Quarks, fractals, non-linearities, and related elliptic operators

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1 Introduction

This paper surveys some recent results in the theory of function spaces. First we deal with quarkonial decompositions of function spaces and related Taylor expansions of distributions. These subatomic representations are of interest for their own sake (at least we hope so). But they are also an efficient tool in some applications, which we are going to describe afterwards:

(i) Function spaces on fractals and a spectral theory of related fractal elliptic operators,

(ii) mapping properties of some special non-linear operators and their use in connection with a regularity theory of related semilinear elliptic differential equations.

Our presentation will be somewhat sketchy. We outline motivations and basic ideas, describe interrelations, and hint on further possibilities. We do not give most general formulations. As for further details, systematic treatments, and proofs one has to consult the cited papers and the references given there. In other words, what follows is by no means a balanced report of the state of art and its roots, but at the best a guide where more information (also with respect to the omitted literature) can be found.

2 Function spaces: the Weierstrassian approach

2.1 Motivation

2.1.1 Holomorphic functions

Let $f(x)$, where $x = (x_1, x_2)$, be a holomorphic function with respect to the complex variable $z = x_1 + ix_2$ in a connected domain Ω in $\mathbb{R}^2$. Let K_j be open circles centred at $x^j \in \Omega$ and of radius r_j with $0 < r_j < 1$ such that for some $N \in \mathbb{N}$ at most N of these circles have a non-empty intersection,

$$\Omega = \bigcup_{l=1}^{\infty} K_l \quad \text{and} \quad \operatorname{dist}(K_j, \partial\Omega) > r_j \quad \text{where} \quad j \in \mathbb{N}. \tag{1}$$

We assume in addition that the circles K_j are chosen in such a way that there is a resolution of unity by C^∞ functions $\psi_j(x)$ with

$$\operatorname{supp}\psi_j \subset K_j, \quad 1 = \sum_{j=1}^{\infty} \psi_j(x) \quad \text{if} \quad x \in \Omega, \tag{2}$$

and

$$|D^\gamma \psi_j(x)| \le c_\gamma \, r_j^{-|\gamma|} \quad \text{where} \quad \gamma \in \mathbb{N}_0^n, \; j \in \mathbb{N}. \tag{3}$$

Here $c_\gamma > 0$ are suitable constants which are independent of $j \in \mathbb{N}$. With $z = x_1 + ix_2$ and $z^j = x_1^j + ix_2^j$ it follows from the classical Taylor expansion,

$$\begin{aligned} f(x) &= \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \varrho_j^k (z - z^j)^k \psi_j(x) \\ &= \sum_{j=1}^{\infty} \sum_{\gamma \in \mathbb{N}_0^2} \lambda_j^\gamma (x - x^j)^\gamma \psi_j(x), \quad x \in \Omega, \end{aligned} \tag{4}$$

where $\varrho_j^k \in \mathbb{C}$ are the Taylor coefficients with respect to the off-points $z^j \in \Omega$. Since by (1) for some $c_j \ge 0$,

$$|\varrho_j^k| \le c_j (2r_j)^{-k} \quad \text{if} \quad j \in \mathbb{N} \quad \text{and} \quad k \in \mathbb{N}_0;$$

$$(z - z^j)^k = (x_1 - x_1^j + i(x_2 - x_2^j))^k = \sum_{|\gamma| = k} a_\gamma (x - x^j)^\gamma \tag{5}$$

with $\sum_{|\gamma|=k} |a_\gamma| = 2^k$, and

$$\lambda_j^\gamma = \varrho_j^k a_\gamma \quad \text{where} \quad |\gamma| = k, \tag{6}$$

both the complex and the real representation in (4) converge absolutely at any $x \in \Omega$. One may ask whether this *Weierstrassian approach* to holomorphic functions has a counterpart for non-smooth functions, function spaces and (tempered) distributions.

2.1.2 Function spaces

Compared with 2.1.1 elements of function spaces on $\mathbb{R}^n$ are usually defined in the *Riemannian spirit*, that means in qualitative terms. For example, f belongs to the Sobolev space $W_p^k(\mathbb{R}^n)$ with $1 < p < \infty$ and $k \in \mathbb{N}_0$ if, by definition,

$$D^\gamma f \in L_p(\mathbb{R}^n) \quad \text{where} \quad |\gamma| \le k.$$

Similarly, Hölder-Zygmund spaces and classical Besov spaces are defined via differences $\Delta_h^M f(x)$. Recall that all the spaces

$$B^s_{pq}(\mathbb{R}^n) \quad \text{and} \quad F^s_{pq}(\mathbb{R}^n) \quad \text{with} \quad 0 < p \le \infty \ , \ 0 < q \le \infty \ , \ s \in \mathbb{R}, \tag{7}$$

are introduced in qualitative terms, [11, 12, 13], how differently the diverse definitions may look like. We assume that the reader is familiar with basic notation of these spaces. We only recall the special cases:

$$\mathcal{C}^s = B^s_{\infty\,\infty}, \quad s \in \mathbb{R}, \quad \text{(Hölder-Zygmund spaces)},$$

$$H^s_p = F^s_{p,2}, \quad 0 < p < \infty, \ s \in \mathbb{R}, \quad \text{(Hardy-Sobolev spaces)}.$$

It is our aim to introduce the spaces (7) in a *Weierstrassian spirit*: We ask for representations of the real version in (4) with suitable building blocks of type $(x - x^j)^\gamma \psi_j(x)$ where otherwise all information can be extracted from the coefficients λ_j^γ. In contrast to holomorphic functions, distributions in $\mathbb{R}^n$ are only local. Hence it is quite clear that the circles K_j in 2.1.1 must be replaced by a sequence of lattices with mesh lengths tending to zero. The first choice in $\mathbb{R}^n$ may be

$$\left\{ 2^{-j} m : m \in \mathbb{Z}^n \right\}, \quad j \in \mathbb{N}_0, \tag{8}$$

where $\mathbb{Z}^n$ is the lattice of all points $x \in \mathbb{R}^n$ with integer-valued components. Let $\psi(x)$ be a suitable non-negative compactly supported C^∞ function and let $\psi^\gamma(x) = x^\gamma \psi(x)$, then the wavelet procedure

$$\psi^\gamma(2^j x - m) \, ; \quad \gamma \in \mathbb{N}_0^n, \ j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n, \tag{9}$$

seems to be a reasonable substitute of the functions $(x - x^j)^\gamma \psi_j(x)$ in (4). Let Δ be the Laplacian in $\mathbb{R}^n$ and let A be either B or F. Recall the well-known lifting property

$$(\mathrm{id} - \Delta)^m A^s_{pq}(\mathbb{R}^n) = A^{s-2m}_{pq}(\mathbb{R}^n) \tag{10}$$

where

$$m \in \mathbb{N}, \quad 0 < p \leq \infty, \quad 0 < q \leq \infty, \quad s \in \mathbb{R}. \tag{11}$$

