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Isometries of special manifolds†

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ABSTRACT

We describe special Kähler geometry, special quaternionic manifolds, and very special real manifolds and analyze the structure of their isometries. The classification of the homogeneous manifolds of these types is presented.

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1 Introduction

Quaternionic spaces with a transitive solvable group of motions have been classified by Alekseevskiĭ twenty years ago [1]. More recently it was discovered that these so-called *normal* quaternionic spaces and their classification are related to supergravity coupled to abelian gauge fields [2, 3, 4].

Supergravity theories are invariant under local (i.e. space-time dependent) supersymmetry transformations. Under such transformations bosonic (commuting) and fermionic (anticommuting) fields transform into each other with parameters that are themselves anticommuting Lorentz spinors. Extended supersymmetry implies that we are dealing with N independent supersymmetry transformations, each described by a separate anticommuting spinorial parameter. The number of supersymmetry generators ('supercharges') is thus equal to N times the dimension of the (smallest) spinor representation. For realistic supergravity this number of supercharge components cannot exceed 32. As 32 is the number of components of a Lorentz spinor in $d = 11$ space-time dimensions, it follows that realistic supergravity theories can only exist for dimensions $d \leq 11$. A characteristic feature of the algebra containing the supersymmetry generators is that it contains the space-time translation operators. Therefore local supersymmetry leads to space-time diffeomorphisms and the invariant actions are generalizations of the Einstein-Hilbert action.

By its very nature supersymmetry implies the presence of both bosonic and fermionic fields carrying integer and half-integer spin, respectively. The graviton, the particle described by the space-time metric, has spin 2. The 'gravitino', the particle associated with the fermionic gauge field of supersymmetry, has spin 3/2. Massless spin-3/2 particles are described by the Rarita-Schwinger action. Obviously the number of gravitini must be equal to N. All other particles carry spin less than 3/2. In a four-dimensional space-time they are usually described by vector, spinor and spinless fields and in the action they appear in generalizations of the Yang-Mills, Dirac and Klein-Gordon actions.

The type of supergravity is thus characterised by the numbers d and N. For instance, for the physical $d = 4$ dimensional space-time, one can have supergravity theories with $1 \leq N \leq 8$. For any N and d there is a *pure* supergravity theory, having physical states with spins ranging from 0 to 2. If the number of supercharge components does not exceed 16, one can have couplings with (supersymmetric) matter, which has spin $s \leq 1$. For the purpose of our work, we shall be dealing with 8 supercharge components, corresponding to $N = 4$ in three, $N = 2$ in four and five, and $N = 1$ in six space-time dimensions.

Of particular interest for geometry are the spinless fields, denoted by $\phi^i(x)$, which define a map from the d-dimensional Minkowskian space-time, with coordinates x^μ with $\mu = 1, ..., d$, to some 'target space'. In the physics literature, such a model is called a nonlinear sigma model. The kinetic terms in the action read

$$S = -\tfrac{1}{2} \int d^dx \, \sqrt{\det g(x)} \, g^{\mu\nu}(x) \, \frac{\partial\phi^i(x)}{\partial x^\mu} \, \frac{\partial\phi^j(x)}{\partial x^\nu} \, G_{ij}(\phi(x)) \ , \tag{1.1}$$

where $g^{\mu\nu}(x)$ refers to the space-time metric. $G_{ij}(\phi)$ defines the target-space metric corresponding to the invariant line element

$$\mathrm{d}s^2 = G_{ij}(\phi) \, \mathrm{d}\phi^i \mathrm{d}\phi^j \ . \tag{1.2}$$

A crucial point is that supersymmetry severely restricts the possible target-space geometries. As clearly exhibited in table 1, the more supercharge components one has, the more restrictions one finds. For large N and/or d the target-space manifold is a unique symmetric space. When the number of supercharge components equals 16, the target-

Table 1: Restrictions on target-space manifolds according to the type of supergravity theory. The rows are arranged such that the number κ of supercharge components is constant. $\mathcal{M}$ refers to a general Riemannian manifold, SK to 'special Kähler', VSR to 'very special real' and Q to quaternionic manifolds.

κ	d=3	d=4	d=5	d=6		
2	$N=1$ $\mathcal{M}$					
4	$N=2$ Kähler	$N=1$ Kähler				
6	$N=3$ Q					
8	$N=4$ $Q \oplus Q$	$N=2$ $SK \oplus Q$	$N=2$ $VSR \oplus Q$	$N=1$ $\emptyset \oplus Q$		
10	$N=5$ $\frac{Sp(2,n)}{Sp(2)\otimes Sp(n)}$					
12	$N=6$ $\frac{SU(4,n)}{S(U(4)\otimes U(n))}$					
16	N=8 $\frac{SO(8,n)}{SO(8)\otimes SO(n)}$	$N=4$ $\frac{SU(1,1)}{U(1)} \otimes \frac{SO(6,n)}{SO(6)\otimes SO(n)}$	... ...	... ..	$\rightarrow$ $\rightarrow$	$N=1$ $d=10$
18	$N=9$ $\frac{F_{4(-20)}}{SO(9)}$					
20	N=10 $\frac{E_{6(-14)}}{SO(10)\otimes SO(2)}$	$N=5$ $\frac{SU(5,1)}{U(5)}$	... ...			
24	N=12 $\frac{E_{7(-5)}}{SO(12)\otimes SO(3)}$	$N=6$ $\frac{SO^*(12)}{U(6)}$	$N=6$ $\frac{SU^*(6)}{USp(6)}$	... ...		
32	N=16 $\frac{E_{8(+8)}}{SO(16)}$	$N=8$ $\frac{E_{7(+7)}}{SU(8)}$	$N=8$ $\frac{E_{6(+6)}}{USp(8)}$	... ...	$\rightarrow$ $\rightarrow$	$N=1$ $d=11$

space geometry is fixed once the number of matter multiplets is given (for $d=4$ each of them must contain a spin-1 field). This row continues to $N=1$, $d=10$. Beyond 16 supercharge components there is no freedom left. The row with 32 supercharge components continues to $N=1$, $d=11$. On the other hand in $d=3$, $N=1$ any Riemannian manifold can occur. For $N=2$ in $d=3$, and $N=1$ in $d=4$, supersymmetry induces a natural complex structure, and the manifold is Kählerian. For higher N one finds quaternionic manifolds; the three complex structures are again closely related to the supersymmetry transformations. For the purpose of this paper the row with $\kappa=8$ supercharge components is important, in particular the entries with $d=3,4$ and 5. Supersymmetry has already fixed a lot of the structure of these manifolds, but it is the highest value of N

where they are not yet restricted to symmetric spaces.

The rows in the table are arranged such that the number of supercharge components is constant. E.g. for $\kappa = 16$, in $d = 4$ one has 4 charges, each with 4 spinor components, while in $d = 10$, there is one charge with 16 spinor components. These theories can be related by 'dimensional reduction', according to which a higher-dimensional theory is truncated to a lower-dimensional one by suppressing the dependence on some of the space-time coordinates. This relationship forms an important ingredient in the approach outlined below. Generally, the procedure of dimensional reduction leaves the supersymmetries preserved.

From now on we concentrate on the case of 8 supercharge components, and the relevant supergravity theories have $N = 2$ supersymmetry in $d = 5$ and $d = 4$ and $N = 4$ supersymmetry in $d = 3$ space-time dimensions [5, 6, 7]. In $d = 4$ the target space factorizes into a quaternionic and a Kähler manifold of a particular type [6], called *special* [8]. The definition of these special Kähler manifolds will be the subject of section 2. The reduction to $d = 3$ leads for both factors to a quaternionic manifold. The interesting results on quaternionic manifolds follow from analyzing how they emerge in $d = 3$ as a result of dimensional reduction from the first factor in $d = 5$ (the 'very special real' manifolds) via the corresponding $d = 4$ special Kähler manifold. Actually, we could have started from $d = 6$, but then there are no scalars, and thus no target-space manifold in the sense described above.

As a consequence of extended supersymmetry, the target space must be a real, a Kähler or a quaternionic manifold, depending on whether the supergravity space-time dimension is $d = 5$, $d = 4$ or $d = 3$, respectively. By means of ordinary dimensional reduction, which preserves supersymmetry, one can thus relate real, complex and quaternionic spaces. More specifically, one defines two maps, the **r** and the **c** map, induced by the dimensional reduction of the corresponding supergravity theories, which act as

$$\mathbb{R}_{n-1} \xrightarrow{\mathbf{r}} \mathbb{C}_n\ , \qquad \mathbb{C}_n \xrightarrow{\mathbf{c}} \mathbb{H}_{n+1}\ , \tag{1.3}$$

Here $n-1$, n and $n+1$ denote the real, complex and quaternionic dimension of the real, Kähler and quaternionic spaces, respectively. The existence of these maps is based on the fact that dimensional reduction preserves supersymmetry.

However, the images of these maps do not comprise all the special Kähler and all the quaternionic manifolds. The real manifolds occuring in $d = 5$, $N = 2$ supergravity theories coupled to spin-1 fields are called *very special* real manifolds [9]. Dimensional reduction of the $d = 5$ Lagrangians leads to a subclass of the special Kähler manifolds, which will be called very special Kähler manifolds. Reduction of the actions containing the special Kähler manifolds in $d = 4$ to $d = 3$ space-time dimensions leads to a class of quaternionic manifolds, which will be called 'special quaternionic manifolds'. A subclass of the latter, which constitutes the image of the **c** map acting on the very special Kähler manifolds, are the 'very special quaternionic' manifolds.

