Optimal Sobolev embeddings

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1 Introduction

Sobolev inequalities constitute an important part of functional analysis with wide field of applications, mainly to the theory of partial differential equations and to mathematical physics. There are many forms of Sobolev inequalities; their common feature is that certain information on a function u is derived from known data on its gradient ∇u (or on a higher-order gradient). By the "data" we usually mean the membership of the function to a certain function space or class.

One form of the classical Sobolev inequality asserts that, given $1 < p < n$ and setting $p^* = np/(n-p)$, there exists $C > 0$ such that

$$\left(\int_\Omega |u(x)|^{p^*}\,dx\right)^{1/p^*} \le C\left(\int_\Omega |(\nabla u)(x)|^p\,dx\right)^{1/p}, \qquad u \in C_0^1(\Omega). \tag{1.1}$$

(Here and throughout the paper, Ω is a bounded domain in $\mathbb{R}^n$, $n \ge 2$. For our convenience, with no loss of generality, we shall everywhere below assume that $|\Omega| = 1$. As usual, c, C will denote various positive constants independent of appropriate quantities and not necessarily the same at each occurrence.)

In case when $p > n$ and Ω is a Lipschitz domain, a function whose gradient belongs to $L^p(\Omega)$ is known to be Hölder continuous, namely,

$$\sup_{x,y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|^{1-n/p}} \le C\left(\int_\Omega |(\nabla u)(x)|^p\,dx\right)^{1/p}, \qquad u \in C_0^1(\Omega).$$

The limiting case $p = n$ is the most interesting (and the most difficult) one. It is known that, for every $q < \infty$,

$$\left(\int_\Omega |u(x)|^q\,dx\right)^{1/q} \le C\left(\int_\Omega |(\nabla u)(x)|^n\,dx\right)^{1/n}, \qquad u \in C_0^1(\Omega), \tag{1.2}$$

but standard examples show that, although $np/(n-p) \to \infty$ when $p \to n_-$, one cannot take the L^∞-norm on the left side of (1.2). In particular, there

AMS Class.: 46E30, 46E35, 47G10, 26D10.

does not exist an optimal (largest) L^q-norm on the left hand side of (1.2). However, a finer result than (1.2) is available if we are willing to replace the environment of Lebesgue spaces by a broader one. Perhaps the most natural next step in this direction is to consider the context of Orlicz spaces. We say that A is a *Young function* when A is convex and increasing on $[0,\infty)$ and

$$\lim_{t\to 0_+} t/A(t) = \lim_{t\to\infty} A(t)/t = \infty.$$

The quantity

$$\|u\|_A = \|u\|_{L_A(\Omega)} = \inf\left\{\lambda > 0; \int_\Omega A\Big(\frac{|u(x)|}{\lambda}\Big)\,dx \le 1\right\}$$

is called the *Luxemburg norm* of u and the set $L_A = L_A(\Omega)$ of all functions u such that $\|u\|_A < \infty$ is called the *Orlicz space* generated by A.

Now, independently of one another, Pokhozhaev [Po], Trudinger [Tr] and Yudovich [Y] have shown that there is a constant C such that

$$\|u\|_{\exp L^{n'}} \le C\left(\int_\Omega |(\nabla u)(x)|^n\,dx\right)^{1/n}, \qquad u \in C_0^1(\Omega), \tag{1.3}$$

where $\|u\|_{\exp L^{n'}}$ is the norm in the Orlicz space $\exp L^{n'}$, generated by any Young function A which is equivalent for large t to $\exp t^{n'}$, $n' = n/(n-1)$. This space is essentially smaller than any L^q-space with finite q, but, naturally, it is essentially larger than L^∞.

Hempel, Morris and Trudinger [HMT] showed that the $\exp L^{n'}$-norm on the left hand side of (1.3) cannot be replaced by any essentially larger Orlicz norm. It however turns out that a further essential improvement of the norm on the left hand side of (1.3) is still possible if we allow a different context of function spaces than that of Orlicz spaces. Namely, we can replace the $\exp L^{n'}$-norm in (1.3) by a larger, classical Lorentz norm to get, for $u \in C_0^1(\Omega)$,

$$\left(\int_0^1 u^*(t)^n\big(\log(e/t)\big)^{-n}\frac{dt}{t}\right)^{1/n} \le C\left(\int_\Omega |(\nabla u)(x)|^n\,dx\right)^{1/n}, \tag{1.4}$$

where $u^*(t) = \inf\{\lambda > 0; |\{x \in \Omega; |u(x)| > \lambda\}| \le t\}$, $t \in [0,1]$, is the *nonincreasing rearrangement* of u. This inequality can be easily derived from classical capacitary estimates of Maz'ya (see [M], pages 105, 109); it also appears in [Pe] (cf. [CPu] for more details), and it was stated explicitly by Hansson ([H]) and by Brézis and Wainger ([BW]).