Together with the lifting (10) the outcome is perfect: The building blocks (9) are completely sufficient for the Weierstrassian approach to the spaces in (7) (s large, $p < \infty$ in the F-case). In other words, we ask for representations of type

$$f(x) = \sum_{\gamma \in \mathbb{N}_0^n} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}^{\gamma} \psi^{\gamma}(2^j x - m), \quad x \in \mathbb{R}^n. \tag{12}$$

2.2 Sequence spaces

2.2.1 Basic notation

As above we use standard notation: $\mathbb{R}^n$ (Euclidean n-space); $\mathbb{N}$ (natural numbers); $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $\mathbb{Z}^n$ (explained above); $\mathbb{N}_0^n$ multi-indices; $\mathbb{C}$ (complex numbers). Furthermore, let $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}_0^n$.

Cubes in $\mathbb{R}^n$, centred at $2^{-\nu}m$ with sides parallel to the axes and of length $2^{-\nu}$, where $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, are denoted by $Q_{\nu m}$. Let $\chi_{\nu m}^{(p)}$ be the p-normalised characteristic function of $Q_{\nu m}$, this means

$$\chi_{\nu m}^{(p)}(x) = 2^{\nu n/p} \quad \text{if } x \in Q_{\nu m} \quad \text{and} \quad \chi_{\nu m}^{(p)}(x) = 0 \quad \text{if } x \notin Q_{\nu m}, \tag{13}$$

where $\nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n$, and $0 < p \leq \infty$. Of course,

$$\| \chi_{\nu m}^{(p)} \,|\, L_p(\mathbb{R}^n) \| = 1, \tag{14}$$

where $L_p(\mathbb{R}^n)$ are the usual Lebesgue spaces of p-integrable complex-valued functions in $\mathbb{R}^n$, quasi-normed in the natural way.

2.2.2. Definition. Let $0 < p \leq \infty,\ 0 < q \leq \infty$, and

$$\lambda = \{\lambda_{\nu m} \in \mathbb{C} :\ \nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n\}. \tag{15}$$

Then

$$b_{pq} = \left\{ \lambda :\ \|\lambda \,|\, b_{pq}\| = \left(\sum_{\nu=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p \right)^{q/p} \right)^{1/q} < \infty \right\} \tag{16}$$

and

$$f_{pq} = \left\{ \lambda : \, \|\lambda \,|\, f_{pq}\| = \left\| \left(\sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m} \chi_{\nu m}^{(p)}(\cdot)|^q \right)^{1/q} \,|\, L_p(\mathbb{R}^n) \right\| < \infty \right\} \tag{17}$$

(with the usual modification if $p = \infty$ and/or $q = \infty$).

2.2.3. Remark. By (14) holds

$$b_{pp} = f_{pp} = \ell_p, \quad 0 < p \le \infty. \tag{18}$$

Furthermore, using Hölder's inequality, it follows

$$b_{p,\min(p,q)} \subset f_{pq} \subset b_{p,\max(p,q)}. \tag{19}$$

2.3 Function spaces

2.3.1 Quarks

Let ψ be a C^∞ function in $\mathbb{R}^n$ with a compact support (say, near the origin, but this is unimportant) and

$$\sum_{m \in \mathbb{Z}^n} \psi(x-m) = 1 \quad \text{if } x \in \mathbb{R}^n. \tag{20}$$

Let $0 < p \le \infty$ and

$$s > \sigma_p = n \left(\frac{1}{p} - 1 \right)_+ . \tag{21}$$

Recall $\psi^\gamma(x) = x^\gamma \psi(x)$. Then

$$(\gamma qu)_{\nu m}(x) = 2^{-\nu(s-n/p)} \psi^\gamma(2^\nu x - m) \tag{22}$$

are called (s,p)-γ-quarks, related to $Q_{\nu m}$ where $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. This coincides essentially with the normalised building blocks in (12), such that

$$\|(\gamma qu)_{\nu m} \,|\, B^s_{pp}(\mathbb{R}^n)\| \sim \|(\gamma qu)_{\nu m} \,|\, H^s_p(\mathbb{R}^n)\| \sim 1, \tag{23}$$

where the equivalence $\sim$ is independent of ν and m (but may depend on γ). As suggested by (12) we complement (15) by

$$\lambda^\gamma = \{\lambda^\gamma_{\nu m} \in \mathbb{C} : \, \nu \in \mathbb{N}_0, \; m \in \mathbb{Z}^n\}, \qquad \gamma \in \mathbb{N}^n_0. \tag{24}$$

As usual, $S(\mathbb{R}^n)$ stands for the Schwartz space of all rapidly decreasing (complex-valued) C^∞ functions on $\mathbb{R}^n$. Its dual $S'(\mathbb{R}^n)$ is the collection of all tempered distributions.

2.3.2. Proposition. *Let* $0 < p \le \infty,\ s > \sigma_p,\ 0 < q \le \infty$ *and* $\mu \ge \mu_0(s,p)$. *Let* $(\gamma qu)_{jm}$ *be given by* (22). *If*

$$\sup_\gamma 2^{\mu|\gamma|} \|\lambda^\gamma \mid b_{pq}\| < \infty, \tag{25}$$

then

$$f = \sum_{\gamma \in \mathbb{N}_0^n} \sum_{j=0}^\infty \sum_{m \in \mathbb{Z}^n} \lambda^\gamma_{jm}\, (\gamma qu)_{jm}(x) \tag{26}$$

converges in $S'(\mathbb{R}^n)$.

2.3.3. Remark. By (19) one can replace b_{pq} in (25) by f_{pq}. These are the quarkonial decompositions of the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ we are looking for. In the *Weierstrassian spirit* one can even take representations of type (25), (26) as starting point.

