^{n-1} this transform is complex Clifford algebra valued, and its construction makes explicit use of the Clifford algebra.

In order to make use of this spherical Fourier transform it is worth looking at the analogue of the convolution operator over the sphere. To do this let us first look at the standard convolution operator on R^{n-1}.

Formally for $f, g : R^{n-1} \to A_n(C)$ the convolution $f *_{R^{n-1}} g$ is defined to be $\int_{R^{n-1}} f(y-v)g(v)dv^{n-1}$. On noting that

$$\psi(u) - \psi(x) = (x+e_1)^{-1}(u-x)(u+e_1)^{-1},$$

then under the inverse Cayley transform this integral becomes

$$\int_{S^{n-1}} f((x+e_1)^{-1}(u-x)(u+e_1)^{-1})g(\psi(u))\|u+e_1\|^{-2n+2}d\sigma(u).$$

Consequently, for $\alpha, \beta : S^{n-1} \to A_n(C)$ we can formally define their convolution to be

$$\int_{S^{n-1}} J(\psi, x)h_\alpha((x+e_1)^{-1}(u-x)(u+e_1)^{-1}))J(\psi, u)\beta(u)d\sigma(u),$$

where $h_\alpha(\psi(x)) = J(\psi, x)^{-1}\alpha(x)$.

We shall denote this convolution over the sphere by $\alpha *_{S^{n-1}} \beta$. We denote the spherical Fourier transform, $F_{S^{n-1}}(\alpha)$, of a function $\alpha : S^{n-1} \to A_n(C)$, by $\overset{\star}{\alpha}$, whenever the transform is well defined.

Clearly,

$$F^{-1}_{S^{n-1}}(\overset{\star}{\alpha}) = \alpha,$$

where the inverse spherical transform, $F^{-1}_{S^{n-1}}$ is defined via the formula

$$F^{-1}_{S^{n-1}}(\overset{\star}{\alpha}) = d(n)J(\psi, x)\int_{S^{n-1}} e^{-i<\psi(x),\psi(u)>}\ \bar{J}\ (\psi, u)\alpha(u)d\sigma(u),$$

with $d(n)$ a dimensional constant.

Also, it is well known that

$$F_{R^{n-1}}(f \star_{R^{n-1}} g) = F_{R^{n-1}}(f)F_{R^{n-1}}(g).$$

It follows that

$$F_{S^{n-1}}(\alpha \star_{S^{n-1}} \beta) = F_{S^{n-1}}(\alpha)J(\psi,)^{-1}F_{S^{n-1}}(\beta),$$

whenever these terms are well defined.

The previous formula defines the product

$$\alpha ._{S^{n-1}}\beta = \alpha J(\psi,)^{-1}\beta$$

for functions $\alpha, \beta : S^{n-1} \to A_n(C)$. It is a simple matter to verify that this product is associative. Consequently, it may be observed that the spherical convolution $\star_{S^{n-1}}$ is also an associative 'convolution'.

The function $e^{i<y,v>}$ monogenically extends to the function $e^{i<y,v>}e^{iy_1e_1v}$ in a neighbourhood of R^{n-1} within R^n. Similarly, the function $e^{-i<y,v>}$ monogenically extends to the function $e^{-i<y,v>}e^{-iy_1e_1v}$ on a neighbourhood of R^{n-1} within R^n. Under the Cayley map these monogenic functions transform themselves to

$$J(\psi, x`)e^{i<\psi(x),\psi(u)>}e^{i(1-x_1)(1+x_1)^{-1}e_1\psi(u)}\tilde{J}(\psi, u),$$

and

$$J(\psi, x`)e^{-i<\psi(x),\psi(u)>}e^{-i(1-x_1)(1+x_1)^{-1}e_1\psi(u)}\tilde{J}(\psi, u),$$

where $x` \in R^n$.

These functions are defined in a neighbourhood of $S^{n-1}\backslash\{-e_1\}$, within R^n, and they are also left monogenic functions.