All $N = 2$ supergravity theories with spin-1 fields in five space-time dimensions are characterized by cubic polynomials of n variables. Hence it is clear that cubic polynomials define a series of related 'very special' real, Kähler and quaternionic manifolds. The cubic polynomials that correspond to homogeneous manifolds have been classified in [10]. They were denoted by $L(q, P)$ and $L(4m, P, \dot{P})$, where $q \neq 4m$, m, P and $\dot{P}$ are integers restricted by[1] $q \geq -1$ and $m, P, \dot{P} \geq 0$, and they cover all the homogeneous non-symmetric quaternionic spaces classified in [1].[2] In particular, they include the spaces denoted in [1] by $V(p,q)$ and $W(p,q)$ with q positive, which were related to $(q+1)$-dimensional Clifford modules. We found, however, that all the $L(q,P)$ and $L(4m,P,\dot{P})$ can be described in a common framework based on Clifford algebras. For instance, the case of $q=-1$, where the dimension of the underlying Clifford algebra vanishes, can be naturally incorporated in the general analysis. Actually, some of the polynomials did not appear in Alekseevskiĭ's original work. When q is a multiple of 4, P has to be replaced by the symmetric pair of integers $(P,\dot{P})$; furthermore the case $L(-1,P)$ was missing. Meanwhile these results have been confirmed in the mathematical literature [11].

Here we intend to explain and summarize the results of [4] on the structure of the continuous isometries of these spaces[3]. These isometries of the target-space metric extend in fact to symmetries of the full supergravity action.

The recent new interest of physicists in special geometry and the symmetry structure of target spaces is to a large extent motivated by string theory. The supersymmetric ground states arising from string theory give rise to an effective field theory of the supergravity type. The moduli space of these ground states is usually isomorphic to the moduli space of the restricted target-space manifolds discussed above. Exploiting these relations and making use of the known supersymmetry properties of the ground states, one for instance derives that the moduli space of Calabi-Yau manifolds must exhibit special geometry [13, 2, 14, 8, 15, 16].

In section 2 we explain the notion of special Kähler manifolds. Then in section 3 we discuss the isometries of special manifolds. First, in section 3.1, we specify the isometries of special Kähler manifolds. This will involve the concept of symplectic transformations, which is a rather important concept in special Kähler manifolds, and at the heart of recent developments. In section 3.2, the **c** map is explained which leads to special quaternionic manifolds, whose isometries are also discussed. Subsequently we turn to very special manifolds in section 4. We first explain the **r** map and introduce the isometries of very special real manifolds. This forms the starting point for discussing the isometries of very special Kähler and very special quaternionic manifolds.

Then we concentrate on homogeneous spaces. We shall see that Alekseevskiĭ's homogeneous non-symmetric spaces are all very special quaternionic, and in section 5 we give their classification comprising all known homogeneous quaternionic manifolds. Section 6 then discusses their isometries. It starts from the symmetries of representations of real Clifford algebras, to determine subsequently the isometries of the homogeneous very special real manifolds, special Kähler manifolds and quaternionic manifolds. In a final section we summarise all these results.

[1] At the end of this text we generalise this to $q=-2$ and $q=-3$ to include symmetric special Kähler and quaternionic manifolds that are not of the very special type.

[2] Actually, this was already clear from previous work, which had revealed that Alekseevskiĭ's non-symmetric spaces were all in the image of the **c** map and that the **c** map is in fact closely related to Alekseevskiĭ's classification method.

[3] For an alternative summary of these results, see [12].

2 Special Kähler manifolds

We first briefly introduce the special Kähler manifolds in the context of supergravity. Subsequently we cast the results in a more abstract form based on symplectic vectors, which was originally discussed in the context of the moduli space of Calabi-Yau three-folds. We close the section by presenting a few characteristic examples of special Kähler manifolds.

2.1 Vector multiplets coupled to supergravity

The scalar sector of the $N = 2$ supergravity-Yang-Mills theory in four space-time dimensions defines the 'special Kähler manifolds'. Without supergravity we have $N = 2$ supersymmetric Yang-Mills theory, whose spinless fields parametrize a similar type of Kähler manifolds. The vector potentials, which describe the spin-1 particles, are accompanied by complex scalar fields and doublets of spinor fields, all taking values in the Lie algebra associated with the gauge group. The presence of two independent supersymmetries implies that the action is encoded in a holomorphic prepotential $F(X)$, where X denotes the complex scalar fields [6]. Two different functions $F(X)$ may correspond to equivalent equations of motion and to the same geometry. The relation is made by certain symplectic transformations that we discuss shortly. When coupling n of these so-called vector multiplets to supergravity, one again has a holomorphic prepotential $F(X)$, this time of $n+1$ complex fields, but now it must be a *homogeneous* function of degree two [6]. The physical scalar fields of this system parameterize an n-dimensional complex hypersurface, defined by the condition that the imaginary part of $X^I\,\bar F_I(\bar X)$ must be a constant, while the overall phase of the X^I is irrelevant in view of a local (chiral) invariance.[4]

$$i(\bar X^I F_I - \bar F_I X^I) = 1\,. \tag{2.1}$$

The embedding of this hypersurface can be described in terms of n complex coordinates z^A by letting X^I be proportional to some holomorphic sections $Z^I(z)$ of the projective space $P\mathbb{C}^{n+1}$ [17]. The n-dimensional space parametrized by the z^A $(A = 1,\ldots,n)$ is a Kähler space; the Kähler metric $g_{A\bar B} = \partial_A\partial_{\bar B}K(z,\bar z)$ follows from the Kähler potential

$$\begin{aligned} &K(z,\bar z) = -\log\Big[i\bar Z^I(\bar z)\,F_I(Z(z)) - iZ^I(z)\,\bar F_I(\bar Z(\bar z))\Big]\,, \qquad \text{where} \\ &X^I = e^{K/2}Z^I(z)\,, \qquad \bar X^I = e^{K/2}\bar Z^I(\bar z)\,. \end{aligned} \tag{2.2}$$

The resulting geometry is known as *special* Kähler geometry [6, 8]. The curvature tensor associated with this Kähler space satisfies the characteristic relation [18]

$$R^A{}_{BC}{}^D = \delta^A_B\delta^D_C + \delta^A_C\delta^D_B - e^{2K}\mathcal{W}_{BCE}\,\bar{\mathcal{W}}^{EAD}\,, \tag{2.3}$$

where

$$\mathcal{W}_{ABC} = iF_{IJK}(Z(z))\,\frac{\partial Z^I(z)}{\partial z^A}\frac{\partial Z^J(z)}{\partial z^B}\frac{\partial Z^K(z)}{\partial z^C}\,. \tag{2.4}$$

A convenient choice of inhomogeneous coordinates z^A are the *special* coordinates, defined by

$$z^A = X^A/X^0, \qquad A = 1,\ldots,n, \tag{2.5}$$

[4]Here and henceforth we use the convention where $F_{IJ\cdots}$ denote multiple derivatives with respect to X of the holomorphic prepotential.

or, equivalently,

$$Z^0(z) = 1\,, \qquad Z^A(z) = z^A\,. \tag{2.6}$$

The kinetic terms of the spin-1 gauge fields in the action are proportional to the symmetric tensor

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + 2i\frac{\mathrm{Im}(F_{IK})\,\mathrm{Im}(F_{JL})\,X^K X^L}{\mathrm{Im}(F_{KL})\,X^K X^L}\,. \tag{2.7}$$

This tensor describes the field-dependent generalization of the inverse coupling constants and so-called θ parameters.

As mentioned above, the field equations corresponding to two supersymmetric Yang-Mills actions characterized by different functions $F(X)$, can be identical up to a symplectic transformation. In that case the two functions describe equivalent classical field theories. These symplectic transformations act as $Sp\,(2n+2,\mathbb{R})$ rotations on the vectors (X^I, F_J) and also on the Yang-Mills field-strength tensors. However, on the field strengths they generically rotate electric into magnetic fields and vice versa. Such rotations, which are called duality transformations because in four space-time dimensions electric and magnetic fields are dual to each other (in the sense of Poincaré duality), cannot be implemented on the vector potentials, at least not in a local way. Therefore, the use of these symplectic transformations is only legitimate for zero gauge coupling constant. From now on we deal exclusively with Abelian gauge groups. The symplectic transformations take the form

$$\begin{pmatrix} \tilde{X}^I \\ \tilde{F}_I \end{pmatrix} = \mathcal{O} \begin{pmatrix} X^I \\ F_I \end{pmatrix} \tag{2.8}$$

and generically lead to another prepotential $\tilde{F}(\tilde{X})$, whose first derivatives with respect to $\tilde{X}$ correspond to the $\tilde{F}_I$. Here $\mathcal{O}$ is a real $(2n+2)$-by-$(2n+2)$ matrix satisfying the symplectic condition

$$\mathcal{O}\,\Omega\,\mathcal{O}^{\mathrm{T}} = \Omega\,, \tag{2.9}$$

where

$$\Omega = \begin{pmatrix} 0 & \mathbb{1}_{n+1} \\ -\mathbb{1}_{n+1} & 0 \end{pmatrix}\,. \tag{2.10}$$

Hence we may find a variety of descriptions of the same theory in terms of different functions F. If a symplectic transformation leads to the same function F, then we are dealing with an invariance of the equations of motion (but not necessarily of the action as not all transformations can be implemented locally on the gauge fields). This invariance reflects itself in the isometries of the target-space manifold. We return to this in section 3.1.