2.3.4. Definition. (i) Let $0 < p \le \infty,\ s > \sigma_p,\ 0 < q \le \infty$ and $\mu \ge \mu_0(s,p)$. Then

$$B^s_{pq}(\mathbb{R}^n) = \{f \in S'(\mathbb{R}^n) \text{ can be represented by (26), (25)}\}, \tag{27}$$

where $(\gamma qu)_{jm}$ are the (s,p)-γ-quarks given by (22). Furthermore

$$\|f \mid B^s_{pq}(\mathbb{R}^n)\| = \inf \left[\sup_\gamma 2^{\mu|\gamma|} \|\lambda^\gamma \mid b_{pq}\| \right] \tag{28}$$

where the infimum is taken over all admissible representations.
(ii) Let $0 < p < \infty,\ s > \sigma_p,\ \min(1,p) \le q \le \infty$ and $\mu \ge \mu_0(s,p,q)$. Then

$$\begin{aligned} F^s_{pq}(\mathbb{R}^n) &= \{f \in S'(\mathbb{R}^n) \text{ can be represented by (26), (25)} \\ &\qquad\qquad \text{with } f_{pq} \text{ in place of } b_{pq}\}, \end{aligned} \tag{29}$$

where $(\gamma qu)_{jm}$ are the (s,p)-γ-quarks given by (22). Furthermore,

$$\|f \mid F^s_{pq}(\mathbb{R}^n)\| = \inf \left[\sup_\gamma 2^{\mu|\gamma|} \|\lambda^\gamma \mid f_{pq}\| \right] \tag{30}$$

where the infimum is taken over all admissible representations.

2.3.5. Comments. As said above this is the constructive Weierstrassian approach of introducing the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. These spaces had been studied in detail in [1, 10, 11, 12, 13]. Quarkonial (or subatomic) decompositions of these spaces had been treated first in [14], Sec 14. Some clumsy formulations given there had been improved afterwards in [6] and [15]. Furthermore, the restrictions $s > \sigma_p$ and (in (ii)) $q \geq \min(1,p)$ can be removed if one uses arguments of type (10). Maybe the best formulation covering all cases can be found in [15], Sec. 1.5. Furthermore the problem of convergence of (26), (25) can be treated more directly similarly as in [18], 2.2.6. We repeat that we omit detailed references of the literature. They can be found in the quoted books and papers.

2.3.6 Cauchy formula

We stressed the analogy of (26) with the classical Taylor expansion (4). It is well-known that for given f the Taylor coefficients ϱ_j^k can be calculated by some Cauchy integrals. Of course they depend linearly on f. The representation (26) is not unique. But for given s, p, q and also μ one finds functions $\Psi_{jm}^\gamma \in S(\mathbb{R}^n)$ (depending on s, p, q, μ) such that

$$\lambda_{jm}^\gamma = (f, \Psi_{jm}^\gamma)\,; \quad \gamma \in \mathbb{N}_0^n,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{31}$$

(dual pairing in $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$), are optimal coefficients. This means,

$$\sup_\gamma 2^{\mu|\gamma|}\, \|\lambda^\gamma \,|\, b_{pq}\| \sim \|f \,|\, B_{pq}^s(\mathbb{R}^n)\|, \tag{32}$$

where λ_{jm}^γ are given by (31) and the equivalence constants in (32) are independent of $f \in B_{pq}^s(\mathbb{R}^n)$. Similarly for $F_{pq}^s(\mathbb{R}^n)$. We refer to [14], 14.16, on p. 104. Curiously enough the linear dependence of the coefficients λ_{jm}^γ on f is out of interest for the linear problems we have in mind (spectral theory of fractal PDEs) but it is at the heart of the matter of the non-linear problems we are going to discuss later on (the Q-method).

2.3.7 The role of μ

If s, p, q are fixed as in part (i) or in part (ii) in 2.3.4 then there is a number $\mu_0 > 0$ such that (27)–(30) make sense for any μ with $\mu \geq \mu_0$. Of course, (28) and (30) depend on the chosen μ (equivalent quasi-norms). However, the related equivalence constants depend also exponentially on μ. We discussed this phenomenon in [14], 14.6, p. 97. Hence there is a bargain: If μ is large, then one has rapid decay in (28), (30), but bad equivalence constants.

2.4 Further possibilities

2.4.1 Spaces on $\mathbb{R}^n$

If s is not restricted as in 2.3.4 then representations of type (26), (25) cannot be expected in general. Some moment conditions for the underlying elementary building blocks are needed. Maybe the best way to ensure this is to generalise (22) by

$$(\gamma qu)^L_{\nu m}(x) = 2^{-\nu(s-n/p)}((-\Delta)^{(L+1)/2}\psi^\gamma)(2^\nu x - m) \tag{33}$$

where $(L+1)/2 \in \mathbb{N}_0$. Then the counterparts of (26), (25) look a little bit more complicated, but they work for all spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. We refer to [14], Sec. 14, for details. An improved formulation may be found in [15], 1.5. In that paper we replaced $\psi^\gamma(x) = x^\gamma\psi(x)$ by the *Gausslets*

$$G^\gamma(x) = x^\gamma e^{-|x|^2/2}, \quad \gamma \in \mathbb{N}^n_0,\ x \in \mathbb{R}^n. \tag{34}$$

Including normalising factors the counterpart of (33) is given by

$$(\gamma G)^L_{\nu m}(x) = 2^{-\nu(s-n/p)}((-\Delta)^{(L+1)/2}G^\gamma)(2^\nu x - m) \tag{35}$$

with $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. It is the main aim of [15] to prove representations of type (26), (25) based on the (s,p)-γ-Gausslets (35).

2.4.2 Spaces on domains

Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$. Spaces of type

$$\widetilde{F}^s_{pq}(\Omega) = \{f \in F^s_{pq}(\mathbb{R}^n) : \operatorname{supp} f \subset \overline{\Omega}\}, \tag{36}$$

similarly $\widetilde{B}^s_{pq}(\Omega)$, and their special cases, have a long history, see [11]. In [16] we developed a theory of these spaces, especially quarkonial representations. These results and techniques can be used to study quarkonial decompositions of (weighted) spaces on some special hyperbolic manifolds (with the Poincaré n-ball as proto-type). This, in turn, is the basis of a related spectral theory. We refer to [17].

2.4.3 Taylor expansions of distributions

Let $\langle x\rangle = (1+|x|^2)^{1/2},\ \alpha \in \mathbb{R}$, and

$$B^s_{pq}(\mathbb{R}^n, \langle x\rangle^\alpha) = \{\langle x\rangle^\alpha f \in B^s_{pq}(\mathbb{R}^n)\}, \tag{37}$$

obviously quasi-normed. As for these spaces we refer to [1] and the literature quoted there. There is no problem to find quarkonial representations for these weighted spaces. Since

$$S'(\mathbb{R}^n) = \bigcup_{s\in\mathbb{R}} B^s_{pp}(\mathbb{R}^n, \langle x\rangle^s) \tag{38}$$

for any $0 < p \leq \infty$, the *Taylor expansions* can be extended to any tempered distribution. We refer for details to [18].