In [S2, Mc] and elsewhere it is pointed out that $F_{R^{n-1}}(Ge_1) = \frac{-1}{2}ie_1\frac{\zeta}{\|\zeta\|}$, where $\zeta \in R^{n-1}$ and G is the Clifford-Cauchy kernel introduced earlier. It follows that

$$F_{S^{n-1}}(Gx) = \frac{-1}{2}J(\psi, u)ie_1\frac{\psi(u)}{\|\psi(u)\|}.$$

Using the fact that $G(\psi(x) - \psi(u)) = J(\psi, x)^{-1}G(x-y)\tilde{J}(\psi, u)^{-1}$, see for instance [PQ], one can see that for each $\alpha \in L^2(S^{n-1}, A_n(C))$ one has that

$$F_{S^{n-1}}(P.V.\int_{S^{n-1}} G(x-u)u\alpha(u)d\sigma(u)) = \frac{-1}{2}J(\psi, x)ie_1\frac{\psi(x)}{\|psi(x)\|}J(\psi, u)^{-1}\overset{\star}{\alpha}(u)$$

$$= \frac{-1}{2}i\,\widetilde{\psi(x)}\,\|\psi(x)\|^{-1}e_1\overset{\star}{\alpha}.$$

As $\widetilde{\psi(x)} = \psi(x)$ then the last expression becomes

$$\frac{-1}{2}ie_1\frac{\psi(x)}{\|\psi(x)\|}\overset{\star}{\alpha}.$$

Consequently, the spherical Fourier transform of the Plemelj-Sokhotzkij formulae over the sphere,

$$F_{S^{n-1}}(\pm\frac{1}{2} + P.V.\int_{S^{n-1}} G(v-u)v\alpha(v)d\sigma(v)),$$

are $\frac{1}{2}(1 \pm e_1\frac{\psi(x)}{\|\psi(x)\|})\overset{\star}{\alpha}(x)$.

From [S1] it may be noted that

$$L^2(S^{n-1}, A_n(C)) = H^2(D, A_n(C)) \oplus H^2(R^n\backslash cl(D)),$$

where D is the open unit disc and $cl(D)$ is its topological closure. Moreover, $H^2(D, A_n(C))$ is the Hardy space of $A_n(C)$ valued left monogenic functions on D which extend continuously in the L^2 sense to the boundary, S^{n-1} of D, and $H^2(R^n\backslash cl(D), A_n(C))$ is the Hardy space of $A_n(C)$ valued left monogenic functions defined on $R^n\backslash cl(D)$ which extend continuously in the L^2 sense to S^{n-1}.

In [QR] the Cayley transformation is used to show that this decomposition of the Hilbert space $L^2(S^{n-1}, A_n(C))$ is canonically equivalent to the decomposition

$$L^2(R^{n-1}, A_n(C)) = H^2(R^{n,+}, A_n(C)) \oplus H^2(R^{n,-}, A_n(C)),$$

where these spaces are the Hardy 2 spaces of left monogenic functions in upper and lower half space in R^n and extend continuously in the L^2 sense to their common boundary, R^{n-1}.

In [Mc] and elsewhere Fourier analysis is used to show that the Plemelj- Sokhotzjy formulae are in fact projection operators of $L^2(R^{n-1}, A_n(C))$ onto $H^2(R^{n,\pm}, A_n(C))$ respectively. The following result follows automatically:

Theorem 6 *The operators*

$$\Lambda^+_{S^{n-1}} : L^2(S^{n-1}, A_n(C)) \to H^2(D, A_n(C)) : \overset{\star}{\alpha}(x) \to \frac{1}{2}(1 + ie_1 \frac{\psi(x)}{\|\psi(x)\|}) \overset{\star}{\alpha}(x)$$

and

$$\Lambda^-_{S^{n-1}} : L^2(S^{n-1}, A_n(C)) \to H^2(R^n \backslash cl(D)) : \overset{\star}{\alpha}(x) \to \frac{1}{2}(1 - ie_1 \frac{\psi(x)}{\|\psi(x)\|}) \overset{\star}{\alpha}(x)$$

are projection maps.