Now, the norm on the left hand side of (1.4) is essentially larger than that of (1.3) and therefore (1.4) is a better estimate than (1.3) but we have to pay a tax for this improvement: while the definition of the norm in an Orlicz space involves solely superposition of functions and integration, the new function norm that appears in (1.4) contains a further nontrivial operation: the nonincreasing rearrangement of a function.

Many of the function norms that have been mentioned so far are particular examples of (quasi-)norms in *Lorentz-Zygmund spaces* $L_{p,q;\alpha}(\Omega)$, which were introduced and studied by Bennett and Rudnick in [BR]: for $0 < p, q \le \infty$ and $\alpha \in \mathbb{R}$, define

$$\|u\|_{p,q;\alpha} = \|u\|_{L_{p,q;\alpha}(\Omega)} := \left\| u^*(t) t^{(1/p)-(1/q)} \big(\log(e/t)\big)^{\alpha} \right\|_{L^q(0,1)}.$$

For $\alpha = 0$, $L_{p,q;\alpha}(\Omega)$ coincides with the usual Lorentz space $L_{p,q}(\Omega)$. The following particular examples of Lorentz-Zygmund spaces are of interest: $\exp L^{n'}(\Omega) = L_{\infty,\infty;-1/n'}(\Omega)$, $L^p(\Omega) = L_{p,p;0}(\Omega)$, and the space determined by the norm on the left hand side of (1.4) is just $L_{\infty,n;-1}(\Omega)$.

Lorentz-Zygmund spaces and Orlicz spaces are two independent classes of function spaces having a nontrivial overlap but also a nontrivial intersection (for example, Lebesgue spaces belong to both). A more interesting example is $\exp L^{n'}$, which is an Orlicz space as well as it is a Lorentz-Zygmund space. In (1.3), $\exp L^{n'}(\Omega)$ is optimal as an Orlicz space, but it is not optimal as a Lorentz-Zygmund space, since it can be replaced by $L_{\infty,n;-1}(\Omega)$, which is essentially smaller (see Theorem 3.2 below).

We shall focus on the question when Sobolev inequalities are optimal within various classes of function spaces. We shall derive certain quite general results and apply them to concrete situations. A special attention will be paid to limiting cases such as those described by (1.3) and (1.4).

We will also consider inequalities involving the m^{th} order gradient, $\nabla^m u$, of a function u in $C_0^m(\Omega)$, defined in terms of the usual first order gradient $\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$ and the Laplacian $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ as follows:

$$\nabla^m u = \begin{cases} \Delta^k u & \text{when } m = 2k, \\ \nabla(\Delta^k u) & \text{when } m = 2k+1, \end{cases}$$

where $\Delta^j u = \Delta(\Delta^{j-1} u)$, $j = 2, \dots, \left[\frac{m}{2}\right]$. In order to obtain the analogues of (1.1), (1.3) and (1.4) for $\nabla^m u$ we have to replace n by n/m throughout (for the resulting inequalities see [M], [Pe], [Sob], [Str], [H] and [BW]).

In general, we shall consider two rearrangement-invariant Banach function norms ϱ_R and ϱ_D, defined on $\mathfrak{M}_+(0,1)$, the set of nonnegative measurable functions on $(0,1)$, for which there is a constant C such that

$$\varrho_R\big(u^*(t)\big) \le C\varrho_D\big(|\nabla^m u|^*(t)\big), \qquad u \in C_0^m(\Omega). \tag{1.5}$$

We would like to know that ϱ_R cannot be effectively increased nor ϱ_D effectively decreased.

For the reader's convenience we denote by ϱ_R the *range* norm and by ϱ_D the *domain* norm of the Sobolev inequality. The function spaces corresponding to ϱ_R and ϱ_D will be frequently called the *range space* or the *domain space*, respectively.

