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COMPLEX CONTACT HOMOGENEOUS SPACES AND QUATERNION-KAHLER SYMMETRIC SPACES

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Introduction.

1. The main purpose of this talk is to recall, by a new method connected with the notion of coadjoint representation, the determination of the compact complex contact homogeneous spaces and the connection of these spaces with the compact quaternion-Kähler symmetric spaces (Boothby and Wolf).

The determination of the proper **compact** complex homogeneous spaces has being given, long time ago, by Boothby under the hypothesis that these spaces are **simply connected** ; we prove that this hypothesis is unnecessary and that these spaces are a priori Kähler homogeneous spaces and simply connected.

2. The main technical instrument used here is the Schouten bracket on the skew symmetric real or complex contravariant p-tensors (or shortly p-tensors). The corresponding theory has been developed by Nijenhuis [9].

We introduce the notions of complex Poisson and Jacobi manifolds. A complex Jacobi manifold is foliated (in the generalized sense of Sussmann-Stefan) by complex contact manifolds or complex locally conformal symplectic (clcs) manifolds.

Let G be a connected complex Lie group of complex dimension $n, \mathcal{G}$ its Lie algebra, $\mathcal{G}^*$ the dual space. From the coadjoint representation of G on $\mathcal{G}^*$, we deduce the notion of quotient coadjoint map on the associated complex projective space PC^{n-1}. A complex Jacobi structure is defined naturally on such a space. Each proper complex contact homogeneous space G/H is a covering of a leaf S of odd complex dimension of this Jacobi structrure, with correspondance of the contact structure of W and S.

I. Complex Poisson structures and complex Jacobi structures.

3. Complex Poisson manifold.

Let W be a connected complex manifold, of complex dimension m and of complex structure operator I. We set $N^c(W) = C^\infty(W; C)$. Let x^α be a complex chart of W, $\{x^i\} = \{x^\alpha\}, x^{\bar{\alpha}} (i = 1, \ldots, 2m\,; \bar{\alpha} = \alpha + m)$ the associated complete chart.

a) Let $\Lambda^{(2,0)}$ be a complex 2-tensor of type $(2,0)$. We set $\bar{\Lambda}^{(2,0)} = \Lambda^{(0,2)}$ where $\bar{\cdot}$ denotes the complex conjugation. A complex Poisson structure is defined on W by a 2-tensor $\Lambda^{(2,0)}$ satisfying

$$[\Lambda^{(2,0)}, \Lambda^{(2,0)}] = 0 \tag{3.1}$$

and

$$[\Lambda^{(0,2)}, \Lambda^{(2,0)}] = 0. \tag{3.2}$$

In particular (3.2) is satisfied if $\Lambda^{(2,0)}$ is a holomorphic 2-tensor.

b) The real 2-tensor $\Lambda = \Lambda^{(2,0)} + \Lambda^{(0,2)}$ defines on W a real Poisson structure with characteristic field P and thus a Sussmann foliation denoted again by P . If $M\Lambda = i\Lambda^{(2,0)} - i\Lambda^{(0,2)}$ this real 2-tensor defines again a Poisson structure of field P. It follows that I acts on P.

Let S be the leaf of (W, Λ) passing through $x \in W$; I induces on S an integrable almost complex structure, that is a complex structure and S is a complex manifold such that $dim_c S = 2p$, the tensor $\Lambda^{(2,0)}$ defining on S a transitive complex Poisson structure, that is a complex symplectic structure ; by inversion of $\Lambda^{(2,0)}|_S$, we obtain on S a holomorphic closed 2-form F_S such that $F_S^p \neq 0$ everywhere. One has

Proposition. - *A complex Poisson manifold $W, \Lambda^{2,0}$ admits a (generalized) foliation for which the leaves have complex symplectic structures defined by $\Lambda^{2,0}$.*

4. Complex Jacobi manifolds. [8]

a) Let $\pi : K \to W$ a holomorphic fiber bundle in complex straight lines on W, $\Gamma(K)$ the space of the sections of K. We denote by σ_U a holomorphic local section of K without zero over a domain U of W. If $s \in \Gamma(K)$, one has $s|_U = u_U \sigma_U$ (with $u_U \in N^C(U)$). If V is another domain of W such that $U \cap V \neq \emptyset$, we have for another section σ_V

$$\sigma_V = h_{UV} \sigma_U$$

where holomorphic functions h_{UV} on $U \cap V$ are transition functions of K and $u_V = h_{UV}^{-1} u_U$. A point $\tilde{x}$ of K such that $x = \pi\tilde{x} \in U$ may be represented by $\tilde{x} = x_U \sigma_U(x)$, where $x_U \in \mathbf{C}$. For V such that $\tilde{x} \in \pi^{-1}(U \cap V)$, one has $x_V = h_{UV}^{-1}(x) x_U$.

If x^α is a complex chart of W of domain U, it defines a complex chart $\{x^0 = x_U, x^\alpha\}$ of K of domain $\pi^{-1}(U)$. If $\{x^{0'} x^{\beta'}\}$ is a similar chart for V, one has on $U \cap V$

$$x^{0'} = h_{UV}^{-1} x^0.$$

The canonical holomorphic vector field $\tilde{Z}^{1,0}$ of K is the generator of the complex Lie algebra of dim .1 defined by the homotheties with complex scalar of K. It has components $\tilde{Z}^0 = -x^0, \tilde{Z}^\alpha = 0$ in the chart in consideration. We set $\tilde{Z}^{(0,1)} = \bar{\tilde{Z}}^{(1,0)}$.