In [S2, LMcQ] it is noted that $ie_1 v \frac{1}{2}(1 + \frac{ie_1 v}{\|v\|}) = \|v\| \frac{1}{2}(1 + \frac{ie_1 v}{\|v\|})$. Consequently, $i\frac{1}{2}(1 + \frac{ie_1 v}{\|v\|}) = v^{-1}\|v\|\frac{1}{2}(1 + \frac{ie_1 v}{\|v\|})$. Consequently, [S2, LMcQ], the monogenic function

$$e^{-i<y,v>} e^{-y_1 e_1 v} \frac{1}{2}(1 + \frac{ie_1 v}{\|v\|}) = e^{-i<y,v>} e^{-y_1\|v\|} \frac{1}{2}(1 + \frac{ie_1 v}{\|v\|})$$
$$= e^{-v^{-1}\|v\|<y,v>} e^{-y_1\|v\|} \frac{1}{2}(1 + \frac{ie_1 v}{\|v\|})$$

is well defined on upper half space, and similarly the monogenic function

$$e^{-i<y,v>} e^{-y_1 e_1 v} \frac{1}{2}(1 - \frac{ie_1 v}{\|v\|}) = e^{-i<y,v>} e^{y_1\|v\|} \frac{1}{2}(1 - ie_1 v\|v\|)$$
$$= e^{-v^{-1}\|v\|<y,v>} e^{-y_1\|v\|} \frac{1}{2}(1 - \frac{ie - 1v}{\|v\|})$$

is well defined on lower half space. Consequently, we obtain:

Theorem 7 *(A): The function*

$$E_i(x, u) = J(\psi, x`) e^{-i<\psi(x),\psi(u)> + (x_1+1)(x_1-1)^{-1}\|\psi(u)\|} \frac{1}{2}(1 + \frac{e_1\psi(u)}{\|\psi(u)\|}) \tilde{J}(\psi, u)$$

is well defined on $D \times S^{n-1} \backslash \{-e-1\}$, and monogenic on D.
(B): The function

$$E_o(x, u) = J(\psi, x`) e^{-i<\psi(x),\psi(u)> - (x_1+1)(x_1-1)^{-1}\|\psi(x)\|} \frac{1}{2}(1 - \frac{ie_1\psi(u)}{\|\psi(u)\|} \tilde{J}(\psi, u)$$

is well defined on $(R^n \backslash cl(D)) \times S^{n-1} \backslash \{-e_1\}$, and monogenic on $R^n \backslash cl(D)$.
(C): The transform

$$E_i : H^2(D, A_n(C)) \to H^2(D, A_n(C)) : E_i(\omega) = \int_{S^{n-1}} E_i(x, u)\omega(u) d\sigma(u)$$

where $\omega \in H^2(D, A_n(C))$, *is an isomorphism.*
(D): The transform

$$E_o : H^2(R^n\backslash cl(D), A_n(C)) \to H^2(R^n\backslash cl(D), A_n(C)) :$$

$$E_o(\kappa) = \int_{S^{n-1}} E_o(x,u)\kappa(u)d\sigma(u)$$

where $\kappa \in H^2(R^n\backslash cl(D), A_n(C))$, *is an isomorphism.*

In [LMcQ] it is observed that

$$\int_{R^{n-1}} e^{-i<y,v>-y_1e_1v}\frac{1}{2}(1-\frac{ie_1v}{\|v\|})d\sigma(v)$$

$$= \int_{S^{n-2}}(\int_0^\infty e^{-R(y_1e_1w-i<y,w>}R^{n-2}dR\frac{1}{2}(1+iw)dS^{n-2},$$

where $w = \frac{v}{\|v\|}$. The radial integral in this expression may be observed to be the Laplace transform of R^{n-2}. In [LMcQ] it is shown that on evaluating this simple Laplace transform one obtains the plane wave decomposition formula for the Clifford-Cauchy kernel, G, in upper half space. One may easily, and similarly, obtain the similar formula for lower half space. These formulae first appeared in [S3]. On applying the inverse Cayley transformation one obtains:

$$G(x+e_1) = c(n)G(x-e_1)$$

$$\times \int_{S^{n-2}}((x_1+1)(x_1-1)^{-1}e_1\psi(s) - i < \psi(x), \psi(s)>)^{-n+1}\xi(s)e_1dS^{n-2},$$

where $x \in D$, $c(n)$ is a dimensional constant, $\psi(s) = w$ and $\xi(s) = \frac{1}{2}(1+\psi(s))$.