We will make no difference between *Sobolev inequalities* (between norms) and *Sobolev embeddings* (between corresponding function spaces).

We shall present a survey of recent results on the subject of the optimality of Sobolev embeddings, many of which have been obtained in collaboration with other authors, namely A. Cianchi, D.E. Edmunds, W.D. Evans, R. Kerman and B. Opic.

We assume throughout that $m, n \in \mathbb{N}$, $n \ge 2$ and $1 \le m \le n-1$. For two positive quantities A and B we write $A \approx B$ when $c \le A/B \le C$.

2 Reduction to Hardy operators

Our first step is the reduction of (1.5) (an inequality involving gradients) to an inequality involving more manageable Hardy-type integral operators acting on monotone functions on $(0,1)$.

To illustrate how the Hardy-type operators arise, let us consider the cases $m=1$ and $m=2$ for smooth radial functions $u(x) = u(|x|)$ supported in the ball

$$B = \left\{ x \in \mathbb{R}^n ; |x| < \kappa_n = \frac{\Gamma((n/2)+1)^{1/n}}{\pi^{1/2}} \right\}$$

of unit measure centred at the origin. Setting $r = |x|$, one has

$$|(\nabla^1 u)(r)| = |(\nabla u)(r)| = |u'(r)|,$$

with $u(r) = \int_r^{\kappa_n} u'(s)\,ds$ or

$$u(\kappa_n r^{1/n}) = \frac{\kappa_n}{n}\int_r^1 f(t)t^{-1/n'}\,dt, \quad f(t) = u'(s),\ s = \kappa_n t^{1/n}.$$

Similarly,

$$(\nabla^2 u)(r) = (\Delta u)(r) = u''(r) + \frac{n-1}{r}u'(r), \quad u(\kappa_n) = u'(\kappa_n) = 0,$$

so that

$$u(r) = \frac{1}{2-n}\left(r^{2-n}\int_0^r (\nabla^2 u)(s)s^{n-1}\,ds + \int_r^{\kappa_n}(\nabla^2 u)(s)s\,ds\right)$$

or

$$u(\kappa_n r^{1/n}) = \frac{\kappa_n^2}{n(2-n)}\left(r^{(2/n)-1}\int_0^r f(t)\,dt + \int_r^1 f(t)t^{(2/n)-1}\,dt\right),$$

with $f(t) = (\nabla^2 u)(s)$, $s = \kappa_n t^{1/n}$.

For general u, the connection with Hardy operators is made by a version of the Pólya-Szegö inequality when $m = 1$ and by a convolution inequality of O'Neil when $m > 1$. This connection is sharp when u is radially decreasing. It should be clear from the example above that the case $m = 1$ is different from the others, involving, as it does, one Hardy operator rather than a pair of dual Hardy operators.

Definition 2.1. Let $(\mathcal{R}, \mu)$ be a measure space such that $\mu(\mathcal{R}) = 1$ and let $\mathfrak{M}_+(\mathcal{R}, \mu)$ be the set of nonnegative μ-measurable functions on $\mathcal{R}$. A *Banach function norm* ϱ on $\mathfrak{M}_+(\mathcal{R}, \mu)$ is defined by the following six axioms (as usual, χ stands for the characteristic function):

(A_1) $\varrho(f) \ge 0$ with $\varrho(f) = 0$ if and only if $f = 0$ a.e.;
(A_2) $\varrho(cf) = c\varrho(f), \quad c \ge 0$;
(A_3) $\varrho(f+g) \le \varrho(f) + \varrho(g)$;
(A_4) $0 \le f_n \nearrow f$ implies $\varrho(f_n) \nearrow \varrho(f)$;
(A_5) $\varrho(\chi_{\mathcal{R}}) < \infty$;
(A_6) there exists $C > 0$ such that

$$\int_{\mathcal{R}} f(x)\,d\mu \le C\varrho(f), \qquad f \in \mathfrak{M}_+(\mathcal{R}, \mu).$$

If moreover
(A_7) $\varrho(f) = \varrho(g)$ for every f, g such that $f^* = g^*$,
then ϱ is called a *rearrangement-invariant (r.i.) norm.*

The *associate norm* of an r.i. norm ϱ on $\mathfrak{M}_+(\mathcal{R}, \mu)$ is the functional

$$\varrho'(g) = \sup_{\varrho(h)=1}\int_{\mathcal{R}} gh\,d\mu, \qquad g, h \in \mathfrak{M}_+(\mathcal{R}, \mu).$$