3 Fractals and spectra

3.1 Fractals

3.1.1 Distinguished fractals

Let Γ be a compact subset of $\mathbb{R}^n$ and let $0 < d < n$. Then Γ is called a *d-set* if there is a Radon (or Borel) measure μ in $\mathbb{R}^n$ with

$$\operatorname{supp}\mu = \Gamma \quad \text{and} \quad \mu(B(\gamma, r)) \sim r^d, \tag{39}$$

where $B(\gamma, r)$ stands for a ball centred at $\gamma \in \Gamma$ and of radius $0 < r < 1$. Here (and in the sequel) the equivalence means that each side in (39) can be estimated from above by some constant multiplied with the other side where this constant is independent of γ and r. The notion of d-sets is well-known both in fractal geometry (see [14] for references, especially to the books by Falconer, [4, 5], and Mattila, [8]) and in the theory of function spaces, where [7] is the standard reference.

We are interested in a generalisation of (39). Let $\Psi(r)$ be a positive monotone (decreasing or increasing) function on $(0, 1]$ with

$$\Psi(2^{-j}) \sim \Psi(2^{-2j}), \quad j \in \mathbb{N}. \tag{40}$$

Let Γ be again a compact set in $\mathbb{R}^n$ and let $0 < d < n$. Then Γ is called a (d, Ψ)*-set* if there is a Radon (or Borel) measure μ in $\mathbb{R}^n$ with

$$\operatorname{supp}\mu = \Gamma \quad \text{and} \quad \mu(B(\gamma, r)) \sim r^d\Psi(r)\,; \quad 0 < r < 1. \tag{41}$$

We refer to [2, 3].

3.1.2 Examples and properties

If $b \in \mathbb{R}$ and $0 < c < 1$ then

$$\Psi(r) = |\log cr|^b, \quad 0 < r \le 1, \tag{42}$$

are typical examples of the above function Ψ. Furthermore, one finds in any case positive numbers b, c_1, c_2 and $0 < c < 1$ such that

$$c_1 |\log cr|^{-b} \le \Psi(r) \le c_2 |\log cr|^b. \tag{43}$$

3.1.3 Limiting case

Let $\Psi(r)$ be monotonically decreasing and let $\Psi(r) \to \infty$ if $r \to 0$. Then (41) with $d = n$ makes sense and the outcome is called an (n, Ψ)-set. In this case and also in all the other cases with $0 < d < n$ we have always

$$|\Gamma| = 0 \quad \text{(Lebesgue measure)}. \tag{44}$$

3.1.4 Self-similar and pseudo self-similar sets

Under suitable conditions for $x^l \in \mathbb{R}^n$ the contractions

$$A_l x = rx + x^l; \quad 0 < r < 1; \; l = 1, \ldots, N, \tag{45}$$

generate self-similar sets Γ in $\mathbb{R}^n$. It is one of the basic facts of fractal geometry that for any d with $0 < d < n$ there are self-similar d-sets, [4, 5, 8]. The contraction factor of the jth iteration of the maps in (45) is r^j. Varying this contraction factor r^j slightly, say, by $r^j j^\varkappa$ with $\varkappa \in \mathbb{R}$, then the resulting fractals are called pseudo self-similar, see [3] for a more precise version. As proved there, for any d and Ψ in 3.1.1 and 3.1.3 there are pseudo self-similar (d, Ψ)-sets.

3.2 Function spaces on and related to fractals

3.2.1 Traces

We repeat a few assertions in simplified versions which may be found in [14]. Let Γ be a d-set in $\mathbb{R}^n$ with $0 < d < n$ according to 3.1.1. First we clarify under which conditions traces of $f \in B^s_{pq}(\mathbb{R}^n)$ on Γ make sense. Let $\varphi \in S(\mathbb{R}^n)$. Then $\mathrm{tr}_\Gamma \varphi$ denotes the pointwise trace of φ on Γ. There is a constant $c > 0$ such that for all $\varphi \in S(\mathbb{R}^n)$,

$$\|\mathrm{tr}_\Gamma \varphi \,|\, L_p(\Gamma)\| \le c \, \|\varphi \,|\, B^{(n-d)/p}_{p,1}(\mathbb{R}^n)\|, \quad 1 < p < \infty. \tag{46}$$

Of course, $L_p(\Gamma)$ has the usual meaning with respect to the measure μ in (39). By completion, tr_Γ can be extended from $S(\mathbb{R}^n)$ to $B^{(n-d)/p}_{p,1}(\mathbb{R}^n)$. The outcome is perfect:

3.2.2. Proposition. *Let* $1 < p < \infty$ *and let* Γ *be the above* d*-set with* $0 < d < n$. *Then*

$$L_p(\Gamma) = \mathrm{tr}_\Gamma B^{(n-d)/p}_{p,1}(\mathbb{R}^n). \tag{47}$$

3.2.3 Spaces on $\boldsymbol{\Gamma}$

Let $s > 0$ and $1 \le q \le \infty$. By 3.2.2,

$$B^s_{pq}(\Gamma) = \mathrm{tr}_\Gamma B^{(n-d)/p+s}_{pq}(\mathbb{R}^n), \tag{48}$$

obviously normed, are Besov spaces on Γ. It is quite clear that the embedding

$$\mathrm{id} : B^s_{pq}(\Gamma) \to L_p(\Gamma) \tag{49}$$

is compact. It comes out that a precise knowledge of the related entropy numbers $e_k(\mathrm{id})$ is of crucial importance for the spectral theory of fractal (pseudo) differential operators. Basic information for entropy numbers may be found in [1].

3.2.4. Theorem. *Let* id *be given by* (49). *Then*

$$e_k(\mathrm{id}) \sim k^{-s/d} \quad \textit{where} \quad k \in \mathbb{N}. \tag{50}$$

3.2.5. Comment.

As said we rely on [14].

The proof of (50) *is just the point where the quarkonial decompositions described in Sec. 2 enter the scene.*

In very rough terms: (28) with $p = q$ (this is sufficient) reduces the involved spaces to sequence spaces ℓ_p with some weights. In other words:

Quarkonial decompositions allow to reduce problems of type (50) *to corresponding problems between* (*weighted*) *sequence spaces of type* ℓ_p.