Similarly, by taking the inverse Cayley transformation of lower half space one obtains

$$G(x+e_1) = c(n)G(x-e_1)$$

$$\times \int_{S^{n-2}}((x_1+1)(x_1-1)^{-1}e_1\psi(x) - i < \psi(x), \psi(s)>)^{-n+1}\rho(s)e_1dS^{n-2},$$

where $x \in R^n\backslash cl(D)$, $c(n)$ is a dimensional constant, again $\psi(s) = w$ and $\rho(s) = \frac{1}{2}(1 - \psi(s))$.

In [LmcQ] and [LMcS] the authors consider the sector domain

$$S_\mu = \{y \in R^n :| y_1 < \tan\mu\|y_2e_2 + \ldots + y_ne_n\|, 0 < \mu < \frac{\pi}{2}\}.$$

This domain is bounded by two cones, a cone C_μ^+ sitting in upper half space, and a cone C_μ^- sitting in lower half space. These cones intersect only at the origin. The domain sitting above C_μ^- is denoted by S_μ^+ while the domain sitting below C_μ^+ by S_μ^-. Moreover, $C_\mu^+ \cap C_\mu^- = C_\mu$. The inverse Cayley transformation of these cones and all of these domains is worked out in [QR]. In [LMcQ, LMcS] a pair of functions (T;
theta) are introduced, where T is a right monogenic function on the sector domain S_μ and satisfies $\|T(x)\| < C(T)\|x\|^{-n+1}$, for some constant $C(T) \in R^+$. Moreover, $\theta : R^+ \to A_n(C)$ is a bounded and continuous function satisfying

$$\theta(R) - \theta(r) = \int_{A(r,R)\cap R^{n-1}} T(y)e_1dy^{n-1},$$

where $A(r,R)$ is the annulus, or spherical shell, centred at the origin of radii, R and r respectively. Again the transformations T_ψ and θ_ψ of T and θ under the inverse Cayley transformation are worked out in [QR]. In [LMcS] it is noted that $T = T_1 + T_2$, where T_1 is a right monogenic function on S_μ^+ and satisfies $\|T_1(y)\| < C(T_1)\|y\|^{-n+1} and T_2$ is a right monogenic function on S_μ^- and also satisfies $\|T_2(y)\| < C(T_2)\|y\|^{-n+1}$. Moreover, [LMcQ], $\theta = \theta_1 + \theta_2$, where $\theta_j : R^+ \to A_n(C)$ for $j = 1, 2$, and

$$\theta_j(R) - \theta_j(r) = \int_{A(R,r)\cap R^{n-1}} T_j(y)e_1 dy^{n-1}.$$

The conformal covariance of these decompositions are described in [QR].

In [LMcQ] the projection operators described here over R^{n-1}, and the Clifford-Fourier transforms described earlier over $H^2(R^{n,+}, A_n(C))$ and
$H^2(R^{n,-}, A_n(C))$ are employed to show that for each such function T Fourier transforms to a bounded holomorphic function $\aleph_T$ defined on the sector domain, $S_\mu(C^{n-1})$, defined by

$$\{u + iv \in C^{n-1} : u, v \in R^{n-1}, (u + iv)^2 \in C\backslash(0, \infty), \|v\| < Re((u + iv)^2)^{\frac{1}{2}} \tan \mu\}.$$

Moreover, [LMcQ], each bounded holomorphic function defined on the domain $S_\mu(C^{n-1})$ is the Fourier transform of such a right monogenic function T.

The set $\psi(S_\mu(C^{n-1}))$ is an open subset of the complex sphere $S_C^{n-1} = \{\zeta \in C^n : \zeta^2 = -1\}$. Moreover, it may now be observed that the spherical Fourier transform of the right monogenic function T_ψ is a holomorphic function, η, defined on $\psi(S_\mu(C^{n-1}))$ and $\|\eta(\zeta)\| < C(\eta)\|\zeta\|^{-n+1}$, where $C(\eta) \in R^+$. Moreover, the inverse spherical Fourier transform gives such a right mongenic function, T_ψ.

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