In the special case when $\mathcal{R} = (0,1)$ and μ is the Lebesgue measure, we have

$$\varrho'(g) = \varrho'_d(g^*), \tag{2.1}$$

where the *"down" associate norm*, ϱ'_d, is given at g by

$$\varrho'_d(g) = \sup_{\varrho(h)=1} \int_0^1 g(t)h^*(t)\,dt, \qquad g,h \in \mathfrak{M}_+(0,1).$$

Then ϱ' and ϱ'_d are Banach function norms, ϱ' is moreover rearrangement-invariant, and the *duality principle*

$$\varrho'' = \varrho$$

holds (see [BS], Chapter 1, Theorem 2.7). Moreover, Hölder's inequality

$$\int_{\mathcal{R}} fg\,d\mu \le \varrho(f)\varrho'(g)$$

is true for every $f,g \in \mathfrak{M}_+(\mathcal{R},\mu)$.

In this paper, $\mathcal{R}$ will be mostly either Ω or the interval $(0,1)$, and μ will be the corresponding Lebesgue measure. In any case, we shall assume throughout that μ is atom-free.

In the sequel we shall denote by P the integral mean operator

$$Pg(t) := \frac{1}{t}\int_0^t g(s)\,ds, \quad g \in \mathfrak{M}_+(0,1),\ t \in (0,1).$$

Now we can state the general version of the reduction theorem.

Theorem 2.2. *Let ϱ_R be an r.i. norm on $\mathfrak{M}_+(0,1)$. Then, when $m = 1$, a necessary and sufficient condition that* (1.5) *hold with ϱ_R and a Banach function norm ϱ_D* (*not necessarily rearrangement-invariant*) *on $\mathfrak{M}_+(0,1)$ is the existence of $K > 0$ for which*

$$\varrho_R\left(\int_t^1 f(s)s^{-1/n'}\,ds\right) \le K\varrho_D(f), \qquad f \in \mathfrak{M}_+(0,1). \tag{2.2}$$

When $n \ge 3$ and $2 \le m \le n-1$, a necessary and sufficient condition that (1.5) *hold for ϱ_R and another r.i. norm ϱ_D on $\mathfrak{M}_+(0,1)$ is the existence of $K > 0$ for which*

$$\varrho_R\left(\int_t^1 (Pf^*)(s)s^{(m/n)-1}\,ds\right) \le K\varrho_D(f^*), \tag{2.3}$$

for every $f \in \mathfrak{M}_+(0,1)$, $f(1-) = 0$.

Remark 2.3. (i) A short argument involving Fubini's theorem yields (2.3) equivalent to

$$\varrho_R\left(t^{(m/n)-1}\int_0^t f^*(s)\,ds\ +\int_t^1 f^*(s)s^{(m/n)-1}\,ds\right)\le K\varrho_D(f^*),$$

where $f\in\mathfrak{M}_+(0,1)$, $f(1-)=0$.

(ii) It is important that ϱ_D and ϱ_R in Theorem 2.2 are *norms*. The situation is different when we allow quasinorms (cf. [EKP]).

The proof of the sufficiency part of Theorem 2.2 follows for $n\ge 3$ and $m\ge 2$ from O'Neil's convolution inequality ([O])

$$[P(f*g)^*](t)\le t(Pf^*)(t)(Pg^*)(t)+\int_t^\infty f^*(s)g^*(s)\,ds, \tag{2.4}$$

and, when $m=1$, from the following version of the Pólya-Szegö inequality, proved in [CPi] (cf. also [Ta], p. 203):

$$\int_0^t\Big[-y^{1/n'}\frac{du^*}{dy}\Big]^*(s)\,ds\ \le C\int_0^t|\nabla u|^*(s)\,ds,\quad t\in\mathbb{R}_+,$$

where $u\in C_0^1(\mathbb{R}^n)$. The necessity part of Theorem 2.2 is proved by a reduction of (1.5) to spherically symmetric functions. Details of the proof can be found in [EKP, Section 3].