Let $K_0 = \tilde{W}$ be a connected component of K **without its zero section.** Associate to $s \in \Gamma(K_0)$ the function $\tilde{u}$ defined on K_0 by :

$$\tilde{u}|_{\pi^{-1}(U)} = (x^0)^{-1} \pi^* u_U, \tag{4.1}$$

where $\tilde{u} \in N^c(\tilde{W}) = \tilde{N}^c$; $\tilde{u}$ satisfies

$$\mathcal{L}(\tilde{Z}^{1,0})\tilde{u} = \tilde{u} \qquad \mathcal{L}(\tilde{Z}^{0,1})\tilde{u} = 0. \tag{4.2}$$

Let $\tilde{N}_1^c$ be the subspace of $\tilde{N}^c$ for which the $\tilde{u}$ satisfy (4.2). One has an isomorphism of complex spaces $\tau : s \in \Gamma(K) \to \tilde{u} \in \tilde{N}_1^c$.

b) Consider on $K_0 = \tilde{W}$ a **complex Poisson structure** $\tilde{\Lambda}^{(2,0)}$ such that

$$[\tilde{\Lambda}^{(2,0)}, \tilde{\Lambda}^{(2,0)}] = 0 \qquad [\tilde{\Lambda}^{(0,2)}, \tilde{\Lambda}^{(2,0)}] = 0 \tag{4.3}$$

satisfying the homogeneity conditions

$$\mathcal{L}(\tilde{Z}^{(1,0)})\tilde{\Lambda}^{(2,0)} = -\tilde{\Lambda}^{(2,0)} \qquad \mathcal{L}(\tilde{Z}^{(0,1)})\tilde{\Lambda}^{(2,0)} = 0 \tag{4.4}$$

For $\tilde{\Lambda}^{(2,0)}$, we set over $\pi^{-1}(U)$

$$\tilde{\Lambda}^{(2,0)}|_{\pi^{-1}(U)} = x^0 \tilde{\Gamma}_U^{(2,0)} \qquad \tilde{\Gamma}_U^{(2,0)} = \Lambda_U^{(2,0)} + \tilde{Z}^{(1,0)}|_{\pi^{-1}(U)} \wedge E_U^{(1,0)}$$

where, according to the homogeneity conditions, $\Lambda_U^{(2,0)}$, $E_U^{(1,0)}$ are projectable on U ; the projections are denoted by the same notation. We have on $U \cap V$

$$\Lambda_V^{(2,0)} = h_{UV} \Lambda_U^{(2,0)} \qquad E_V^{(1,0)} = h_{UV} E_U^{(1,0)} + [\Lambda_U^{(2,0)} \ , \ h_{UV}]. \tag{4.5}$$

The relations (4.3) may be translated by

$$[\Lambda_U^{(2,0)}, \Lambda_U^{(2,0)}] = 2E_U^{(1,0)} \wedge \Lambda_U^{(2,0)} \qquad [E_U^{(1,0)}, \Lambda_U^{(2,0)}] = 0 \tag{4.6}$$

and by

$$[\Lambda_U^{(2,0)}, \Lambda_U^{(2,0)}] = 0 \qquad [E_U^{(0,1)}, \Lambda_U^{(2,0)}] = 0 \qquad [E_U^{(0,1)}, E_U^{(1,0)}] = 0. \tag{4.7}$$

We will see that the interpretation of (4.7) is simple. Conversely if a system $(\Lambda_U^{(2,0)}\ ,\ E_U^{(1,0)})$ associated to a covering of W satisfies (4.5), (4.6), (4.7) for holomorphic functions h_{UV} without zeros, we may construct the initial situation. If such is the case, we have defined on W **a complex Jacobi structure** which determines on K_0 **an associated complex Poisson structure.**

A complex Jacobi structure defines on $\Gamma(K)$ a structure of complex Lie algebra given by the following bracket : if $s, t \in \Gamma(K), [s,t]$ is given by

$$[s,t]|_U = \{i(\Lambda_U^{(2,0)})(du_U \wedge dv_U) + i(E_U^{(1,0)})(u_U dv_U - v_U du_U)\}\sigma_U \tag{4.8}$$

The isomorphism between $\Gamma(K)$ and $\tilde{N}_1^c$ gives an isomorphism of the Lie algebra $(\tilde{N}_1^c, \{\ \})$ with the Lie algebra $(\Gamma(K), [\ ])$

$$\{\tilde{u},\ \tilde{v}\} = i(\tilde{\Lambda}^{(2,0)})(d\tilde{u} \wedge d\tilde{v}) \tag{4.9}$$

c) At one section s corresponds **complex Hamiltonian field** $X_s^{(1,0)}$ given by

$$X_s^{(1,0)}|_U = u_U E_U^{(1,0)} + [\Lambda_U^{(2,0)},\ u_U] \qquad \text{(see (4.5))} \tag{4.10}$$

If $X_s^{(0,1)} = \bar{X}_s^{(1,0)}$ consider the real field $X_s = X_s^{(1,0)} + X_s^{(0,1)}$. The values at x of the fields X_s determine a real plane P_x. To the complex Jacobi structure we associate the field P of the real planes defined by the P_x, which is said to be the characteristic field of the structure.