2.2 Symplectic formulation of special geometry

We now give an alternative and more abstract formulation of special geometry. This formulation was first given in the context of a treatment of the moduli space of Calabi-Yau three-folds [8, 14, 16]. The connection between these rather different topics hinges on string theory. The Calabi-Yau manifolds arise as ground-state configurations for certain string theories, whose low-energy field theories take the form of an $N=2$ supergravity theory. The scalar-field sector is locally isomorphic to the moduli space of the Calabi-Yau manifolds, so that both exhibit special geometry [13]. Here we give a self-contained derivation based on the material presented above.

Based on the previous exposition it makes sense to define a $(2n+2)$-component vector $V \equiv (X^I, F_J) \in \mathbb{C}^{2n+2}$, which transforms under $Sp\,(2n+2,\mathbb{R})$ according to $V \to \tilde{V} = \mathcal{O}V$. The constraint (2.1) can then be written as[5]

$$\langle \bar{V}, V\rangle \equiv \bar{V}^T \Omega V = -i \; . \tag{2.11}$$

Holomorphic sections $v(z)$, which describe the holomorphic dependence on the coordinates z^A, follow from $V = e^{K/2} v$, where K is the Kähler potential (2.2), which in this notation is defined by

$$e^{-K(z,\bar{z})} = i\langle \bar{v}(\bar{z}), v(z)\rangle \, . \tag{2.12}$$

Here the (X^I, F_J) are the basic objects; these $2n+2$ quantities are parametrized in terms of the n complex coordinates z^A. We do not impose the condition that the F_I can be written as the derivatives of a homogeneous holomorphic function, so that the dependence of the F_I on z^A is not necessarily induced via their dependence on the X^I. This starting point is motivated by the fact that there are situations where the holomorphic prepotential does not exist, although the sections are well defined [19]. The sections v are only defined projectively, i.e., they are uniquely defined modulo

$$v(z) \longrightarrow e^{f(z)}\, v(z) \, . \tag{2.13}$$

Under holomorphic transformations the Kähler potential changes by a Kähler transformation

$$K(z,\bar{z}) \to K(z,\bar{z}) - f(z) - \bar{f}(\bar{z}) \, ,$$

and the original $Sp\,(2n+2,\mathbb{R})$ vector V changes by a phase transformation

$$V \to e^{\frac{1}{2}(f(z)-\bar{f}(\bar{z}))}\, V \, .$$

The holomorphicity of the sections v is expressed by

$$\begin{aligned} \mathcal{D}_{\bar{A}} V &\equiv \Big[\partial_{\bar{A}} - \tfrac{1}{2}(\partial_{\bar{A}} K)\Big] V = 0 \, , \\ \mathcal{D}_{A} \bar{V} &\equiv \Big[\partial_{A} - \tfrac{1}{2}(\partial_{A} K)\Big] \bar{V} = 0 \, . \end{aligned} \tag{2.14}$$

Here we recognize $\mathcal{D}$ as the Kähler derivative, which is covariant with respect to the projective transformations (2.13). For the nonvanishing derivatives of V and $\bar{V}$ we thus define

$$\begin{aligned} U_A = \mathcal{D}_A V &\equiv \Big[\partial_A + \tfrac{1}{2}(\partial_A K)\Big] V \, , \\ \bar{U}_{\bar{A}} = \mathcal{D}_{\bar{A}} \bar{V} &\equiv \Big[\partial_{\bar{A}} + \tfrac{1}{2}(\partial_{\bar{A}} K)\Big] \bar{V} \, . \end{aligned} \tag{2.15}$$

On the constraint (2.1) covariant and ordinary derivatives coincide, so that application of holomorphic and anti-holomorphic derivatives leads to

$$\langle \bar{V}, U_A\rangle = \langle \bar{U}_{\bar{A}}, V\rangle = \langle \bar{V}, \mathcal{D}_A U_B\rangle = 0 \, . \tag{2.16}$$

[5]This inner product arises naturally in the treatment of Calabi-Yau manifolds, where V corresponds to the periods of a certain harmonic form. The inner product can be defined in the dual cohomology basis and the symplectic group corresponds to redefinitions of that basis that leave the inner product (the intersection numbers) invariant.

When acting on co- and contravariant vectors the covariant derivatives contain the Levi-Civita connection associated with the Kähler metric $g_{A\bar{B}}$, whose nonzero components are $\Gamma^A_{BC} = g^{A\bar{A}}\partial_B g_{C\bar{A}}$ and their complex conjugates. Obviously the Levi-Civita connection is not present in $\mathcal{D}_{\bar{B}}U_A$ and we derive

$$\mathcal{D}_{\bar{B}}U_A = \mathcal{D}_{\bar{B}}\mathcal{D}_A V = (\partial_{\bar{B}}\partial_A K)\, V = g_{A\bar{B}}\, V \ , \tag{2.17}$$

where we used the constraint (2.14). The above relation implies that we are dealing with a so-called Kähler-Hodge manifold. Furthermore, using $\mathcal{D}_{\bar{B}}\mathcal{D}_A\langle\bar{V}, V\rangle = \partial_{\bar{B}}\partial_A\langle\bar{V}, V\rangle = 0$ gives

$$g_{A\bar{B}} = i\langle U_A, \bar{U}_{\bar{B}}\rangle \ , \ \ \text{or,} \quad \langle U_A, \bar{U}^B\rangle = -i\delta^B_A \ . \tag{2.18}$$

As the metric is covariantly constant, one easily establishes that

$$\langle \mathcal{D}_A U_B, \bar{U}_{\bar{C}}\rangle = 0\,, \tag{2.19}$$

making use of (2.17) and (2.16).

In (2.7) we defined the tensor $\mathcal{N}$ in terms of derivatives of the prepotential $F(X)$. However, it is possible to find an expression for this tensor without referring to the prepotential [19]. Namely we note the following two properties, which are direct consequences of the definition (2.7) and of (2.16),

$$\mathcal{N}_{IJ}X^J = F_I\,, \qquad \mathcal{N}_{IJ}\mathcal{D}_{\bar{A}}\bar{X}^J = \mathcal{D}_{\bar{A}}\bar{F}_I\,, \tag{2.20}$$

They enable us to express $\mathcal{N}$ as the ratio of two $(n+1)$-by-$(n+1)$ matrices. In addition, we use that the matrix $\mathcal{N}$ is symmetric. This symmetry requires one further condition,

$$\langle U_A, U_B\rangle = 0\,, \tag{2.21}$$

which follows from multiplying the second equation above by $\mathcal{D}_{\bar{B}}\bar{X}^I$ and using the symmetry of $\mathcal{N}$. Acting on (2.21) with $g^{A\bar{C}}\mathcal{D}_{\bar{C}}$ and using (2.17) yields

$$\langle V, U_A\rangle = \langle V, \mathcal{D}_A V\rangle = \langle V, \partial_A V\rangle = 0 \ . \tag{2.22}$$

When a prepotential $F(X)$ exists, the latter condition is trivially satisfied[6]. Strictly speaking it follows only for $n \neq 1$, because (2.21) is trivially satisfied when $n = 1$. In [27] it is argued that in this case (2.22) should be imposed as an extra requirement. Combining (2.21) and (2.22) it follows that

$$\langle V, \mathcal{D}_A U_B\rangle = \langle V, \mathcal{D}_A\mathcal{D}_B V\rangle = \langle V, \partial_A\partial_B V\rangle = 0 \ . \tag{2.23}$$

Let us now define the following $(2n+2)$-by$(2n+2)$ matrix $\mathcal{V}$ consisting of the row vectors

$$\mathcal{V} = \begin{pmatrix} V \\ \bar{U}^A \\ \bar{V} \\ U_A \end{pmatrix} \ , \tag{2.24}$$

[6]In discussions on the moduli space of Calabi-Yau three-folds, one usually argues that the F_I are locally determined by the X^I, from which the existence of a holomorphic prepotential $F(X)$ follows [8, 14, 16]. However, we stress that examples are known where the function F does not exist, simply because the X^I are not independent [19]. Furthermore, in the context of Calabi-Yau manifolds, (2.21) follows directly from the decomposition of the cohomology basis, while in our treatment it amounts to an extra condition (following from the symmetry of $\mathcal{N}$).

and transforming from the right under $Sp(2n+2,\mathbb{R})$. From the above results it is easy to see that $\mathcal{V}$ satisfies the symplectic condition

$$\mathcal{V}\,\Omega\,\mathcal{V}^{\mathrm{T}} = i\Omega \,, \tag{2.25}$$

so that $\mathcal{V}$ is isomorphic to an element of $Sp(2n+2,\mathbb{R})$. Therefore $\mathcal{V}$ is invertible and we can define $Sp(2n+2)$ connections $\mathcal{A}_A$ and $\mathcal{A}_{\bar{A}}$ such that

$$\mathcal{D}_A\mathcal{V} = \mathcal{A}_A\mathcal{V}\,, \qquad \mathcal{D}_{\bar{A}}\mathcal{V} = \mathcal{A}_{\bar{A}}\mathcal{V}\,. \tag{2.26}$$