The well-known estimate

$$|u(x)|\le C\int_{\mathbb{R}^n}\frac{|(\nabla^m u)(y)|}{|x-y|^{n-m}}\,dy,\qquad u\in C_0^m(\mathbb{R}^n),$$

where $C>0$ depends only on m and n (see [Z], Remark 2.8.6), combined with (2.4), yields, for $0<t<1$,

$$u^*(t)\le C\Big[t^{(m/n)-1}\int_0^t|\nabla^m u|^*(s)\,ds\ +\int_t^1|\nabla^m u|^*(s)s^{(m/n)-1}\,ds\Big].$$

(The constant C is given explicitly in [Ad], p. 390.)

Corollary 2.4. *Let* $m=1$. *Then* (2.3) *is sufficient for* (1.5). *If moreover*

$$\varrho_R\left(\int_t^1(Pf^*)(s)s^{-1/n'}\,ds\right)\le C\varrho_R\left(\int_t^1 f^*(s)s^{-1/n'}\,ds\right), \tag{2.5}$$

then (2.3) *is also necessary for* (*and hence equivalent to*) (1.5).

Proof. That (2.3) implies (1.5) follows from the preceding remarks. Next, Theorem 2.2 yields that (1.5) is equivalent to (2.2). The more so, (1.5) implies (2.2) restricted to monotone functions, that is,

$$\varrho_R\left(\int_t^1 f^*(s)s^{-1/n'}\,ds\right) \leq K\,\varrho_D(f^*), \qquad f \in \mathfrak{M}_+(0,1).$$

If now (2.5) is satisfied, then the last estimate is equivalent to (2.3) (with $m = 1$). □

It would be of interest to be able to decide that a given r.i. norm ϱ_R satisfies (2.5). A sufficient condition can be expressed by means of the lower Boyd index.

Given an r.i. norm ϱ on $\mathfrak{M}_+(0,1)$, the *lower Boyd index* i_ϱ is given by

$$i_\varrho = \lim_{t\to 0_+} \frac{\log(1/t)}{\log h_\varrho(t)},$$

where

$$h_\varrho(t) = \sup_{f\not\equiv 0} \frac{\varrho(E_t f)}{\varrho(f)}, \quad E_t f(s) = f(st), \qquad f \in \mathfrak{M}_+(0,1),\ 0 < s,t < 1.$$

Theorem 2.5. *Let ϱ_R be an r.i. norm on $\mathfrak{M}_+(0,1)$. Then* (2.5) *holds whenever the lower index i_R of ϱ_R satisfies*

$$i_R > n/(n-m). \tag{2.6}$$

Proof. Fix $g \in \mathfrak{M}_+(0,1)$, with $\varrho_R'(g) = 1$. Then, by Fubini's theorem and an elementary change of variable,

$$\begin{aligned}\int_0^1 g^*(t)\int_t^1 &(Pf^*)(s)s^{(m/n)-1}\,ds\,dt\\ &\leq \int_0^1 s^{-m/n}\int_0^1 g^*(t)\int_{st}^1 f^*(y)y^{(m/n)-1}\,dy\,dt\,ds.\end{aligned}$$

Taking the supremum over g, we obtain

$$\begin{aligned}\varrho_R\left(\int_t^1 (Pf^*)(s)s^{(m/n)-1}\,ds\right) &\leq \int_0^1 s^{-m/n}\varrho_R\left(\int_{st}^1 f^*(y)y^{(m/n)-1}\,dy\right)ds\\ &\leq \left(\int_0^1 s^{-m/n}h_{\varrho_R}(s)\,ds\right)\varrho_R\left(\int_t^1 f^*(y)y^{(m/n)-1}\,dy\right).\end{aligned}$$

But (see [B]), $i_R > n/(n-m)$ is equivalent to $\int_0^1 s^{-m/n}h_{\varrho_R}(s)\,ds < \infty$. □

Equipped with these facts, we shall now investigate the optimality of (1.3) and (1.4).

3 The optimality of (1.4) in the context of Lorentz-Zygmund spaces

Given two (quasi-)normed spaces X, Y, we say that X *is embedded into* Y, and write $X \hookrightarrow Y$, when $X \subset Y$ and there is a $C > 0$ such that $\|f\|_Y \le C\|f\|_X$ for every f.

Definition 3.1. The *fundamental function* of an r.i. norm ϱ on $\mathfrak{M}_+(\mathcal{R}, \mu)$ is the function φ_ϱ, defined at $t \in [0,1]$ by

$$\varphi_\varrho(t) := \varrho(\chi_E), \qquad \text{where} \quad \mu(E) = t.$$

The function φ_ϱ is *quasiconcave*, that is, nondecreasing on $[0,1]$, satisfying $\varphi_\varrho(t) = 0$ if and only if $t = 0$, and such that the function $t/\varphi_\varrho(t)$ is nondecreasing on $(0,1)$.