We will show that P determines a foliation of W.

A complex Hamiltonian field of $(\tilde{W}, \tilde{\Lambda}^{(2,0)})$ associated to $\tilde{u} \in \tilde{N}^c$ is given by $\tilde{X}_{\tilde{u}}^{(1,0)} = [\tilde{\Lambda}_{\tilde{s}}^{(2,0)}, \tilde{u}]$. If $\tilde{X}_{\tilde{u}}^{(0,1)} = \bar{\tilde{X}}_{\tilde{u}}^{(1,0)}$, the real fields $\tilde{X}_{\tilde{u}} = \tilde{X}_{\tilde{u}}^{(1,0)} + \tilde{X}_{\tilde{u}}^{(0,1)}$ generate the characteristic field $\tilde{P}$ of $(\tilde{W}, \tilde{\Lambda}^{(2,0)})$ and it is easy to verify that $\pi_* \tilde{P} = P$. If $s \in \Gamma(K), \tilde{u} = \tau s$, one has $\pi_* \tilde{X}_{\tilde{u}}^{(1,0)} = X_s^{(1,0)}$ and thus $\pi_* X_{\tilde{u}} = X_s$. If $s, t \in \Gamma(K)$ and $\tilde{u} = \tau s$, $\tilde{v} = \tau t$, $[\tilde{X}_{\tilde{u}}, \tilde{X}_{\tilde{v}}]$ belongs to $\tilde{P}$ and, by projection $[X_s, X_t]$ belongs to P. The field P being invariant by the fields X_s , it follows from the Theorem of Sussmann [11] that P defines a foliation of W. If $\tilde{x} \in \tilde{W}$ and $x = \pi\tilde{x} \in W$ one has for the leaves $\tilde{S}(\tilde{x})$ of $\tilde{P}$ and $S(x)$ of P : $\pi\tilde{S}(\tilde{x}) = S(x)$. If V is a vector tangent to a leaf S of P, IV is also tangent. Therefore I induces on S a complex structure, the complex dimension of S being even or odd. The restriction to S induces **a transitive complex Jacobi structure.** For such a structure, the relations (4.7) express that $\Lambda_U^{(2,0)}$, $E_U^{(1,0)}$ are holomorphic. One has the following theorem.

Theorem. - *A complex Jacobi manifold admits a foliation in leaves which are complex manifolds on which the Jacobi structure induces by restriction transitive complex Jacobi structures. For a leaf of odd complex dimension, the induced Jacobi structure is a complex contact structure. For a leaf of even complex dimension, the induced Jacobi structure is a complex locally conformally symplectic (l.c.s) structure.*

II. Quotient coadjoint map of a complex Lie algebra.

5. Coadjoint map and Poisson structure.

Let $\mathcal{G}$ be a complex Lie algebra of complex dimension $n, \mathcal{G}^*$ the dual space, $\Xi^{(1,0)} : \xi \in \mathcal{G}^* \to \xi \in \mathcal{G}^*$ the vector field generator of the group Δ of the complex homotheties of $\mathcal{G}^*$. $\langle , \rangle$ is the bilinear from of duality. We introduce on $\mathcal{G}^*$ the 2-tensor $\tilde{\Lambda}^{(2,0)}$ given by

$$i(\tilde{\Lambda}^{(2,0)})(X \wedge Y) = \langle\ \Xi^{(1,0)}, [X,Y]\rangle \qquad (X, Y \in \mathcal{G}) \tag{5.1}$$

The 2-tensor $\tilde{\Lambda}^{(2,0)}$ is holomorphic on $\mathcal{G}^*$ and , according to its definition, satisfies

$$[\tilde{\Lambda}^{(2,0)}, \tilde{\Lambda}^{(2,0)}] = 0 \qquad [\tilde{\Lambda}^{(2,0)}, \bar{\tilde{\Lambda}}^{(2,0)}] = 0 \tag{5.2}$$

If we set $\Xi^{(0,1)} = \bar{\Xi}^{(1,0)}$, it is easy to verify

$$\mathcal{L}(\Xi^{(1,0)})\ \tilde{\Lambda}^{(2,0)} = -\tilde{\Lambda}^{(2,0)} \qquad \mathcal{L}(\bar{\Xi}^{(1,0)})\ \tilde{\Lambda}^{(2,0)} = 0 \qquad (\mathcal{L} = \text{Lie derivative}) \tag{5.3}$$

Therefore $\tilde{\Lambda}^{(2,0)}$ defines on $\mathcal{G}^*$ a complex Poisson structure satisfying with respect to $\Xi^{(1,0)}$ the homogeneity conditions. Let G be a connected complex Lie group admitting $\mathcal{G}$ as Lie algebra. The group G acts naturally on $\mathcal{G}^*$ by its **coadjoint representation** $g \in G \to Ad^*(g^{-1})$. To an element X of $\mathcal{G}$ corresponds the element $\tilde{X}^{(1,0)}$ of $Ad^*\mathcal{G}$ given by

$$\tilde{X}^{(1,0)} = [\tilde{\Lambda}^{(2,0)}, \tilde{u}_X] \tag{5.4}$$

where $\tilde{u}_X = -\langle \Xi^{(1,0)}, X\ \rangle$; $\tilde{\Lambda}^{(2,0)}$ and $\Xi^{(1,0)}$ are invariant under $Ad^*\mathcal{G}$. The leaves of $(\mathcal{G}^*\ , \tilde{\Lambda}^{(2,0)})$ are the orbits of Ad^*G which are Hamiltonian complex symplectic homogeneous spaces. We have here a version of the famous theorem of Kirillov-Kostant-Souriau.