The values of these two connections can be computed from $\mathcal{A}_A = i\langle\mathcal{D}_A\mathcal{V},\mathcal{V}^T\rangle\Omega$ and $\mathcal{A}_{\bar{A}} = i\langle\mathcal{D}_{\bar{A}}\mathcal{V},\mathcal{V}^T\rangle\Omega$. They are completely determined from the results above, with the exception of the contributions proportional to the symmetric tensor

$$C_{ABC} \equiv -i\langle\mathcal{D}_A U_B, U_C\rangle = -i\langle V, \mathcal{D}_A\mathcal{D}_B\mathcal{D}_C V\rangle = -i\langle V, \partial_A\partial_B\partial_C V\rangle \,, \tag{2.27}$$

the last expression being due to (2.22) and (2.23). The previous results constrain $\mathcal{D}_A U_B$ to be only proportional to $\bar{U}^A$:

$$\mathcal{D}_A U_B = C_{ABC}\,\bar{U}^C \,. \tag{2.28}$$

Incidentally, we note the following equation, which follows from combining (2.28) and (2.21),

$$\langle\mathcal{D}_A\mathcal{D}_B V, \mathcal{D}_C\mathcal{D}_D V\rangle = 0 \,. \tag{2.29}$$

The connections $\mathcal{A}$ are now determined and read

$$\begin{aligned}
\mathcal{A}_A &= \begin{pmatrix} 0 & 0 & 0 & \delta_A^C \\ 0 & 0 & \delta_A^B & 0 \\ 0 & 0 & 0 & 0 \\ 0 & C_{ABC} & 0 & 0 \end{pmatrix} , \\
\mathcal{A}_{\bar{A}} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{C}_{\bar{A}}^{BC} \\ 0 & g_{\bar{A}C} & 0 & 0 \\ g_{\bar{A}B} & 0 & 0 & 0 \end{pmatrix} .
\end{aligned} \tag{2.30}$$

Both these connections are nilpotent; the product of more than three of them vanishes: $\mathcal{A}_A\mathcal{A}_B\mathcal{A}_C\mathcal{A}_D = 0$ and likewise for the fourth power of the $\mathcal{A}_{\bar{A}}$. This feature was relevant in e.g. [20, 21]. The conditions (2.26) imply that the combined connection consisting of $\mathcal{A}$ and the Kähler and Levi-Civita connections must be flat. Calculating the integrability conditions

$$[\mathcal{D}_C - \mathcal{A}_C, \mathcal{D}_{\bar{D}} - \mathcal{A}_{\bar{D}}]\,\mathcal{V} = 0 \,; \qquad [\mathcal{D}_C - \mathcal{A}_C, \mathcal{D}_D - \mathcal{A}_D]\,\mathcal{V} = 0 \,, \tag{2.31}$$

using e.g.

$$[\mathcal{D}_C, \mathcal{D}_{\bar{D}}]\,Z_B = Z_A R^A{}_{BC\bar{D}} + Z_B(\bar{m} - m)g_{C\bar{D}} \,. \tag{2.32}$$

for a vector Z_A transforming under the Kähler symmetry as $Z_A \to e^{mf(z)+\bar{m}\bar{f}(\bar{z})}Z_A$, we obtain the following consequences. First of all the Riemann curvature (all other curvature components vanish for a Kähler manifold) is given by

$$R^A_{BC}{}^D = \delta^A_B\delta^D_C + \delta^A_C\delta^D_B - C_{BCE}\bar{C}^{ADE} \,, \tag{2.33}$$

which may be compared to (2.3). Secondly, the tensor C_{ABC} satisfies the following two conditions,

$$\mathcal{D}_{\bar{A}} C_{BCD} = \mathcal{D}_{[A} C_{B]CD} = 0 \ . \tag{2.34}$$

From these equations one deduces that $\mathcal{W}_{ABC} = e^{-K} C_{ABC}$ is independent of $\bar{z}$. Furthermore, C_{ABC} can be written as the third covariant derivative of some scalar function.

This completes the discussion of special geometry. In summary special Kähler manifolds can be defined starting from a complex $(2n+2)$-component vector V subject to the constraint (2.11). One demands the existence of symplectic holomorphic sections $v(z)$ proportional to V, such that $V = \exp(\frac{1}{2}K(z,\bar{z}))\, v(z)$, and identifies K as the Kähler potential. The holomorphic sections are defined projectively, as expressed in (2.13). In addition, one demands the symmetry of $\mathcal{N}$ defined by (2.20), or equivalently, imposes (2.21)[7]. Then all the above result follow.

If, as in the previous subsection, the sections F_I depend on X^I, they should be homogeneous of first degree in X^I and (2.21) is solved by $F_I = \partial_I F$ for a holomorphic prepotential $F(X) = \frac{1}{2} F_I X^I$. Then the expression (2.4) for $\mathcal{W}_{ABC}$ follows directly from the definition (2.27) of C_{ABC}, and one can also determine the following expression for C_{ABC} [20],

$$C_{ABC} = -\tfrac{i}{2} \mathcal{D}_A \mathcal{D}_B \mathcal{D}_C \Big[\Big(F_{IJ}(X(z,\bar{z})) - \bar{F}_{IJ}(\bar{X}(z,\bar{z})) \Big) \; X^I(z,\bar{z})\, X^J(z,\bar{z}) \Big] \, . \tag{2.35}$$

To derive this result one makes use of

$$\mathcal{D}_A \mathcal{D}_B X^I(z,\bar{z}) = C_{ABC}\, g^{C\bar{C}} \mathcal{D}_{\bar{C}} \bar{X}^I(z,\bar{z}) \ ,$$

which is implied by (2.28).

2.3 Examples of special Kähler manifolds

We give here some examples of functions $F(X)$ and their corresponding target spaces, which will be useful later on:

$$F = i[(X^0)^2 - (X^1)^2] \qquad \frac{SU(1,1)}{U(1)} \tag{2.36}$$

$$F = (X^1)^3/X^0 \qquad \frac{SU(1,1)}{U(1)} \tag{2.37}$$

$$F = \sqrt{X^0 (X^1)^3} \qquad \frac{SU(1,1)}{U(1)} \tag{2.38}$$

$$F = i X^I \eta_{IJ} X^J \qquad \frac{SU(1,n)}{SU(n) \otimes U(1)} \tag{2.39}$$

$$F = d_{ABC} X^A X^B X^C / X^0 \qquad \text{‘very special Kähler’} \tag{2.40}$$

The first three functions give rise to the manifold $SU(1,1)/U(1)$. However, the first one is not equivalent to the other two as the manifolds have a different value of the curvature [22]. The latter two are, however, equivalent by means of a symplectic transformation

[7] Imposing instead (2.22) deals also with the $n = 1$ case if one wants to restrict the manifolds to those for which an $N = 2$ supergravity action has been found [27].

(2.8). In the fourth example η is a constant non-degenerate real symmetric matrix. In order that the manifold has a non-empty positivity domain, the signature of this matrix should be $(+-\cdots-)$. So not all functions $F(X)$ allow an non-empty positivity domain. The last example, defined by a real symmetric tensor d_{ABC}, defines a class of special Kähler manifolds, which we will denote as 'very special' Kähler manifolds. This class of manifolds is important in the applications discussed below.

3 Isometries of special manifolds

3.1 Special Kähler : duality transformations

The formulation of special Kähler manifolds, as given in section 2 shows clearly the possibility of symplectic reparametrizations, as expressed in (2.8). Explicitly, such a transformation is of the form

$$\begin{aligned} \tilde{X}^I &= U^I{}_J X^J + V^{IJ} F_J \ , \\ \tilde{F}_I &= W_{IJ} X^J + Z_I{}^J F_J \ . \end{aligned} \tag{3.1}$$

When we start from a prepotential $F(X)$, the F_I are the derivatives of F, so that the first line expresses the dependence of the new coordinates $\tilde{X}$ on the old coordinates X. If this transformation is invertible (the full symplectic matrix itself is always invertible), the $\tilde{F}_I$ are again the derivatives of an new function $\tilde{F}(\tilde{X})$ of the new coordinates,

$$\tilde{F}_I(\tilde{X}) = \frac{\partial \tilde{F}(\tilde{X})}{\partial \tilde{X}^I} \ . \tag{3.2}$$

The integrability condition which implies this statement is equivalent to the condition that $\begin{pmatrix} U & V \\ W & V \end{pmatrix} \in Sp\,(2n+2, \mathrm{I\!R})$. Hence we obtain a new, but equivalent, formulation of the theory, and thus of the target-space manifold, in terms of the function $\tilde{F}$. The manifold was expressed in terms of the vector (X, F), and these transformations thus give a reparametrization of the same manifold. The diffeomorphisms are other ways to reparametrize the manifold. The total group of reparametrizations is thus

$$D_{pseudo} = Diff(\mathcal{M}) \times Sp(2(n+1), \mathrm{I\!R}) \ . \tag{3.3}$$

The elements of this group were called 'pseudo-symmetries' in [23]. As explained earlier, the field equations for the two actions based on functions F and $\tilde{F}$ are equivalent, but this equivalence involves duality transformations, i.e., rotations between electric and magnetic fields, which cannot be implemented locally on the underlying vector potentials. Consequently the relationship cannot be made explicit on the full Lagrangian.