Conversely, every quasiconcave function φ on $[0,1]$ is a fundamental function of certain r.i. space(s). Among these, of particular importance are the *endpoint Lorentz space* $\Lambda_\varphi(\mathcal{R})$, given by

$$\|f\|_{\Lambda_\varphi(\mathcal{R})} = \int_0^1 f^*(t)\, d\varphi(t),$$

and the *endpoint Marcinkiewicz space* $M_\varphi(\mathcal{R})$, given by

$$\|f\|_{M_\varphi(\mathcal{R})} = \sup_{0<t<1} (Pf^*)(t)\varphi(t).$$

If X is an r.i. space with the fundamental function φ, then

$$\Lambda_\varphi(\mathcal{R}) \hookrightarrow X \hookrightarrow M_\varphi(\mathcal{R}). \tag{3.1}$$

We shall make use of the following embedding theorem from [Sh], Proposition 3.1 (for more general version cf. e.g. [OP], Theorem 4.6 and [EOP], Theorem 6.3; most of the results can be obtained also from various earlier ones on weighted embeddings of classical Lorentz spaces—cf. e.g. [St], [Sor], or [CPSS] and the references given there).

Theorem 3.2. *Let $0 < p, p_1, p_2, q, r \le \infty$ and let $\alpha, \beta \in \mathbb{R}$.*

(i) *If $p_1 > p_2$, then the embedding*

$$L_{p_1,q;\alpha}(\Omega) \hookrightarrow L_{p_2,r;\beta}(\Omega)$$

holds.

(ii) *The embedding*

$$L_{p,q;\alpha}(\Omega) \hookrightarrow L_{p,r;\beta}(\Omega)$$

holds if and only if one of the following conditions is satisfied:

$$\begin{aligned} &0 < q \le r \le \infty, \qquad p = \infty, \qquad \alpha + \frac{1}{q} \ge \beta + \frac{1}{r}; \\ &0 < q \le r \le \infty, \qquad 0 < p < \infty, \qquad \alpha \ge \beta; \\ &0 < r < q \le \infty, \qquad \alpha + \frac{1}{q} > \beta + \frac{1}{r}. \end{aligned}$$

Our aim is to use Theorem 3.2 and some ideas from [EOP] to show that the range norm in (1.4) cannot be improved in the context of Lorentz-Zygmund norms, but the domain norm can be replaced by the norm of any of the spaces

$$L_{n,r;(1/n)-(1/r)}(\Omega), \qquad 1 \le r \le n. \tag{3.2}$$

By Theorem 3.2, for every two distinct values of $r \in [1, n]$, the corresponding spaces in (3.2) are incomparable (we say that two (quasi)-normed spaces X, Y are *incomparable* if neither of the embeddings $X \hookrightarrow Y$, $Y \hookrightarrow X$ holds).

Theorem 3.3. (i) *Assume that $r \in [1, n]$. Then there exists a $C > 0$ such that for every $u \in C_0^1(\Omega)$*

$$\left(\int_0^1 u^*(t)^n \big(\log(e/t)\big)^{-n} \frac{dt}{t}\right)^{1/n} \le C\left(\int_0^1 \Big(|(\nabla u)|^*(t)\Big)^r t^{(r/n)-1} \big(\log(e/t)\big)^{(r/n)-1}\, dt\right)^{1/r}.$$

(ii) *Assume that for some $0 < p, q \le \infty$ and $\alpha \in \mathbb{R}$*

$$\|u\|_{L_{p,q;\alpha}(\Omega)} \le C\left(\int_\Omega |(\nabla u)(x)|^n\, dx\right)^{1/n}.$$

Then, necessarily,

$$L_{\infty,n;-1}(\Omega) \hookrightarrow L_{p,q;\alpha}(\Omega).$$

Proof. (i) follows immediately from Corollary 2.4 and [EOP], Theorem 4.2 (ii) with $p_2 = n$, $r \in [1, n]$, $q_2 = \infty$, $s = n$, $\gamma = -1$, $\delta = (1/n) - (1/r)$, and $\alpha = \beta = 0$. The proof of (ii) can be obtained from Theorem 3.2 and [EOP], Theorem 10.4 (ii), via a tedious verification of conditions for various parameters involved. Details are omitted. ☐

We conclude that, in (1.4), the range norm is optimal among Lorentz-Zygmund norms, but there is no optimal Lorentz-Zygmund domain norm.