6. Quotient coadjoint map.

a) **A complex ray** of $\mathcal{G}^*$ is a complex straight line passing through 0 and without $\{0\}$. Consider $\mathcal{G}_0^* = \mathcal{G}^* - \{0\}$ with the field $\Xi^{(1,0)}$ of the homotheties and the 2-tensor $\tilde{\Lambda}^{(2,0)}$. The quotient space of $\mathcal{G}_0^*$ by the group Δ is isomorphic to the complex projective space $(PC)^{n-1}$. Conversely $\mathcal{G}_0^*$ may be considered as a holomorphic bundle space in rays over $(PC)^{n-1}$. We denote by p the projection $\xi \in \mathcal{G}_0^* \to \zeta \in (PC)^{n-1}$.

Let σ_U be a holomorphic section of $\mathcal{G}_0^*$ over the domain U of $(PC)^{n-1}$. If $\xi \in \mathcal{G}_0^*$ and $\zeta = p\xi$, we set $\xi = z^0\sigma_U(\zeta)$. If $\{z^\lambda\}$ $(\lambda, \mu, \ldots = 1, \ldots, n-1)$ is a complex chart of $(PC)^{n-1}$ of domain U, $(^0, z^\lambda)$ is a complex chart of $\mathcal{G}_0^*$ for $p^{-1}(U)$ and $\Xi^{(1,0)}$ has for components in this chart $\Xi^0 = -z^0, \Xi^\lambda = 0$.

According to (5.2), (5.3), $\tilde{\Lambda}^{(2,0)}$ defines by p a complex Jacobi structure on $(PC)^{n-1}$. The elements $\Lambda_4^{(2,0)}$, $E_4^{(1,0)}$ determining for U this structure are holomorphic on U. One has

Proposition. - *Let $(PC)^{n-1}$ the complex projective space defined by the quotient of $\mathcal{G}_0^*$ by the group of the complex homotheties. The canonical complex Poisson structure of $\mathcal{G}_0^*$ determines on $(PC)^{n-1}$ a holomorphic Jacobi structure.*

b) According to (5.4), we have associate to $X \in \mathcal{G}$ the function $\tilde{u}_X = -\langle \Xi^{1,0}, X\rangle$ homogeneous of degree 1 with respect to $\Xi^{(1,0)}$. We obtain on each domain U of $(PC)^{n-1}$ a local holomorphic function u_{UX} defined by $\tilde{u}_X = (z^0)^{-1}p^* u_{UX}$. The complex Hamiltonian field $\tilde{X}^{1,0}$ given by (5.4) gives by projection by p the Hamiltonian field of the Jacobi structure of $(PC)^{n-1}$

$$\hat{X}^{(1,0)}|_U = u_{UX}E_U^{(1,0)} + [\Lambda_0^{(2,0)}, u_{UX}] \tag{6.1}$$

that is holomorphic. It is an infinitesimal automorphism of the Jacobi structure

$$\begin{aligned}\mathcal{L}(\hat{X}^{(1,0)})\Lambda_U^{(2,0)} &= a_{UX}\Lambda_U^{(2,0)} \ , \ \mathcal{L}(X^{(1,0)})E_U^{(1,0)} \\ &= a_{UX}E_U^{(1,0)} + [\Lambda_U^{(2,0)}, a_{UX}], a_{UX} = -i(E_U^{(1,0)})\, du_X\end{aligned} \tag{6.2}$$

p defines a Lie algebra isomorphism between $Ad^*\mathcal{G}$ and the algebra $\hat{\mathcal{G}}$ given by the $\hat{X}^{(1,0)}$

The group G acts on $\mathcal{G}_0^*$ by Ad^* and on $(PC)^{n-1}$ by an action Ψ such that

$$pAd^*(g^{-1}) = \Psi_g p \qquad g \in G \tag{6.3}$$

Ψ is **the quotient coadjoint map.** The complex Jacobi structure of $(PC)^{n-1}$ is invariant under this action.

c) $\mathcal{G}_0^*$ and $(PC)^{n-1}$ admit Kähler structures, non invariant by the action of G. A leaf S (resp. $\tilde{S}$) is holomorphically weakly inbedded in $(PC)^{n-1}$ (resp. $\mathcal{G}_0^*$). Therefore S and $\tilde{S}$ admit Kähler structures. We may prove the following theorem.

Theorem. - *The leaves of $\mathcal{G}_0^*$ and of $(PC)^{n-1}$, orbits of G, admit Kähler structures. If a leaf S of $(PC)^{n-1}$ has an odd complex dimension, it admits a complex contact homogeneous space structure and a leaf $\tilde{S}$ of $\mathcal{G}_0^*$ corresponding to S is generated by complex rays of $\mathcal{G}_0^*$. If a leaf S of $(PC)^{n-1}$ has an even complex dimension, it admits a l.c.s. complex homogeneous space structure and a leaf $\tilde{S}$ corresponding to S is tranverse to the complex rays.*

7. Hamiltonian complex symplectic homogeneous spaces.

a) Let (W, F_W) be a complex symplectic manifold (dim $W = 2p+2; p \geq 0$) ; F_W is a holomorphic closed 2-form such that $F_W^{p+1} \neq 0$ everywhere ; $\Lambda_W^{(2,0)}$ is the holomorphic 2-tensor corresponding to F_W. We suppose that (W, F_W) is a **complex symplectic homogeneous space** G/H, the connected complex Lie group G such that $\dim_c G = M$ acts transitively and effectively on W by holomorphic transformations preserving F_W.