As discussed in section 2.2 the n complex target-space coordinates z^A parametrize the symplectic sections proportional to (X^I, F_J), which are subject to the same symplectic transformations (3.1). In the simplest case, the sections proportional to the X^I are in one-to-one correspondence with the coordinates z^A, up to a projective transformation, while the F_J are the derivatives of the holomorphic prepotential, so that they are determined in terms of the X^I, and therefore in terms of the coordinates z. Then the first line of

(3.1) induces a diffeomorphism on the target-space coordinates. For example, on special coordinates a symplectic transformation yields

$$z^A = \frac{X^A}{X^0} \quad \longrightarrow \quad \tilde{z}^A = \frac{\tilde{X}^A}{\tilde{X}^0} \ . \tag{3.4}$$

This example clearly exhibits how the symplectic transformations induce a corresponding diffeomorphism of the coordinates. In a more general parametrization one encounters a projective term (here corresponding to the division by $\tilde{X}^0$) to ensure that one remains within the initially adopted parametrization. In the case that there is no function $F(X)$, the X^I are not independent and one has to choose another subset consisting of $n+1$ independent components of the symplectic sections.

The question now arises when the symplectic diffeomorphism leaves the action of the spinless fields invariant, so that it will constitute an isometry of the target-space manifold. For simplicity consider the case where a function $F(X)$ exists. By inverting the previous arguments, it is clear that the diffeomorphism induces a symplectic transformation on the sections proportional to X^I. However, this does not necessarily mean that the sections proportional to F_J transform according to (3.1). This can only be the case when (up to a quadratic polynomial with real coefficients)

$$\tilde{F}(\tilde{X}) = F(\tilde{X}) \ . \tag{3.5}$$

In other words the symplectic transformations must lead to the same holomorphic prepotential. Consequently, the corresponding supergravity theory coincides with the original one, so that in particular the spin-0 Lagrangian remains the same and the target-space metric is invariant. When no function $F(X)$ exists, then the condition (3.5) has no meaning. The condition for an isometry is still that the symplectic transformation on the sections is correctly induced by the transformations of the coordinates z, up to a projective transformation.

Henceforth the above isometries are called 'duality symmetries', as they are generically accompanied by duality transformations on the field equations and the Bianchi identities. However, we are not directly interested in fields other than the spin-0 ones, but are only concerned with the symplectic transformations as posssible isometries of the target-space metric. This changes effectively after dimensional reduction to $d=3$, as the spin-1 fields (as well as some components of the space-time metric) are converted to scalar fields and become part of the target space, as we shall discuss in the next section.

As infinitesimal transformations, the symplectic transformations are of the form

$$\mathcal{O} = \mathbb{1} + \mathcal{R} \ ; \qquad \mathcal{R} = \begin{pmatrix} B & -D \\ C & -B^T \end{pmatrix} \qquad \text{with} \ \begin{matrix} C = C^T \ , \\ D = D^T \ . \end{matrix} \tag{3.6}$$

Then the equation governing duality symmetries is the infinitesimal form of (3.5); it reads

$$< V, \mathcal{R} V >= 0 \ . \tag{3.7}$$

In the above we started from the symplectic transformations. A subclass of them are duality symmetries defined by solutions of (3.5) and (3.7). These duality symmetries are symmetries of the full supergravity field equations and their action on the scalar fields defines isometries. The question is whether the duality symmetries comprise all the isometries of the target space, i.e. whether

$$Iso(\mathcal{M}) \subset Sp(2(n+1), \mathbb{R}) \ . \tag{3.8}$$

We investigated this question in [9] for the very special Kähler manifolds, and found that in this case one does obtain the complete set of isometries from the symplectic transformations. For generic special Kähler manifolds no isometries have been found that are not induced by symplectic transformations, but on the other hand there is no proof that these do not exist.

3.2 Special quaternionic

We now consider the so-called **c** map [2] from a special Kähler to a quaternionic manifold. It is induced by reducing an $N = 2$ supergravity action in $d = 4$ space-time dimensions to an action in $d = 3$ space-time dimensions, by suppressing the dependence on one of the (spatial) coordinates. The resulting $d = 3$ supergravity theory can be written in terms of $d = 3$ fields and this rearranges the original fields such that the number of scalar fields increases from $2n$ to $4(n + 1)$, as is indicated in the table 2. The $4(n + 1)$ fields are

Table 2: The **c** map as dimensional reduction from $d = 4$ to $d = 3$ supergravity. The number of fields of various spins is indicated and names are assigned to the scalar fields in $d = 3$.

.

$d = 4$ spins	2	1	0
numbers	1	$n + 1$	$2n$
$d = 3$ spins			
2	1		
0	2	$2(n+1)$	$2n$
	ϕ, σ	A^I, B_I	$z^A, \bar{z}^A$

denoted by $\phi, \sigma, A^I, B_I, z^A$ and $\bar{z}^A$, and parametrize the quaternionic manifold. One may distinguish the following isometries of this quaternionic manifold [24]:

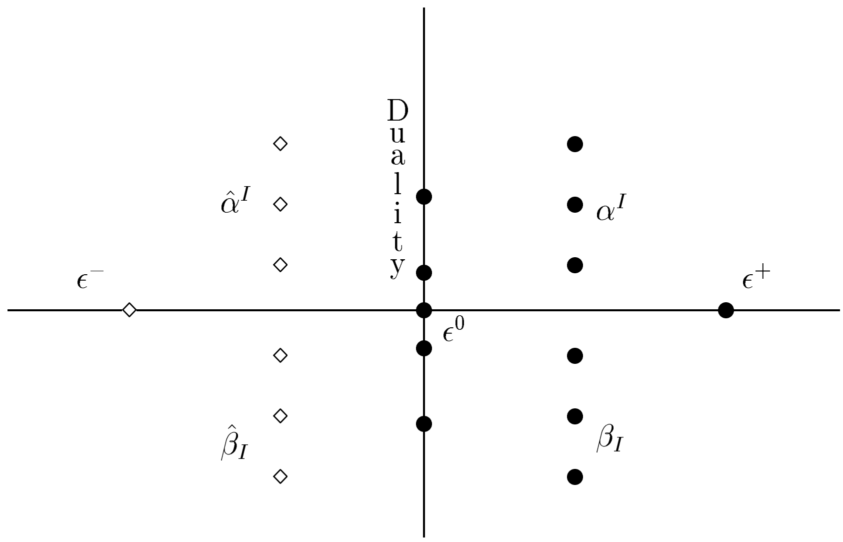
- the duality symmetries discussed previously for the corresponding Kähler manifold. z^A are the coordinates of that Kähler manifold, and their transformation under these symmetries remains the same. (A^I, B_I) transform under the duality transformations as a symplectic vector, while ϕ and σ remain inert.

- shifts and scale transformations which are a consequence of symmetries of the four-dimensional supergravity theory. We have the independent shifts of A^I, B_I and σ, denoted by α^I, β_I and ϵ^+, and the scaling denoted by ϵ^0.

The latter type of symmetries are called *extra* symmetries. We use this terminology to denote symmetries that can be understood from the invariances of the higher-dimensional theory. Such isometries exist for all manifolds of this type. Below we will also find so-called *hidden symmetries*, for which there exists no explanation in terms of the underlying higher-dimensional theory. Such isometries are only present for particular manifolds, depending on whether certain extra conditions are satisfied. These conditions have been determined for all the special Kähler and quaternionic manifolds [18, 25].

From the algebra of these transformations, we can draw a root lattice which consists of the root lattice of the duality symmetry group, extended with one extra dimension as depicted by the filled circles in table 3. The scale symmetry ϵ^0 is the new element of

Table 3: Root lattice of isometries of special quaternionic manifolds. The filled circles represent isometries corresponding to duality symmetries shifts and scalings, which are present in all special quaternionic manifolds. The unfilled squares represent the (hidden) isometries, which only exist for particular manifolds.

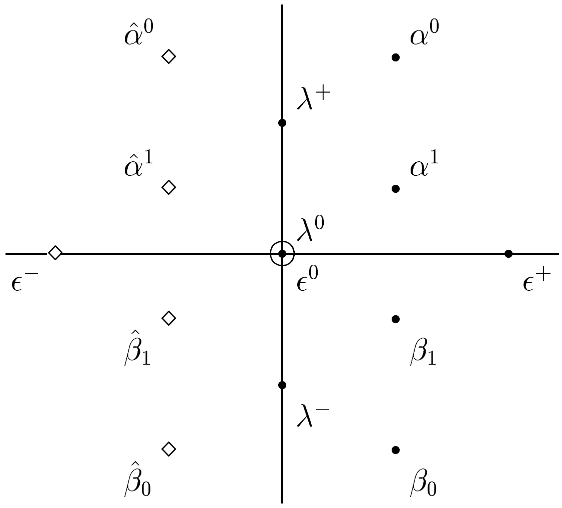


the Cartan subalgebra. Together with the $2n + 3$ shift symmetries this gives at least as many additional symmetries as additional coordinates, as compared to the original special Kähler manifold. These new symmetries provide the isometries such that a homogeneous Kähler manifold gives rise to a homogeneous quaternionic manifold. Note that the shift and scale symmetries combined with the solvable part of the duality group constitute the solvable algebra of the special quaternionic isometry group.

There can also be 'hidden symmetries' in the special quaternionic manifolds [25]. In the root lattice, these appear in the places of the unfilled squares. Their existence depends on the particular function $F(X)$ we started from. If all those indicated by $\hat{\alpha}^I$ and $\hat{\beta}_I$ exist, then and only then ϵ^- exists. This occurs if and only if the space is symmetric.