As for details we refer again to [14]. Historical remarks about entropy numbers may be found in [1], 3.3.5, pp. 126–128. In [1], 3.3, we proved assertions of type (50) for bounded C^∞ domains $\Omega \subset \mathbb{R}^n$ in place of Γ. The proof works for any bounded domain in $\mathbb{R}^n$. It is based on the original Fourier analytical definition of $B^s_{pq}(\mathbb{R}^n)$. However the step from bounded domains Ω

in $\mathbb{R}^n$ to fractals Γ in $\mathbb{R}^n$ destroys the application of this method (at least at first glance). This was the main reason for the author to develop the theory of quarkonial (subatomic) decompositions of function spaces in [14] and as described in Sec. 2. As for the alternative approach to construct frames in function spaces of type $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ via splines and wavelets we refer to [9] and [10], 2.3 (some comments may also be found in [15], 2.4). The advantage of the γ-quarks are their simplicity and the described interrelation with Taylor expansions. They are useful in rather different applications such as entropy numbers, spectral theory, and non-linear problems (the Q-method, discussed below).

3.2.6 The golden triangle

The identification of $L_p(\Gamma)$ with some trace spaces in 3.2.2 is perfect. The other side of the same coin is the question what can be said about functions

$$f^\Gamma \in L_p(\Gamma), \quad 1 < p < \infty, \quad \Gamma: \ d\text{-set}, \tag{51}$$

$0 < d < n$, interpreted as (singular) distributions in $\mathbb{R}^n$. If one looks at f^Γ as a complex measure then there is only one reasonable way to interpret f^Γ as $f \in S'(\mathbb{R}^n)$, given by

$$f(\varphi) = \int_\Gamma f^\Gamma(\gamma)\,(\mathrm{tr}_\Gamma \varphi)(\gamma)\mu(d\gamma), \quad \varphi \in S(\mathbb{R}^n). \tag{52}$$

It comes out that the quality of f can be described in terms of the spaces $B^s_{p\infty}(\mathbb{R}^n)$ which are located in an $(1/p, s)$-diagram in the triangle (fractal country)

$$\left\{ \left(\frac{1}{p}, s\right) : \ 1 < p < \infty, \ n\left(\frac{1}{p} - 1\right) < s < 0 \right\}. \tag{53}$$

Let

$$\begin{aligned} B^{s,\Gamma}_{p\infty}(\mathbb{R}^n) = \{ f \in B^s_{p\infty}(\mathbb{R}^n) : \\ f(\varphi) = 0 \text{ if } \varphi \in S(\mathbb{R}^n) \text{ and } \mathrm{tr}_\Gamma \varphi = 0 \}. \end{aligned} \tag{54}$$

In particular, if $f \in B^{s,\Gamma}_{p\infty}(\mathbb{R}^n)$ then $\operatorname{supp} f \subset \Gamma$. Hence, by (44), any non-trivial distribution f belonging to this space is singular.

3.2.7. Proposition. *Let Γ be the above d-set in $\mathbb{R}^n$ with $0 < d < n$. Then, according to the interpretation* (52),

$$L_p(\Gamma) = B^{-(n-d)/p',\Gamma}_{p\infty}(\mathbb{R}^n) \tag{55}$$

where $1 < p < \infty$ and $1/p + 1/p' = 1$.

3.2.8. Remark. Details, explanations and proofs may be found in [14], Sec. 17, 18. Of course the B-space in (55) lies in the triangle (53).

3.2.9 The operator $\mathbf{tr}^{\Gamma}$

The trace operator tr_{Γ} has the meaning of 3.2.1 – 3.2.3, where Γ is again a d-set in $\mathbb{R}^n$ with $0 < d < n$. Formalising (52) by

$$\mathrm{id}_{\Gamma} : f^{\Gamma} \to f \tag{56}$$

we have

$$\mathrm{tr}^{\Gamma} = \mathrm{id}_{\Gamma} \circ \mathrm{tr}_{\Gamma} \tag{57}$$

with

$$\mathrm{tr}^{\Gamma} : B^{(n-d)/p}_{p,1}(\mathbb{R}^n) \to L_p(\Gamma) \to B^{-(n-d)/p'}_{p\infty}(\mathbb{R}^n) \tag{58}$$

by 3.2.2 and 3.2.7. This, together with (50), paves the way to introduce and to study fractal (pseudo)-differential operators.

3.2.10 Generalisation

So far everything what had been said in 3.2.1 – 3.2.9 is covered by [14]. One can extend all these considerations from d-sets to (d, Ψ)-sets introduced in 3.1. For this purpose one has to generalise the smoothness s in B^s_{pq} by a couple (s, Ψ^a), hence $B^{(s,\Psi^a)}_{pq}$, where $a \in \mathbb{R}$ and Ψ have the same meaning as in 3.1.1 and 3.1.2. Afterwards one has counterparts of Propositions 3.2.2, 3.2.7, Theorem 3.2.4 and interpretation (58). We do not go into detail and refer to [3] and the announcement [2].

3.3 Fractal elliptic operators

3.3.1 The Dirichlet Laplacian

As usual we put

$$H^s(\mathbb{R}^n) = B^s_{2,2}(\mathbb{R}^n), \quad s \in \mathbb{R}, \tag{59}$$

and

$$H^s(\Omega) = H^s(\mathbb{R}^n)\backslash\Omega, \tag{60}$$

for the restriction of $H^s(\mathbb{R}^n)$ to the bounded C^∞ domain Ω in $\mathbb{R}^n$. If $s > 1/2$, then

$$\overset{\circ}{H}{}^s(\Omega) = \{f \in H^s(\Omega) : \mathrm{tr}_{\partial\Omega} f = 0\} \tag{61}$$

makes sense. Let $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$. Then

$$-\Delta : \overset{\circ}{H}{}^1(\Omega) \Longleftrightarrow H^{-1}(\Omega) \tag{62}$$

is an isomorphic map of $\overset{\circ}{H}{}^1(\Omega)$ onto $H^{-1}(\Omega)$. To indicate the vanishing boundary data one calls $-\Delta$ with (62) the *Dirichlet Laplacian.*

3.3.2 The set-up

Let again Ω be a bounded C^∞ domain in $\mathbb{R}^n$ and let Γ be a distinguished fractal according to 3.1.1, that means a d-set or a (d, Ψ)-set, with

$$\Gamma \subset \Omega. \tag{63}$$

We ask under which conditions

$$B = (-\Delta)^{-1} \circ \mathrm{tr}^\Gamma : \overset{\circ}{H}{}^1(\Omega) \to \overset{\circ}{H}{}^1(\Omega) \tag{64}$$

makes sense. Of course, $(-\Delta)^{-1}$ is the inverse of the Dirichlet Laplacian in (62). By (58) with $p = 2$ one needs for (64) that $n - d < 2$, at least in case of d-sets. But this holds also for (d, Ψ)-sets. By (50) one can even expect qualitative assertions of the compactness of B.