Let $\mathcal{G}$ be the Lie algebra of G. We suppose that G acts and W in a **Hamiltonian** manner : for any $X \in \mathcal{G}$ we have a function u_W such that the holomorphic field $X_W^{(1,0)}$ of W given by the right invariant field of G corresponding to X is the Hamiltonian field $X_W^{(1,0)} = [\Lambda_W^{(2,0)}, u_W]$. The space $(W\,,\ F_W) = G/H$ is **a Hamiltonian complex symplectic homogeneous space.**

The Hamiltonian action of G on (W, F_W) admits **a moment map** $\rho : x \in W \to \rho(x) \in \mathcal{G}*$ given by

$$\langle \rho(x),\ X\rangle = -u_W(x) \qquad (\forall\ X \in \mathcal{G},\ x \in W) \tag{7.1}$$

If the symplectic cocycle of ρ is null, ρ is Ad^*-equivariant

$$\rho(gx) = Ad^*(g^{-1})\ \rho(x) \qquad g \in G \tag{7.2}$$

If the cocycle of $\rho \neq 0$ one modify the Poisson structure $\tilde{\Lambda}^{2,0}$ and Ad^* and there is equivariance for the new action : ρ maps W on a leaf $\tilde{S}$ of $\tilde{\mathcal{G}}$ which is a Hamiltonian complex symplectic homogeneous space with $\dim_c \tilde{S} \leq 2p+2$

b) For $X \in \mathcal{G}$ we have the function $u_X = u = -\langle \Xi^{(1,0)}, X\rangle$ on $\mathcal{G}^*$. According to (7.1), one has

$$u_W(x) = -\langle \rho(x),\ X\rangle = u(\rho(x)) \qquad (x \in W)$$

and thus $u_W = \rho^* u$. It is easy to deduce from the definitions of $\tilde{\Lambda}^{(2,0)}$ and ρ

$$\rho_*(X_W^{(1,0)}(x)) = \tilde{X}^{(1,0)}(\rho(x)) \qquad F_W = \rho\, F_{\tilde{S}}$$

where $F_{\tilde{S}}$ is the symplectic 2-form of $\tilde{S}$. In $x \in W$, one has $F_W^{p+1}(x) = \rho^* F_{\tilde{S}}^{p+1}(\rho(x))$, where F_W^{p+1} is $\neq 0$. Therefore $\dim_c \tilde{S} = 2p+2$ and ρ defines W as a covering of $\tilde{S}$. The leaf $\tilde{S}$ being Kählerian, W is Kählerian. It is clear that $\tilde{S}$ is never compact.

Theorem. - *Let $(W, F_W = G/H$ be a Hamiltonian complex symplectic homogeneous space. The moment map defines (W, F_W) as a covering of a leaf $\tilde{S}$ of $\mathcal{G}^*$ and gives a correspondance of the invariant symplectic structures of W and $\tilde{S}$. Such a space is a non compact Kähler manifold with $c_1(W) = 0$ (first Chern class).*

8. Complex contact homogeneous spaces.

a) On a complex manifold W of complex dimension $m = 2p+1$ $(p \geq 1)$ **a complex contact structure** is defined, from a covering of W, by local holomorphic 1-forms $\{\omega_U\}$ satisfying $\omega_U \wedge (d\omega_U)^p \neq 0$ evrywhere and such that, for $U \cap V \neq \emptyset$ one has

$$\omega_V = h_{UV}^{-1} \omega_U \tag{8.1}$$

where the h_{UV} are holomorphic functions without zeros ; the structure is a transitive complex Jacobi structure given, for each U, by holomorphic elements $(\Lambda_U^{(2,0)}\ ,\ E_U^{(1,0)})$.

Let $\pi : K \to W$ be the holomorphic fiber bundle in complex straight lines defined by the system $\{h_{UV}\}$ as transition functions, and $\tilde{Z}^{(1,0)}$ its fundamental field. If $\{x^\alpha\}(\alpha = 1, \dots, 2p+1)$ is a complex chart of W of domain U, we have for $\pi^{-1}(U)$ a complex chart $\{x^\rho\} = \{x^0, x^\alpha\}$ of K for which $\tilde{Z}^{(1,0)}$ has for components $\tilde{Z}^0 = -x^0$ $\tilde{Z}^\alpha = 0$. For $\pi^{-1}(U \cap V)$, one has $x^{0'} = h^{-1}_{UV}(x)x^0$.