As an example consider the $(n = 1)$ special Kähler space $\frac{SU(1,1)}{U(1)}$. The isometries group $SU(1,1)$ is represented by λ^0 as 1-dimensional Cartan subalgebra, a positive root λ^+ and a negative one λ^-. It leads to the special quaternionic space with isometries given in table 4. This is an 8 dimensional space: $\frac{G_2}{SU(2)\otimes SU(2)}$.

Table 4: Root lattice of G_2.



4 Isometries of very special manifolds

The very special manifolds are defined by a 3-index symmetric tensor in n dimensions: d_{ABC}. The real manifold is $n-1$ dimensional, being defined in terms of n coordinates h^A subject to the constraint $d_{ABC}h^A h^B h^C = 1$.

The reduction of supergravity from $d = 5$ to $d = 4$induces an '**r** map' from the very special real manifolds to very special Kähler manifolds. This is depicted in table 5. This

Table 5: The **r** map induced by dimensional reduction from $d = 5$ to $d = 4$ supergravity. The number of fields of integer spins is indicated.

$d = 5$ spins	2	1	0
numbers	1	n	$n-1$
$d = 4$ spins			
2	1		
1	1	n	
0	1	n	$n-1$

map leads to a scalar manifold with $2n$ real coordinates or n complex ones. It can be followed by a **c** map to obtain $n + 1$ quaternions, as was explained before.

4.1 Isometries of very special real manifolds

Let us now consider the isometries of these manifolds [5]. They are determined by the matrices $\tilde{B}$ which satisfy[8]

$$d_{D(AB}\tilde{B}^{D}{}_{C)} = 0 \ . \tag{4.1}$$

4.2 Isometries of very special Kähler manifolds

The duality transformations of the corresponding Kähler manifold are defined by the symplectic matrix

$$\begin{pmatrix} B & -D \\ C & -B^T \end{pmatrix} ; \qquad \begin{aligned} B &= \begin{pmatrix} \beta & a_B \\ b^A & \tilde{B}^A_B + \frac{1}{3}\beta\,\delta^A_B \end{pmatrix} \\ C &= \begin{pmatrix} 0 & 0 \\ 0 & 6d_{ABC}\,b^C \end{pmatrix} \\ D &= \begin{pmatrix} 0 & 0 \\ 0 & 6e^{2K}d^{ABC}\,a_C \end{pmatrix} \ , \end{aligned} \tag{4.2}$$

where d^{ABC} is d_{ABC} where the indices are raised with the metric. They consist of the following isometries:

- $\tilde{B}$ are the solutions of (4.1).
- $\beta\,, b^A$ constitute $n+1$ extra isometries originating from the symmetries of the $d=5$ theory. Hence these isometries exist for all very special Kähler manifolds.
- Hidden symmetries exist for those independent parameters a_A for which $R^A{}_{BC}{}^D\,a_D$ is constant (in special coordinates).

In the root space the isometries appear generically as indicated in table 6. Compared to the very special real space, the corresponding Kähler space has at least as many additional isometries as it has additional coordinates

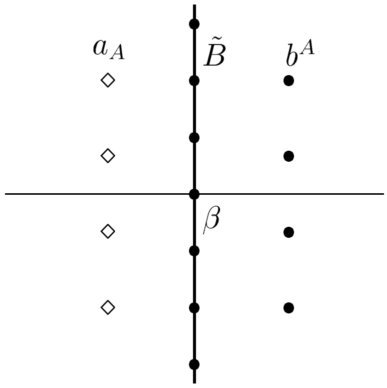
4.3 Isometries of very special quaternionic manifolds

In the very special quaternionic manifold (the image under the **c** map of a very special Kähler manifold), one finds the following isometries:

- all the isometries of the very special Kähler manifold.
- $\alpha^I\,, \beta_I\,, \epsilon^0\,, \epsilon^+$ constitute $2n+4$ extra isometries, as followed already from the general statement made for special quaternionic manifolds.
- $\hat{\beta}_0$ is an isometry present for all very special quaternionic manifolds.

[8] $(\cdots)$ indicates a symmetrisation in the corresponding indices with 'weight one', e.g. $V_{(A}W_{B)} = \frac{1}{2}\,(V_AW_B + V_BW_A)$.

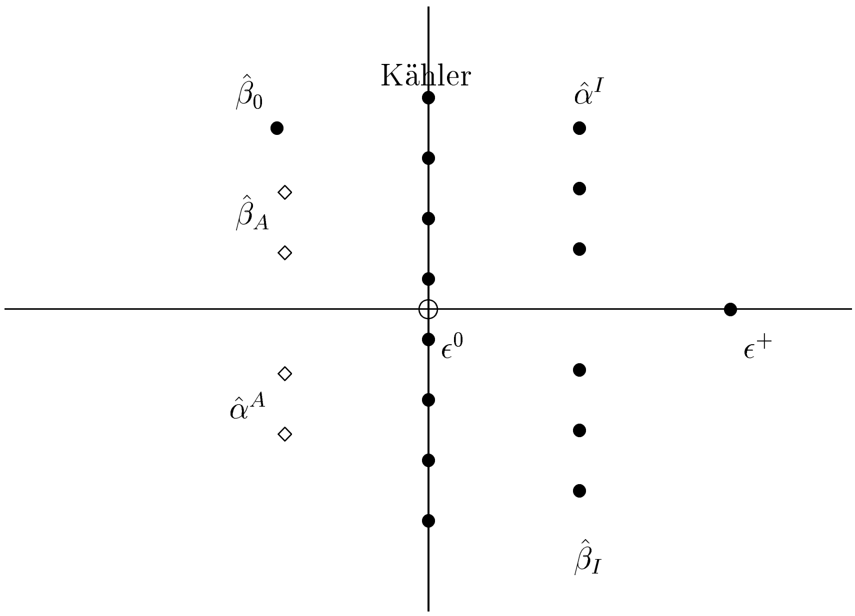
Table 6: Root lattice of the isometries in very special Kähler manifolds.



- Additional hidden symmetries can exist. There are symmetries $\hat{\beta}_A$ under the same condition as for the existence of the a_A in section 4.2. If some of the $\hat{\beta}_A$ are realized, then there are additional ones characterized by $\hat{\alpha}^A = d^{ABC}\hat{\beta}_B\hat{\beta}_C$ (for the exact conditions see section 4.3 of [4]). If the curvature is covariantly constant in all directions (a symmetric manifold) then all $\hat{\beta}_A$ and $\hat{\alpha}^A$ exist and so do the isometries $\hat{\alpha}^0$ and ϵ^-.

The above isometries give rise to the root lattice depicted in table 7.

Table 7: Root lattice of isometries of non-symmetric very special quaternionic manifolds.



5 Classification of homogeneous very special manifolds

In his classification Alekseevskiĭ obtained non-symmetric homogeneous quaternionic manifolds [1] . It was shown by Cecotti [3] that all of these are in fact in the category of very special quaternionic manifolds. This led us to investigate all the homogeneous very special quaternionic manifolds [10]. We proved a theorem [4] that there is a one-to-one mapping between all the homogeneous special quaternionic manifolds and the homogenous special Kähler manifolds[9], and similarly that there is a one-to-one mapping between all the homogeneous very special Kähler and the homogeneous very special real manifolds. Therefore one can start from the very special real manifolds, characterised by the symmetric tensor d_{ABC}.

First we adopt a so-called canonical parametrization [5], in which the tensor takes the form: (with $A = 1$ or $a = 2, \ldots n$)

$$d_{111} = 1 \; ; \qquad d_{11a} = 0 \; ; \qquad d_{1ab} = -\tfrac{1}{2}\delta_{ab} \; . \tag{5.1}$$

Imposing that solutions of (4.1) provide transitive isometries, leads to the equation

$$d_{e(ab}\, d_{cd)e} - \tfrac{1}{2}\, \delta_{(ab}\, \delta_{cd)} = d_{e(ab}\, A_{c)e;d} \tag{5.2}$$

where A is an arbitrary tensor, antisymmetric in its first two indices. This condition can be solved after an elaborate series of steps and we arrived at the following general solution (after some redefinitions, so that we are no longer in the canonical parametrization) [10]. First, decompose the indices A into $A = 1, 2, \mu, i$, with $\mu = 1, \ldots, q+1$ and $i = 1, \ldots, r$, so that $n = 3 + q + r$. Hence we assume $n \geq 2$. The result can then be expressed as follows,

$$d_{ABC}\, h^A h^B h^C = 3\Big\{ h^1\, (h^2)^2 - h^1\, (h^\mu)^2 - h^2\, (h^i)^2 + \gamma_{\mu ij}\, h^\mu\, h^i\, h^j \Big\} \; . \tag{5.3}$$

Here the coefficients $\gamma_{\mu ij}$ are $q+1$ real $r \times r$ gamma matrices that generate a real Clifford algebra of positive signature ($\mathcal{C}(q+1, 0)$). Therefore the solutions are completely classified by specifying a representation of this Clifford algebra. The irreducible representations of these Clifford algebras are unique, except for $q = 4m$, when there are two inequivalent ones, which are, however, interchangeable. Therefore we can denote the solution by $L(q, P)$ for $q \neq 4m$ and $L(4m, P, \dot{P})$, where P is the number of irreducible representations, and

$$L(4m, P, \dot{P}) = L(4m, \dot{P}, P) \; . \tag{5.4}$$

When

$$(\gamma_\mu)_{(ij}\, (\gamma_\mu)_{k\ell)} = \delta_{(ij}\delta_{k\ell)} \; , \tag{5.5}$$

the maximal number of hidden isometries is realized for the real, Kähler and quaternionic case. The corresponding spaces are then symmetric and are listed in table 8. However, this table does not comprise all the symmetric very special spaces. Also the real very special real manifolds $L(-1, P)$ are symmetric, as we shall discuss at the end of this section. One may now consider the question whether table 8 contains all symmetric special Kähler

[9]The theorem presupposes that the homogeneity of the special Kähler manifold is due to isometries satisfying (3.8). For very special Kähler manifolds it was shown that all iosometries are contained in the duality transformations [9]. For other special manifolds no counter examples are known.