3.3.3 Physical background

We interpret the bounded C^∞ domain Ω in the plane $\mathbb{R}^2$ as a membrane, fixed at its boundary $\partial\Omega$ and with the mass density $m(x)$. Vibrations are described by

$$\Delta u(x,t) = m(x) \frac{\partial^2 u}{\partial t^2}(x,t), \quad u(y,t) = 0 \quad \text{in} \quad \partial\Omega \times \mathbb{R}_+ \,. \tag{65}$$

Via $u(x,t) = e^{i\lambda t} v(x)$ the eigenfrequencies λ are given by

$$-\Delta v(x) = \lambda^2 m(x) v(x), \tag{66}$$

where $-\Delta$ is the above Dirichlet Laplacian. What happens if the membrane is crumbling or gets rusty? Assume that the outcome (in an idealised form) can be described replacing $m(x)$ by a measure μ related to a (d, Ψ)-set Γ with $\Gamma \subset \Omega$ according to 3.1.1. Then (66) results in (64), and the positive eigenvalues μ_k of B and the eigenfrequencies λ_k are related by $\lambda_k = \mu_k^{-1/2}$ where $k \in \mathbb{N}$. As for more details we refer again to [14].

3.3.4. Theorem. *Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ and let Γ be a (d,Ψ)-set according to 3.1.1 with*

$$\Gamma \subset \Omega \quad and \quad n-2<d<n. \tag{67}$$

Then B, given by (64), is self-adjoint, non-negative, and compact with null-space

$$N(B)=\{f\in \mathring{H}^1(\Omega):\, \mathrm{tr}_\Gamma f=0\} \tag{68}$$

and

$$0<\mu_k \sim k^{-1}(k\,\Psi(k^{-1}))^{(n-2)/d}, \quad k\in\mathbb{N}, \tag{69}$$

for the positive eigenvalues. Furthermore, B is generated by the quadratic form

$$\int_\Gamma f(\gamma)\overline{g(\gamma)}\mu(d\gamma)=(Bf,g)_{H^1} \tag{70}$$

in $\mathring{H}^1(\Omega)$.

3.3.5. Comments. For details we refer to [3] and its announcement [2]. It generalises both in formulation and proofs a corresponding assertion for d-sets in [14]. The main point is the estimate of μ_k from above in (69). It is based on Carl's inequality

$$\mu_k \le \sqrt{2}\, e_k(B), \quad k\in\mathbb{N}, \tag{71}$$

where $e_k(B)$ are the entropy numbers of B. The estimate of $e_k(B)$ in case of d-sets can be reduced via (58) and (62) to 3.2.4. For (d,Ψ)-sets one finds the corresponding counterpart in the quoted papers.

In other words, the most difficult part of the above theorem is proved via appropriate quarkonial decompositions of the type as described in Sec. 2.

3.3.6. Corollary. *Let Γ be a (n,Ψ)-set according to 3.1.3 with $\Gamma\subset\Omega$. Then the assertions of Theorem 3.3.4 remain valid, now with $d=n$ in (69).*

3.3.7 Further possibilities

As said the above material is taken from [14] and [2, 3]. There one finds further considerations of this type. We mention three of them.

(i) The fractals Γ treated so far are isotropic (there are no distinguished directions in $\mathbb{R}^n$). This fits pretty well to the isotropic operator Δ. One can deal with anisotropic fractals Γ (ferns, grasses, etc.). One can say at least something as in Theorem 3.3.4, where one has now only estimates for the μ_k in (69): *The music of the ferns.* We refer to [14].

(ii) All considerations are qualitative. One needs mapping properties of type (62) (but nothing special related to $-\Delta$, or better, $(-\Delta)^{-1}$) and estimates for the entropy numbers based on quarkonial decompositions. One may also replace $\mu(d\gamma)$ on Γ by $b \in L_r(\Gamma)$ (where $b = 1$ corresponds to $\mu(d\gamma)$). Hence, the method works if one generalises B in (64) by

$$B = b_1 \circ A \circ b_2, \quad b_j \in L_{r_j}(\Gamma) \quad \text{and} \quad A \in \Psi^{-\varkappa}_{1,\varrho}, \tag{72}$$

where $\Psi^{-\varkappa}_{1,\varrho}$ is the Hörmander class of pseudo-differential operators in $\mathbb{R}^n$, here with $\varkappa > 0$ and $0 \le \varrho \le 1$, and r_j are suitably chosen numbers with $1 < r_j \le \infty$.

(iii) In quantum mechanics one asks for the behaviour of the so called negative spectrum of Schrödinger operators of type

$$H = -\Delta + \beta V \quad \text{if} \quad \beta \to \infty \tag{73}$$

(semi-classical limits). Here V is a potential in $\mathbb{R}^n$, assumed to be relatively compact with respect to $-\Delta$, that means $V(-\Delta + \mathrm{id})^{-1}$ is compact, say, in $L_2(\mathbb{R}^n)$. (Of course, $n = 3$ is the case of physical relevance.) What happens if

$$V = b(\gamma)\mu(d\gamma), \quad b \in L_r(\Gamma), \tag{74}$$

where μ is the above measure with respect to a d-set or a (d, Ψ)-set Γ in $\mathbb{R}^n$? With the help of the so called Birman-Schwinger principle one can reduce this question to assertions as treated in Theorem 3.3.4 and Corollary 3.3.6. In [14] we dealt with problems of this type in the case of d-sets. There is no doubt that these considerations can be extended to (d, Ψ)-sets.