Let $\tilde{W} = K_0$ be a connected component of K without null section ; one has $\dim \tilde{W} = 2p+2$; $\tilde{W}$ admits an exact complex symplectic structure given by the holomorphic 1-form $\tilde{\omega}_{\tilde{W}}$

$$\tilde{\omega}_{\tilde{W}}|_{\pi^{-1}(U)} = (x^0)^{-1}\ \pi^* \omega_U \tag{8.2}$$

A complex contact field of W is a holomorphic field $X^{(1,0)}_W$ such that, for any U

$$(\mathcal{L}(X^{(1,0)}_W) + a_U)\ \omega_U = 0$$

where a_U is holomorphic on U. We set $u_U = i(X^{(1,0)}_W)\omega_U$, these local holomorphic functions satisfying $u_V = h^{-1}_{UV} u_U$. We define thus a holomorphic section s of K ; $X^{(1,0)}_W$ is the Hamiltonian complex field associated to s and $a_U = -i(E^{(1,0)}_U)\ du_U$. Under the isomorphism τ (section 4), s corresponds to a holomorphic function $\tilde{u}_{\tilde{W}}$ of $\tilde{W}$. If $X^{(1,0)}_{\tilde{W}}$ is the hamiltonian field associated to $\tilde{u}_{\tilde{W}}$, one has

$$\pi_*\ \tilde{X}^{(1,0)}_{\tilde{W}} = X^{(1,0)}_W \qquad \tilde{X}^0_{\tilde{W}} = x^0\ (E^{(1,0)}_U) du_U \tag{8.3}$$

A **Pfaffian complex manifold** is a complex contact manifold defined by a global holomorphic 1-form ω_W such that $\omega_W \wedge (d\omega_W)^p \neq 0$; the fiber bundle K is then holomorphically trivial. A complex contact structure comes from a Pfaffian structure if and only if the first Chern class $c_1(W)$ is null. The necessary condition is evident, the sufficient condition has been proved by Kobayashi [13].

b) Suppose that W admits a structure G/H of **complex contact homogeneous space** : G acts transitively and near-effectively on W by contact holomorphic transformations and H is the isotropy group at x_0. By near-effective action, one means that each invariant sub group of G included in H is discrete. According to (8.3), we may suppose that G acts by holomorphic transformations preserving $\tilde{Z}^{(1,0)}$ and $\tilde{\omega}_{\tilde{W}}$. Let $\tilde{x}_0$ be a point of $\tilde{W}$ such that $\pi\tilde{x}_0 = x_0$ and $\tilde{\omega}_0$ the orbit of $\tilde{x}_0$ by G. One has

Proposition. - *Let $\mathcal{H}$ be the Lie algebra of H*

1) If there exists $X \in \mathcal{H}$ such that $(i(E^{(1,0)}_U)du_U)$ $(x_0) \neq 0$ for $x_0 \in U$, G acts transitively on $\tilde{W}$.

2) (If for every $X \in \mathcal{H}$, one has $(i(E^{(1,0)}_U)du_U)$ $(x_0) = 0$, $\tilde{W}_0$ of complex dimension $(2p+1)$ is a covering of W ; $\tilde{W}_0$ with the restriction of $\tilde{\omega}_{\tilde{W}}$ is a complex Pfaffian homogeneous space.

In the first case, H is necessarily non discreet. The vector $\tilde{X}^{(1,0)}_{\tilde{W}}$ of $\tilde{W}$ corresponding to X has for components in $\tilde{x}_0$ in the chart $\{x^\rho\} = \{x^0, x^\alpha\} \neq$ $\tilde{X}^\alpha(\tilde{x}_0) = 0$, $\tilde{X}^0(\tilde{x}_0) = -x^0_0 \; i(E^{1,0}_U)(x_0) \neq 0$. The orbit of $\tilde{x}_0$ by the corresponding complex one-parameter group is given on U by $x^\alpha = x^\alpha_0$, $x^0 = -x^0_0 \; (i(E^{1,0}_U)(x_0) \; du_U)(x_0)$ and coincides with $\pi^{-1}(x0)$. If $\tilde{x} \in \tilde{W}$ and $\pi\tilde{x} = x \in W$, there is $g \in G$ such that $x = g \, x_0$ and $\pi^{-1}(gx_0) = \pi^{-1}(x)$ belongs to $\tilde{W}$. Therefore G acts transitively on $\tilde{W}$.

In the second case let H_0 be the connected componed of the unity of H ; $\tilde{x}_0$ is invariant by H_0. The largest subgroup $\tilde{H}$ of G preserving $\tilde{x}_0$ is such that $H_0 \subset \tilde{H} \subset H$ and $\tilde{W}_0$ is isomorphic to $G/\tilde{H}$ and so $\dim_c \tilde{W}_0 = 2p + 1$. The rest is evident.

In the first case the complex contact homogeneous space is said to be proper, in the second non proper. The non proper case corresponds to the case where W admits a covering which is a Pfaffian homogeneous space.

c) In the two case, the action of G on $\tilde{W}$ admits a moment $\rho : \tilde{x} \in \tilde{W} \to \rho(\tilde{x}) \in \mathcal{G}^*$ given by

$$\langle(\rho(\tilde{x}),\; X\,\rangle = -\tilde{\omega}_{\tilde{W}}(\tilde{X}^{(1,0)}_{\tilde{W}})(\tilde{x}) = -\tilde{u}_{\tilde{W}}(\tilde{x}) \tag{8.4}$$