Table 8: Symmetric very special manifolds

	real	Kähler	quaternionic
$L(-1,0)$	$SO(1,1)$	$\left[\frac{SU(1,1)}{U(1)}\right]^2$	$\frac{SO(3,4)}{(SU(2))^3}$
$L(0,P)$	$SO(1,1)\otimes\frac{SO(P+1,1)}{SO(P+1)}$	$\frac{SU(1,1)}{U(1)}\otimes\frac{SO(P+2,2)}{SO(P+2)\otimes SO(2)}$	$\frac{SO(P+4,4)}{SO(P+4)\otimes SO(4)}$
$L(1,1)$	$\frac{S\ell(3,R)}{SO(3)}$	$\frac{Sp(6)}{U(3)}$	$\frac{F_4}{USp(6)\otimes SU(2)}$
$L(2,1)$	$\frac{S\ell(3,C)}{SU(3)}$	$\frac{SU(3,3)}{SU(3)\otimes SU(3)\otimes U(1)}$	$\frac{E_6}{SU(6)\otimes SU(2)}$
$L(4,1)$	$\frac{SU^*(6)}{Sp(3)}$	$\frac{SO^*(12)}{SU(6)\otimes U(1)}$	$\frac{E_7}{SO(12)\otimes SU(2)}$
$L(8,1)$	$\frac{E_6}{F_4}$	$\frac{E_7}{E_6\otimes U(1)}$	$\frac{E_8}{E_7\otimes SU(2)}$

and quaternionic spaces. For the special Kähler manifolds a complete classification of the symmetric manifolds was given in [22]. As it turns out, apart from those already found, there is the image under the **r** map of an empty very special real manifold (it corresponds to the pure supergravity action in $d=5$, i.e. without scalar fields), and there are the complex projective spaces. The latter are not of the very special type. Similarly the symmetric quaternionic spaces that were not yet mentioned, are those in the image under the **c** map of the special Kähler manifolds just mentioned and of the empty special Kähler manifold (pure supergravity in $d=4$); in addition there are the quaternionic projective spaces. This leads to table 9. Note that we extended the notation $L(q,P)$ to include

Table 9: Homogeneous manifolds. In this table, q, P, $\dot P$ and m denote positive integers or zero, and $q\neq 4m$. SG denotes an empty space, which corresponds to supergravity models without scalars. Furthermore, $L(4m,P,\dot P)=L(4m,\dot P,P)$. The horizontal lines separate spaces of different rank. The first non-empty space in each column has rank 1. Going to the right or down a line increases the rank by 1. The manifolds indicated by a $\star$ were not known before our classification, except for the cases $L(0,P,\dot P)$.

	real	Kähler	quaternionic
$L(-3,P)$			$\frac{USp(2P+2,2)}{USp(2P+2)\otimes SU(2)}$
SG_4		SG	$\frac{U(1,2)}{U(1)\otimes U(2)}$
$L(-2,P)$		$\frac{U(P+1,1)}{U(P+1)\otimes U(1)}$	$\frac{SU(P+2,2)}{SU(P+2)\otimes SU(2)\otimes U(1)}$
SG_5	SG	$\frac{SU(1,1)}{U(1)}$	$\frac{G_2}{SU(2)\otimes SU(2)}$
$L(-1,P)$	$\frac{SO(P+1,1)}{SO(P+1)}$	$\star$	$\star$
$L(4m,P,\dot P)$	$\star$	$\star$	$\star$
$L(q,P)$	$X(P,q)$	$H(P,q)$	$V(P,q)$

$q=-2$ and $q=-3$, which have no meaning as Clifford algebra $\mathcal{C}(q+1,0)$. In these cases also the real manifold does not exist, which brings us outside the framework on which this notation was based. We further elucidate this in the next section.

Note that we started from the equation (4.1), which is the condition for symmetries of the full $d = 5$ supergravity theory, leading to what were called duality symmetries in the Kähler case. We mentioned already that for very special Kähler manifolds all the isometries are obtained as duality transformations, i.e. (3.8) holds. On the other hand, the case $L(-1, P)$ exhibits target-space isometries that are not an invariance of the full $d = 5$ supergravity theory. These additional isometries promote the target space to a symmetric space. When applying the **r** map, the non-invariant sector of the supergravity theory become relevant for the scalar sector of the very special Kähler manifold, and therefore these transformations do not constitute isometries of the very special Kähler manifold. As a result we have that the real manifold $L(-1, P)$ is a symmetric space $(\frac{SO(P+1,1)}{SO(P+1)})$, while the corresponding Kähler and quaternionic spaces are non-symmetric (but still homogeneous because the solvable subalgebra corresponds to a transitive group of motions, which leaves the full supergravity action invariant).

6 Isometries of very special homogeneous manifolds

We now apply the above results for very special manifolds to the homogeneous manifolds.

6.1 Symmetries of representations of real Clifford algebras

We start from the symmetries of the Clifford algebra. They consist of the rotation group $SO(q+1)$ (or rather its cover group) and the matrices S satisfying

$$[\gamma_\mu, S] = 0 \; ; \qquad S = -S^T \; . \tag{6.1}$$

This defines $\mathcal{S}_q(P, \dot{P})$, the metric-preserving group in the centralizer of the Clifford algebra. On an irreducible representation, Schur's lemma implies that the solution of the first condition is just the division algebra $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. On reducible representations we obtain the general matrices $\mathbb{R}(P)$, $\mathbb{C}(P)$ or $\mathbb{H}(P)$. The metric-preserving property (second condition) reduces this to $SO(P)$, $SU(P)$ or $USp(2P)$. The real Clifford algebras, their dimensions and the group $\mathcal{S}_q$ are given in table 10 [26].

6.2 Isometries of homogeneous very special real manifolds

Here we have the solutions of (4.1) for the d-symbols given by (5.3). First of all, a scaling symmetry λ is obvious from the form of d_{ABC}. Other solutions are in eigenspaces of of this symmetry

$$\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{X}_{3/2} \tag{6.2}$$

$$\begin{aligned} \mathcal{X}_0 &= \lambda \oplus so(q+1,1) \oplus \mathcal{S}_q(P, \dot{P}) \\ \mathcal{X}_{3/2} &= (spinor, vector) \; , \end{aligned} \tag{6.3}$$

where *spinor* denotes a spinor representation of $\mathcal{C}^+(q+1,1) \simeq \mathcal{C}(q+1,0)$ (of dimension $\mathcal{D}_{q+1}$). In case the space is symmetric, also $\mathcal{X}_{-3/2}$ appears with the same assignment under $\mathcal{X}_0$ as $\mathcal{X}_{3/2}$.

q	$\mathcal{C}(q+1,0)$	$\mathcal{D}_{q+1}$	$\mathcal{S}_q(P,\dot{P})$
-1	$\mathbb{R}$	1	$SO(P)$
0	$\mathbb{R}\oplus\mathbb{R}$	1	$SO(P)\otimes SO(\dot{P})$
1	$\mathbb{R}(2)$	2	$SO(P)$
2	$\mathbb{C}(2)$	4	$U(P)$
3	$\mathbb{H}(2)$	8	$USp(2P)$
4	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	8	$USp(2P)\otimes USp(2\dot{P})$
5	$\mathbb{H}(4)$	16	$USp(2P)$
6	$\mathbb{C}(8)$	16	$U(P)$
7	$\mathbb{R}(16)$	16	$SO(P)$
$n+8$	$\mathbb{R}(16)\otimes\mathcal{C}(n+1,0)$	$16\ \mathcal{D}_n$	as for $q=n$

Table 10: Real Clifford algebras $\mathcal{C}(q{+}1,0)$. Here $\mathbf{F}(n)$ stands for $n\times n$ matrices with entries over the field $\mathbf{F}$, while $\mathcal{D}_{q+1}$ denotes the real dimension of an irreducible representation of the Clifford algebra. $\mathcal{S}_q(P,\dot{P})$ is the metric preserving group in the centralizer of the Clifford algebra in the $(P+\dot{P})\mathcal{D}_{q+1}$-dimensional representation.

The isotropy group is always

$$H = SO(q+1)\otimes\mathcal{S}_q(P,\dot{P})\ . \tag{6.4}$$

Guided by the $SO(q+1,1)$ in the isometry group, we can rewrite the result (5.3) in an $SO(q+1,1)$ invariant form:

$$d_{ABC}h^A\,h^B\,h^C = -\eta_{MN}\,h^M h^N h^1 + \gamma_{M\,ij}\,h^M h^i h^j \tag{6.5}$$

where M takes $q+2$ values (the value 2 or μ from before), η_{MN} is the $SO(q+1,1)$ metric, and

$$\gamma_M = \begin{pmatrix} 0 & \gamma_{M\,ik} \\ \tilde{\gamma}_M{}^{jl} & 0 \end{pmatrix} \tag{6.6}$$

are the corresponding γ-matrices.