4 Composition operators and the Q-method

4.1 Introduction

Let $G(t)$ be a real continuous function on the real line $\mathbb{R}$ with $G(0) = 0$. Then

$$T_G : f \to G(f(x)), \quad f(x) \in L_1^{\mathrm{loc}}(\mathbb{R}^n) \text{ real}, \tag{75}$$

is called a *composition operator*. One asks for mapping properties of T_G between suitable function spaces: boundedness, continuity, Lipschitz continuity, compactness. Of course, $L_p(\mathbb{R}^n)$ with $1 \le p \le \infty$, is the classical choice of underlying function spaces (maybe with different p's in the source and the target space): Nemytskii operators. If one replaces L_p by (real) Sobolev spaces W_p^m or, more general, B^s_{pq} and F^s_{pq} spaces, then the corresponding theory becomes more complicated. Nevertheless, there is an astonishingly comprehensive related theory which may be found in [10] and the references given there. The adopted point of view is the following:

Given a space (or may be two spaces, source and target). For which G is T_G a (bounded, continuous, Hölder or Lipschitz continuous) map?

The other side of the same coin is the following question:

If G is given, find all (real) spaces B^s_{pq}, F^s_{pq} for which T_G is a (bounded, continuous, Hölder or Lipschitz continuous) map.

First non-smooth candidates are

$$G(t) = |t|^{\varkappa}, \quad \varkappa > 0, \tag{76}$$

and, in particular,

$$G(t) = |t| \quad \text{and} \quad G(t) = t_+ = \max(t, 0). \tag{77}$$

Put

$$T : f(x) \to |f(x)| \quad \text{and} \quad T^+ f(x) = f_+(x) = \max(f(x), 0) \tag{78}$$

if $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, real. As far as mapping properties are concerned there is no big difference between T and T^+ since

$$2T^+ = \mathrm{id} + T. \tag{79}$$

Obviously, the operators T and T^+ attracted a lot of attention. However the corresponding theory in the spaces B^s_{pq} and F^s_{pq} (maybe restricted to their real parts) is surprisingly complicated (especially if $s > 1$). What is known so far may be found in [10], 5.4. We complement these results below by describing under which conditions T (and hence also T^+) is bounded. Corresponding results with respect to continuity or even Lipschitz continuity are not available so far (with exception of L_p-spaces). However such continuity assertions seem to be indispensable if one wishes to apply fixed point theorems in connection with existence, uniqueness, regularity of solutions of semi-linear differential equations or integral equations of proto-type

$$(-\Delta + \mathrm{id})u(x) = b(x)|u(x)| + h(x), \quad x \in \mathbb{R}^n, \tag{80}$$

and

$$u(x) = \int_{\mathbb{R}^n} K(x-y)u_+(y)\,dy + h(x), \tag{81}$$

respectively. Here b, K, h are given, and one asks for solutions u. Besides existence (and uniqueness), one wishes to obtain maximal smoothness assertions for u, especially in dependence on the quality of h. If, roughly speaking, $b(x) = \varepsilon > 0$ small, $K \in L_1(\mathbb{R}^n)$ real and real $h \in B^s_{pq}(\mathbb{R}^n)$, then the perfect outcome would be $u \in B^{s+2}_{pq}(\mathbb{R}^n)$ in (80) and $u \in B^s_{pq}(\mathbb{R}^n)$ in (81). Direct application of, say, Banach's contraction theorem would require that T and T^+ in suitable (real) spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ are not only bounded but Lipschitz continuous. But the latter is simply not available (and, in general, not true). This is the point where again quarkonial decompositions as described in 2.3 enter the scene. Let f be given by (26), (25) with $\lambda^\gamma_{jm} = \lambda^\gamma_{jm}(f)$ as in (31). Then we put

$$Qf = \sum_{\gamma \in \mathbb{N}^n_0} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda^\gamma_{jm}(f)|\,|(\gamma qu)_{jm}(x)|. \tag{82}$$

For real f we have by (78)

$$0 \le T^+ f \le Tf \le Qf. \tag{83}$$

The operator Q is the quarkonial twin of T: In the assertions below, Q is bounded in, say, those spaces $B^s_{pq}(\mathbb{R}^n)$ where T is bounded, but in addition it is Lipschitz continuous (in sharp contrast to T). This is the starting point of what we call the *Q-method*. Very roughly: First one replaces the right-hand side of (81) by the auxiliary operator

$$K^Q u(x) = \int_{\mathbb{R}^n} K(x - y(Qu)(y)\,dy + h(x). \tag{84}$$

Suppose real $h \in B^s_{pq}(\mathbb{R}^n)$ (such that Q can be applied), $\|K|L_1\|$ small, $K(y) \ge 0$, then by Banach's contraction theorem, K^Q has a fixed point $u^Q(x) \in B^s_{pq}(\mathbb{R}^n)$. By (83) this function u^Q is a supersolution for (81).

Using the well-known technique of supersolutions and what is known about T, T^+ and $B^s_{pq}(\mathbb{R}^n)$ one arrives at a solution $u \in B^s_{pq}(\mathbb{R}^n)$ of (81) with maximal smoothness. In the following subsections we outline this method and the underlying mapping properties of the operators T and Q. Detailed proofs will be published elsewhere.

4.2 The operators T and Q

4.2.1 Preliminaries

We may always assume that the function ψ in (20) is non-negative. Recall

$$\psi^\gamma(x) = x^\gamma \psi(x) \quad \text{with} \quad x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}.$$

Hence if T or Q maps a given space $B^s_{pq}(\mathbb{R}^n)$ or $F^s_{pq}(\mathbb{R}^n)$ into itself then at least $|x^\gamma|\psi(x)$ must be an element of this space. This question can be reduced to $n = 1$ and $\gamma = 1$. Then

$$|x|\psi(x) \in B^{1+1/p}_{p\infty}(\mathbb{R}), \quad 0 < p \le \infty. \tag{85}$$

If $\psi(0) \neq 0$ then this is the best possible assertion: $B^{1+1/p}_{p\infty}$ cannot be replaced by $B^{1+1/p}_{pq}$ with $q < \infty$, [12], pp. 147–9. Together with some Fubini arguments, see [12], [10], or [16], one arrives at the natural restriction

$$0 < p \le \infty, \quad n\left(\frac{1}{p} - 1\right)_+ < s < 1 + \frac{1}{p}, \tag{86}$$

(not to speak about limiting points in the $(1/p, s)$-diagram). It is somewhat curious but in case of Q one can avoid smoothness restrictions of type (86) if one chooses a function ψ with

$$\operatorname{supp}\psi \subset \{y = (y_1, \ldots, y_n) \in \mathbb{R}^n : y_j > 0\}. \tag{87}$$

But at least so far we have no use for this observation: beyond (86) there is no reasonable interplay between T and Q (not to speak about limiting cases). Although not necessary for some assertions concerning T and for all assertions concerning Q, we restrict ourselves now to the real part $\mathbf{B}^s_{pq}$ of B^s_{pq} (similarly $\mathbf{F}^s_{pq}$).