If $\pi\tilde{x} = x \in U$ and if $\tilde{x}$ has coordinates $(x^0,\; x^\alpha)$, one has

$$\langle(\rho(\tilde{x}),\; X\,\rangle = -(x^0)^{-1}\; u_U(x) \tag{8.5}$$

It is well known that, for an exact symplectic manifold, the moment map is Ad^*-equivariant.$\blacksquare$ It follows that ρ has values in $\mathcal{G}^*_0$, maps $\tilde{W}_0$ on the leaf $\tilde{S}$ of $(\mathcal{G}^*_0, \tilde{\Lambda}^{(2,0)})$ passing through $\rho(\tilde{x}_0)$. By derivation of (8.5) in x^0, with x fixed, one has

$$\rho_*(\tilde{Z}^{1,0}(\tilde{x})) = \;\Xi^{(1,0)}(\rho(\tilde{x})) \tag{8.6}$$

Let $\{z^\lambda\}$ $(\lambda = 1, \ldots, (n-1))$ be a complex chart of $(PC)^{n-1}$ of domain U' , $\{z^0,\; z^\lambda\}$ the chart associated to the fiber bundle $\mathcal{G}^*_0$ on $p^{-1}(U')$; Ξ^λ being null, it follows from (8.6) that $\frac{d\rho^\lambda}{dx^0} = 0$. Therefore $\rho^\lambda(\tilde{x})$ depends only upon $x = \pi\tilde{x}$, where the $\rho^\lambda(\tilde{x})$ are components on $(PC)^{n-1}$ of $p\rho(\tilde{x})$. We see that $p \circ \rho : \tilde{W} \to (PC)^{n-1}$ defines a map $\sigma = W \to (PC)^{n-1}$ such that $p \circ \rho = \sigma \circ \pi$. One has the commutative diagram

$$\begin{array}{ccc} \tilde{W} & \xrightarrow{\rho} & \mathcal{G}^*_0 \\ \big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle p} \\ W & \xrightarrow{\sigma} & (PC)^{n-1} \end{array}$$

σ is equivariant with respect to the quotient coadjoint map and maps W on the leaf S passing through $\sigma(x_0)$ of the Jacobi structure of $(PC)^{n-1}$. One has immediately by p

$$\sigma_*(X^{(1,0)}_W) = \hat{X}^{(1,0)}|_S \tag{8.7}$$

where $\hat{X}(1,0)$ is the Hamiltonian field of $(PC)^{n-1}$ corresponding to the action of the quotient coadjoint map. If we modify the $\{\omega_U\}$ by a holomorphic factor $\neq 0$, we can obtain $x^0 = \rho^* z^0$ for two homologous points and thus identify, for these points and the considered charts, x^0 with z^0.

9. Proper complex contact homogeneous spaces.

a) Let $\tilde{S}$ be the leaf of $(\mathcal{G}_0^*\,,\, \tilde{\Lambda}^{2,0})$ passing through ξ_0. The projection by p of $\tilde{S}$ is the leaf $S = \sigma(G/H)$. It is easy to prove

Proposition. - *The complex contact homogeneous space $W = G/H$ is proper if and only if $\tilde{S}$ is generated by complex rays of $\mathcal{G}_l^*$.*

b) Let $W = G/H$ be a proper complex contact homogeneous space such that $\dim_c W = 2p+1$. According to section 7, G acting transitively on $\tilde{W}$, ρ defines $\tilde{W}$ as a covering of $\tilde{S}$ and $\dim_c \tilde{S} = 2p+2$; $\tilde{S}$ is generated by complex rays of $\mathcal{G}_l^*$. Therefore $p(\tilde{S}) = \sigma(W) = S$ is such that $\dim_c S = 2p+1$ and admits a complex contact structure invariant under G ; S is a contact homogeneous space such that $\tilde{S}$ is an exact complex symplectic homogeneous space ; σ defines W as a covering of S with correspondance of the contact structures ; S being Kählerian, W is necessarily a Kähler manifold. We may prove

Theorem. - *Let $G/H = W$ be a proper complex contact homogeneous space, where G acts transitively and near effectively on W by contact holomorphic transformations. The map σ defines W as a covering of a leaf S of the Jacobi structure of $(PC)^{n-1}$ with correspondance of the invariant contact structures. Such a space is ever a Kähler manifold.*

The leaves of **odd** complex dimension of the complex Jacobi structures of $(PC)^{n-1}$ are **universal models**, up to a covering, of the proper complex contact homogeneous spaces.

10. Other cases.

We have different similar theorems.

a) *Let $(W, \omega_W) = G/H$ be a complex Pfaffian homogeneous space (ω_W invariant) of complex dimension $2p+1$. The map ρ maps W on a Hamiltonian complex symplectic homogeneous space $(\tilde{S}\,,\, \tilde{F}_S) = G/\tilde{K}$, leaf of $(\mathcal{G}_0^*\,,\, \tilde{\Lambda}^{(2,0)})$ of dimension $2p$, such that, for its covering $(S,\, F'_{S'}) = G/\tilde{K}_0$; ($\tilde{K}_0$ connected component of $\tilde{K}$) the symplectic 2-form $F'_{S'}$ has an integer complex cohomology ; W corresponds to a contactification of $(S'\,,\, F'_{S'})$. W is never compact.*

b) Let $W = G/H$ (H non discrete) a Hamiltonian complex l.c.s. symplectic homogeneous space of complex dimension $2p (p \geq 2)$. The map σ defines W as a covering of leaf of even dimension of the Jacobi structure of $(PC)^{n-1}$ with correspondance of l.c.s. structures of W and S. Such a space is ever a Kähler manifold.