Using [EOP], Lemma 9.1, we can improve on (1.4) by replacing the domain space $L^n(\Omega)$ by the sum of "endpoint spaces" in (3.2), $(L^n(\Omega) + L_{n,1;-1/n'}(\Omega))$. We formulate the resulting inequality as a corollary.

Corollary 3.4. *There exists a $C > 0$ such that for every $u \in C_0^1(\Omega)$*

$$\left(\int_0^1 u^*(t)^n \big(\log(e/t)\big)^{-n} \frac{dt}{t}\right)^{1/n} \le C \inf_{f+g=|\nabla u|} \left\{ \left(\int_\Omega |f(x)|^n \, dx\right)^{1/n} + \int_0^1 t^{-1/n'} \big(\log(e/t)\big)^{-1/n'} g^*(t)\, dt \right\}.$$

Remark 3.5. The result of Corollary 3.4 is a special case of [EOP], Theorem 12.6 with $p = n'$, $r = n$, $\theta = -1/n'$ and $\beta = 0$. The sum $L^n(\Omega) + L_{n,1;-1/n'}(\Omega)$ is indeed essentially larger than $L^n(\Omega)$; this follows from Theorem 3.2. It is also of certain interest to note that the fundamental function of $L^n(\Omega) + L_{n,1;-1/n'}(\Omega)$ is equivalent near zero to $t^{1/n}(\log(e/t))^{-1/n'}$, hence it is "better" than that of $L^n(\Omega)$.

Now let us switch to Orlicz spaces.

4 The optimality of (1.3) in the context of Orlicz spaces—Part 1

Given a Young function A, we define its *complementary function* $\widetilde{A}$ by

$$\widetilde{A}(t) := \sup_{s>0}(st - A(s)), \qquad t > 0.$$

Given two Young functions A and B, we shall say that $B \gg A$ when $A(t) \le B(Ct)$ for some $C > 0$ and every large t and, moreover, for every fixed $\lambda > 0$,

$$\limsup_{t\to\infty} \frac{B(\lambda t)}{A(t)} = \infty.$$

As mentioned in the Introduction, Hempel, Morris and Trudinger ([HMT]) proved that $\exp L^{n'}(\Omega)$ is the optimal (that is, the smallest possible) Orlicz range space in (1.3) when the given domain space is $L^n(\Omega)$. The following remarkable general result on the optimality of an Orlicz range space was proved recently by A. Cianchi ([Ci]).

Theorem 4.1. *Let A be a Young function, satisfying*

$$\int_1^\infty \frac{\widetilde{A}(s)}{s^{n'+1}}\, ds = \infty \qquad \textit{and} \qquad \int_0^1 \frac{\widetilde{A}(s)}{s^{n'+1}}\, ds < \infty.$$

Set

$$A_n(t) = \int_0^{t^{n'}} \left(\Phi_n^{-1}(s)\right)^{n'} ds,$$

where Φ_n^{-1} is the inverse function of

$$\Phi_n(t) = \int_0^t \frac{\widetilde{A}(s)}{s^{n'+1}}\, ds.$$

Then

$$\|u\|_{L_{A_n}(\Omega)} \le \|\nabla u\|_{L_A(\Omega)}, \qquad u \in C_0^1(\Omega), \tag{4.1}$$

and (4.1) *no longer holds when A_n is replaced by a Young function B such that $B \gg A_n$.*

However, neither the result of [HMT] nor Cianchi's theorem give any information on the optimality of the Orlicz *domain* space. Our aim in this section is to investigate the optimality of L^n as an Orlicz domain space in (1.3).

Rather surprisingly, it will turn out that $L^n(\Omega)$ is not optimal as an Orlicz domain space in (1.3), and even worse, that such an optimal Orlicz domain space does not exist at all (a similar situation is described by (1.2), in which case there is no optimal Lebesgue range space).

First, we need an auxiliary lemma, which can be obtained by a simple exercise with Luxemburg norms (cf. also [Ci], Lemma 2).