4.2.2. Theorem. *Let*

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad n\left(\frac{1}{p} - 1\right)_+ < s < 1 + \frac{1}{p}. \tag{88}$$

(i) *Then T and T^+, given by* (78), *are bounded operators in $\mathbf{B}^s_{pq}(\mathbb{R}^n)$: there is a constant $c > 0$ such that*

$$\|Tf \mid B^s_{pq}(\mathbb{R}^n)\| \le c\,\|f \mid B^s_{pq}(\mathbb{R}^n)\| \tag{89}$$

and

$$\|T^+ f \mid B^s_{pq}(\mathbb{R}^n)\| \le c\,\|f \mid B^s_{pq}(\mathbb{R}^n)\| \tag{90}$$

for all $f \in \mathbf{B}^s_{pq}(\mathbb{R}^n)$.

(ii) *Then Q, given by* (82), *is a bounded and Lipschitz continuous operator in $\mathbf{B}^s_{pq}(\mathbb{R}^n)$: there is a constant $c > 0$ such that*

$$\|Qf - Qg \mid B^s_{pq}(\mathbb{R}^n)\| \le c\,\|f - g \mid B^s_{pq}(\mathbb{R}^n)\| \tag{91}$$

for all f and g belonging to $\mathbf{B}^s_{pq}(\mathbb{R}^n)$.

4.2.3. Remark. The boundedness of Q is a special case of (91) with $g = 0$. The assertions for Q follow easily from (82), (31) and the quarkonial decompositions. What is known about T (and T^+) may be found in [10], 5.4. The proof of the full theorem will be published elsewhere.

4.2.4. Comment. There is a similar result for $\mathbf{F}^s_{pq}(\mathbb{R}^n)$ with the additional (presumably technical) restriction $s \ne 1/p$ if $p \le 1$ for T and T^+.

4.3 The Q-method

4.3.1 Preliminaries

Let $b(x) = \varepsilon$ in (80). By (79) there is no essential difference if we replace $|u(x)|$ in (80) by $u_+(x)$. Let $G(y)$ be the Green's function of $-\Delta + \mathrm{id}$ in $\mathbb{R}^n$. Then (80) (modified in the indicated way) is equivalent to

$$u(x) = (Bu)(x) = \varepsilon \int_{\mathbb{R}^n} G(x-y)u_+(y)\,dy + H(x) \tag{92}$$

with $H = (-\Delta + \mathrm{id})^{-1}h$. Recall $G(y) > 0$ in $\mathbb{R}^n$. In other words, (92) fits in the scheme of (81). We restrict ourselves here to (80) in the modified version of (92). But the method works also for problems of type (81).

4.3.2. Theorem. *Let*

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad n\left(\frac{1}{p} - 1\right)_+ < s + \lambda < 1 + \frac{1}{p} \tag{93}$$

for some $\lambda \in [0, 2]$. *There is a number* $\varepsilon_0 > 0$ *such that for any* ε *with* $0 < \varepsilon < \varepsilon_0$ *and any* $h \in \mathbf{B}^s_{pq}(\mathbb{R}^n)$,

$$(-\Delta + \mathrm{id})u(x) = \varepsilon u_+(x) + h(x), \quad x \in \mathbb{R}^n, \tag{94}$$

has a unique solution $u \in \mathbf{B}^{s+2}_{pq}(\mathbb{R}^n)$.

4.3.3. Method of proof and comments.
(i) By some embedding and Banach's contraction theorem (94) has always a unique solution $u(x)$ in $L_r(\mathbb{R}^n)$ for some $1 \le r \le \infty$. If $1 < p \le \infty$ then one may even choose $r = p$ and bootstrapping arguments yield the desired result. However this does not work if $p < 1$ and it never works in more general problems of type (81).
(ii) It would be desirable to apply Banach's contraction theorem to the operator B in (92) in $\mathbf{B}^{s+2}_{pq}(\mathbb{R}^n)$. For this purpose one would need that T^+ is bounded and Lipschitz continuous in some spaces $\mathbf{B}^{s+\lambda}_{pq}(\mathbb{R}^n)$ with $\lambda \in [0, 2]$ (the rest is a matter of lifting in this case, but nothing of this type of argument can be used for more general problems related to (81)). However this Lipschitz continuity is not available. As indicated in (84) one deals first with the auxiliary operator

$$(B^Q u)(x) = \varepsilon \int_{\mathbb{R}^n} G(x-y) Q u(y)\, dy + H(x). \tag{95}$$

By Theorem 4.2.2 (ii) with $\mathbf{B}^{s+\lambda}_{pq}(\mathbb{R}^n)$ in place of $\mathbf{B}^s_{pq}(\mathbb{R}^n)$ and Banach's contraction theorem one finds a (unique) solution of (95).
(iii) By $G(y) > 0$ and (83) it follows

$$u_1(x) = \varepsilon \int_{\mathbb{R}^n} G(x-y) T^+ u_0(y)\, dy + H(x) \le u_0(x). \tag{96}$$

Hence u_0 is a *supersolution* of the original problem and by Theorem 4.2.2 (i) combined with lifting $u_1 \in \mathbf{B}^{s+2}_{pq}(\mathbb{R}^n)$. Iteration yields the decreasing sequence

$$u_{j+1}(x) = \varepsilon \int_{\mathbb{R}^n} G(x-y) T^+ u_j(y)\, dy + H(x) \le u_j(x) \tag{97}$$

with

$$\sup_j \|u_j \mid B^{s+2}_{pq}(\mathbb{R}^n)\| < \infty \tag{98}$$

and

$$u_j \to u \quad \text{in} \quad S'(\mathbb{R}^n) \quad \text{if} \quad j \to \infty \tag{99}$$

(even in some $L_r(\mathbb{R}^n)$ with $1 < r \le \infty$). But (98), (99), and the Fatou property of the spaces $B^{s+2}_{pq}(\mathbb{R}^n)$ ensure $u \in \mathbf{B}^{s+2}_{pq}(\mathbb{R}^n)$. It is the solution we are looking for.

4.3.4. Remark. It is quite clear that (80), (81), (94) are model cases. They can be extended in several directions, including boundary value problems and, maybe, obstacle problems where $u_+(y)$ in (81) or (92) is replaced by $(u-\varphi)_+(y)$ where φ is a given function.

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