III. Compacness of the proper complex contact homogeneous spaces.

11. Consequence of the results of Borel-Remmert [3].

Let W be a **compact** complex manifold admitting a Kähler structure. We suppose that a connected complex Lie group acts transitively and effectively on W by holomorphic transformations. Under these hypotheses Borel and Remmert have proved that W **is the product of a complex torus T (the Albanese torus of W) and of a simply connected compact Kähler manifold which is a rational projective manifold M.**

$$W = T \times M. \tag{11.1}$$

For a compact complex manifold, we denote by $A(W)$ the largest connected group of holomorphic transformations of W. It follows from (11.1) that one has

$$A(W) = A(T) \times A(M) \tag{11.2}$$

We denote by $\mathcal{G}$ the Lie algebra of G. The connected component G' of the kernel of the Albanese map $\gamma : W \to T$ is an invariant subgroup of G acting transitively on M. According to Montgomery, there is a compact real subgroup Γ of G' acting transitively on M ; Γ is necessarily semi-simple one may adopt on M a Kähler metric invariant under T. The manifold M being a Kähler homogeneous space, one has $G' = A(M)$. It is then easy to see that G coincides with $A(W)$.

12. Compact proper complex contact homogeneous spaces.

a) Let $W = G/H$ be a compact proper complex contact homogeneous space of complex dimension $2p+1$. We have seen that W is a Kähler manifold. The hypotheses of Borel-Remmert are satisfied and we have

$$W = T \times M$$

where M is a **simply connected compact Kähler manifold**. We have $G = A(T) \times A(M)$. The complex group $G' = A(M)$ is **complex semi-simple.** It follows that the group $Ad^*G = Ad^*G'$ acting on $\mathcal{G}^*$ is **semi-simple**. According to section 6, $Ad^*(G)$ and $\Psi(G)$ acting on $(PC)^{n-1}$ by the quotient coadjoint map is **semi-simple.**

We have seen that W is a covering of a leaf S of the Jacobi structure of $(PC)^{n-1}$, orbit of $\psi(G)$. S which is compact is a Kähler homogeneous space for a real semi-simple group and so is simply connected ; W covering of S coincides with S and T is reduced to a point. One has

Proposition. - *A compact proper complex contact homogeneous space $W = G/H$ is a Kähler homogeneous space and G is semi-simple complex.*

W may be written $W = \Gamma/L$ where Λ is a real compact semi-simple Lie group.

b) Boothby ([1],[2]) and Wolf [12] have determined, by group-theoretical methods, all the complex contact homogeneous spaces $W = G/H$ **under the hypothesis that W is simply connected. Our approach shows that this hypothesis is unnecessary**; on the other hand a simply connected non proper space is **Pfaffian** and so can not be compact.

Boothby processes in two steps : in [1] he studies the simply connected, compact complex contact homogeneous spaces $W = G/H$ such that $c_1(W) \neq 0$; he shows then that these spaces are in fact proper in my sense and he determines these spaces; in particular he shows that the group G is necessarily simple. In [2] he proves that if W is simply connected, one has $c_1(W) \neq 0$ and shows thus he has determined all the **simply connected**, compact, complex contact homogeneous spaces.

c) In the following we denote by T^i a torus of real dimension i. One deduce from our results and from the results of Boothby and Wolf [12] the following statement.

Theorem. - *The proper compact complex contact homogeneous spaces are in one-one correspondance with the compact centerless simple Lie group Λ They are simply connected Kähler homogeneous spaces $W = \Gamma L$ where L is a subgroup unique up to conjugacy and is a local direct product $L_1.T^1$ where T' is a circle group which defines the complex structure. The list of these spaces is the following*

$$SO(n+4)/SO(n) \times SO(3) \times T^1 \ ; \ SU(n+2)/SU(n) \times T^2 \ ; \ Sp(n+1)/Sp(n) \times T^1$$

all of real dimension $4n+2$ $(n = p)$ **and**

$$G_2/SO(3) \times T^1 \ (dim_R\, W = 10)\,; \, F_4/Sp(3) \times T^1 \ (dim_R\, W = 30)$$

$$E_6/SU(6) \times T^1 (dim_R\, W = 42); \ E_7/Sp(12) \times T^1 (dim_R\, W = 66\,) E_8/E_7 \times T^1 (dim_R\, W = 114)$$

d) J.A. Wolf [12] has considered in particular the compact quaternion-Kähler symmetric Riemannian homogeneous spaces M (with a non zero Ricci curvature). These spaces are simply connected, irreducible. By a group theoretical method, he shows that these spaces may be obtained by substitution of $Sp(1)$ to T. It follows that **the proper compact complex contact homogeneous spaces $W = \Gamma/L$ are fiber bundles with a fiber $S^2 \simeq Sp(1)/T^1$ on the quaternion Kähler symmetric spaces M.**

We have the following problem.

Problem. - Find the result of Wolf directly by a differential geometric method.

Let M be a quaternion-Kähler symmetric space. We know that

1) A compact quaternion-Kähler homogeneous space Γ/K is symmetric (Alekseevskii [14]),

2) The twistor space W of M is a proper compact complex contact homogeneous space (according to Salamon [15]) with fibers isometric to the canonical sphere S^2.

This space corresponds to a real simple group. From these results, we may deduce the result of Wolf. Search for a simpler proof.

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