6.3 Isometries of homogeneous special Kähler manifolds

As isometries of the homogeneous very special Kähler manifold we find:

- the isometries discussed for the real manifold.
- the $n+1$ extra isometries which appear for all very special Kähler manifolds, of which β appears in the Cartan subalgebra.
- the condition for hidden isometries has always $q+2$ solutions: a_M.

This leads to the root diagram of table 11. By rotating the axes of this figure, the root

Table 11: Isometries of non-symmetric homogeneous special Kähler manifolds.

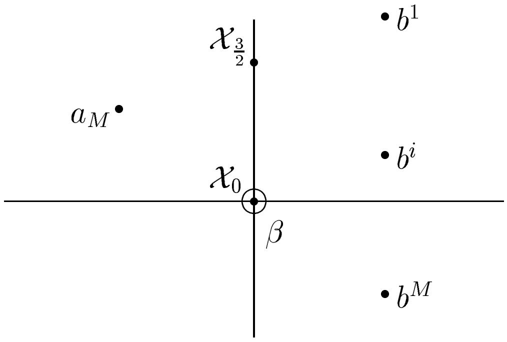


diagram contains again only non-negative roots. As root space we obtain

$$\begin{aligned}
\mathcal{W} &= \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \mathcal{W}_2 \ , \\
\mathcal{W}_0 &= \lambda' \oplus so(q+2,2) \oplus \mathcal{S}_q(P,\dot{P}) \ , \\
\mathcal{W}_1 &= (1, spinor, vector) \ , \\
\mathcal{W}_2 &= (2,0,0) \ ,
\end{aligned} \tag{6.7}$$

In $\mathcal{W}_1$ appears a spinor representation of

$$\mathcal{C}^+(q+2,2) \simeq \mathcal{C}(q+2,1) \simeq \mathcal{C}(q+1,0) \otimes \mathrm{I\!R}(2) \tag{6.8}$$

This representation comprises the roots in $\mathcal{X}_{\frac{3}{2}}$ and the b^i. It thus contains P or $P+\dot{P}$ spinors of dimension $2\mathcal{D}_{q+1}$ each. The isotropy group is

$$H = SO(q+2) \otimes U(1) \otimes \mathcal{S}_q(P,\dot{P}) \ . \tag{6.9}$$

Again in the case that the manifold is symmetric, we find also $\mathcal{W}_{-1}$ and $\mathcal{W}_{-2}$ with the same assignments as $\mathcal{W}_1$ and $\mathcal{W}_2$.

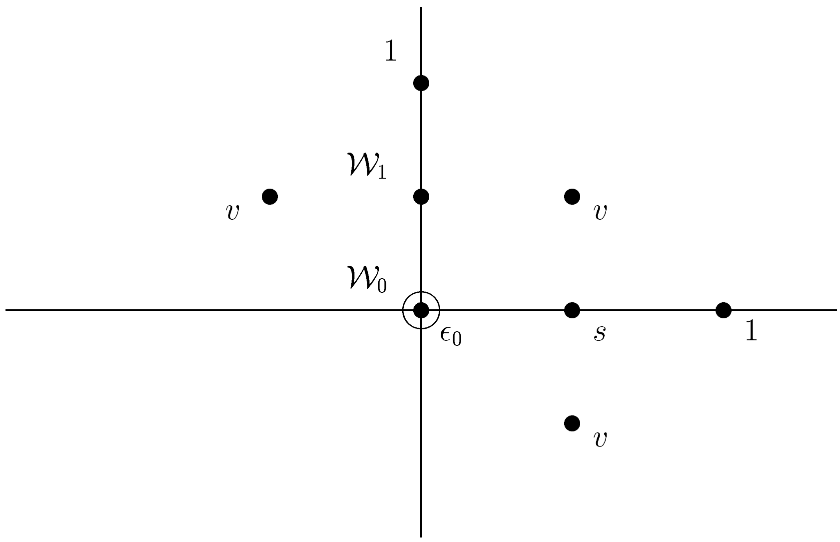
Note that in this classification of the isometries, q appears always in the linear combination $q+2$. Therefore one may consider the case $q=-2$, which made no sense for the real spaces. Then it turns out that we obtain the isometries of the special Kähler manifolds $\frac{U(P+1,1)}{U(P+1)\otimes U(1)}$, which are, however, not very special. This motivates the assignment $L(-2,P)$, used in table 9.

6.4 Isometries of homogeneous quaternionic manifolds

For the homogeneous special quaternionic the isometries are

- isometries of the special Kähler manifold.
- $2n+5$ extra isometries as for all very special quaternionic manifolds, of which ϵ^0 is the extra element in the Cartan subalgebra.
- $q+2$ isometries $\hat{\beta}_M$ and one isometry $\hat{\alpha}^1$.

Table 12: Isometries of non-symmetric homogeneous very special quaternionic manifolds.



These isometries appear in a root diagram as indicated in table 12. Again by a rotation of the axes all roots become non-negative and can be combined as follows:

$$\begin{aligned}
&\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_2 \ , \\
&\mathcal{V}_0 = \epsilon' \oplus so(q+3,3) \oplus \mathcal{S}_q(P,\dot{P}) \ , \\
&\mathcal{V}_1 = (1, spinor, vector) \ , \\
&\mathcal{V}_2 = (2, vector, 0) \ .
\end{aligned} \tag{6.10}$$

The isotropy group is

$$H = SO(q+3) \otimes SU(2) \otimes \mathcal{S}_q(P,\dot{P}) \ . \tag{6.11}$$

The dimension of the irreducible spinor representations is now $4\mathcal{D}_{q+1}$. For the symmetric manifolds we have also $\mathcal{V}_{-1}$ and $\mathcal{V}_{-2}$ with similar assignments as $\mathcal{V}_1$ and $\mathcal{V}_2$.

For the isometries of the quaternionic manifolds the linear combination $q+3$ appears everywhere, so we can consider also the case $q=-3$. Then we obtain the root lattice of the quaternionic homogeneous (and symmetric) spaces, which were called 'of type I' in [1]. These are the quaternionic projective spaces $\frac{USp(2P+2,2)}{USp(2P+2)\otimes USp(2)}$, which thus deserve the name $L(-3,P)$ as indicated in table 9.

7 Summary

The notion of special Kähler manifolds induces also a notion of '*special quaternionic manifolds*', which are those manifolds appearing in the image of the **c** map. Similarly, from the very special real manifolds (characterised by a symmetric tensor d_{ABC}), very special Kähler manifolds are induced as the image of the **r** map and very special quaternionic manifolds as the image of the **c**$\circ$**r** map.

The *homogeneous* very special real manifolds, special Kähler manifolds and quaternionic manifolds are classified as $L(q,P)$ for $q \neq 4m$ or $L(4m,P,\dot{P}) = L(4m,\dot{P},P)$, where P and $\dot{P}$ are non-negative integers. q is also an integer with $q \geq -1$ for the very special real manifolds, $q \geq -2$ for the special Kähler manifolds and $q \geq -3$ for the quaternionic manifolds. These classifications are related to spinor representations of resp. $SO(q+1,1)$, $SO(q+2,2)$ and $SO(q+3,3)$. All those are related by equivalences

$$\mathcal{C}(p,q) \otimes \mathrm{I\!R}(2) \simeq \mathcal{C}(p+1,q+1) \; ; \qquad \mathcal{C}^+(q+r,r) \simeq \mathcal{C}(q+r,r-1) \; . \tag{7.1}$$

The d-symbols of the very special manifolds were defined in terms of realizations of real positive-definite Clifford algebras $\mathcal{C}(q+1,0)$. The special Kähler manifold $SU(1,1)/U(1)$, the image under the **r** map of the empty very special real manifold, its quaternionic image under the **c** map, $\frac{G_2}{SU(2)\otimes SU(2)}$, and the quaternionic manifold $\frac{U(1,2)}{U(1)\otimes U(2)}$, image under the **c** map of the empty special Kähler manifold, are not contained in this scheme. On the other hand, this set-up led to several homogeneous quaternionic manifolds that did not occur in Alekseevskiĭ's classification [1].

For both the real, the Kähler and the quaternionic case, the isometries exhibit a grading with respect to one generator, denoted by λ. For the non-symmetric homogeneous manifolds the root space is

$$\begin{aligned} &G = G_0 \oplus G_1 \oplus G_2 \\ &G_0 = \lambda \otimes \mathcal{O} \otimes \mathcal{S}_q(P,\dot{P}) \\ &G_1 = (1, spinor, vector) \; , \end{aligned} \tag{7.2}$$

where $\mathcal{S}_q(P) = SO(P)$ or $U(P)$ or $USp(2P)$ (see table 10), *spinor* denotes a spinor representation of the group $\mathcal{O}$, which is of dimension $\mathcal{D}_{q+1}$ for real manifolds, $2\mathcal{D}_{q+1}$ for the Kähler case, or $4\mathcal{D}_{q+1}$ for the quaternionic case. The group $\mathcal{O}$ and eigenspace G_2 are given by

	real	Kähler	quatern.
$\mathcal{O}$	$SO(q+1,1)$	$SO(q+2,2)$	$SO(q+3,3)$
G_2	$-$	*singlet*	$(vector, 1)$

(7.3)

In symmetric spaces also G_{-1} and G_{-2} occur.

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