Lemma 4.2. *Let A be a Young function and $0 \ne \alpha \in \mathbb{R}$. Define*

$$E_\alpha(t) = \frac{1}{|\alpha|} t^{-1/\alpha} \int_0^t A(s) s^{(1/\alpha)-1}\, ds, \qquad t \in (0,\infty),$$

and

$$G_\alpha(t) = \frac{1}{|\alpha|} t^{-1/\alpha} \int_t^\infty A(s) s^{(1/\alpha)-1}\, ds, \qquad t \in (0,\infty).$$

Then both E_α *and* G_α *are increasing and, for* $a \in (0, 1/2)$,

$$\left\| t^\alpha \chi_{(0,a)}(t) \right\|_{L_A(0,1)} \approx \begin{cases} \dfrac{a^\alpha}{E_\alpha^{-1}(1/a)} & \text{if } \alpha > 0, \\[2ex] \dfrac{a^\alpha}{G_\alpha^{-1}(1/a)} & \text{if } \alpha < 0; \end{cases}$$

$$\left\| t^\alpha \chi_{(a,1)}(t) \right\|_{L_A(0,1)} \approx \begin{cases} \dfrac{a^\alpha}{G_\alpha^{-1}(1/a)} & \text{if } \alpha > 0, \\[2ex] \dfrac{a^\alpha}{E_\alpha^{-1}(1/a)} & \text{if } \alpha < 0; \end{cases}$$

and

$$\left\| \int_t^1 s^{\alpha-1} \chi_{(0,a)}(s)\, ds \right\|_{L_A(0,1)} \approx \begin{cases} \dfrac{a^\alpha}{A^{-1}(1/a)} & \text{if } \alpha > 0, \\[2ex] \dfrac{a^\alpha}{G_\alpha^{-1}(1/a)} & \text{if } \alpha < 0; \end{cases}$$

Now we are in a position to prove the main theorem of this section (this result was obtained in collaboration with R. Kerman).

Theorem 4.3. *Let* A *be a Young function such that*

$$\|u\|_{\exp L^{n'}(\Omega)} \le C \|\nabla u\|_{L_A(\Omega)}. \tag{4.2}$$

Then there exists another Young function, A_1, *say, such that*

$$A \gg A_1 \tag{4.3}$$

and

$$\|u\|_{\exp L^{n'}(\Omega)} \le C \|\nabla u\|_{L_{A_1}(\Omega)}.$$

Proof. First, we note that $i_\varrho = \infty$ when $\varrho(f) = \|f\|_{\exp L^{n'}(0,1)}$. Hence, by Theorem 2.5,

$$\left\| \int_t^1 s^{-1/n'} (Pf^*)(s)\, ds \right\|_{\exp L^{n'}(0,1)} \leq C \left\| \int_t^1 s^{-1/n'} f^*(s)\, ds \right\|_{\exp L^{n'}(0,1)}.$$

Thus, by Corollary 2.4 with $m = 1$, (4.2) is equivalent to

$$\left\| \int_t^1 s^{-1/n'} (Pf^*)(s)\, ds \right\|_{\exp L^{n'}(0,1)} \leq C\|f\|_{L_A(0,1)}.$$

By duality, this is equivalent to

$$\left\| \int_t^1 s^{-1/n'} (Pf^*)(s)\, ds \right\|_{L_{\widetilde{A}}(0,1)} \leq C\|f\|_{L(\log L)^{1/n'}(0,1)}. \tag{4.4}$$

Using the argument of [Ca], we obtain that (4.4) is equivalent to the same inequality restricted to characteristic functions of intervals $(0,a)$ for $a \in (0,1)$, namely

$$\left\| \int_t^1 s^{-1/n'} (P\chi^*_{(0,a)})(s)\, ds \right\|_{L_{\widetilde{A}}(0,1)} \leq Ca \left(\log(e/a)\right)^{1/n'}.$$

Now, it can be easily seen that, for $a \in (0, 1/2)$,

$$\left\| \int_t^1 s^{-1/n'} (P\chi^*_{(0,a)})(s)\, ds \right\|_{L_{\widetilde{A}}(0,1)} \approx a\|t^{-1/n'} \chi_{(a,1)}(t)\|_{L_{\widetilde{A}}(0,1)},$$

hence, by Lemma 4.2, for $a \in (0,1)$,

$$\left\| \int_t^1 s^{-1/n'} (P\chi^*_{(0,a)})(s)\, ds \right\|_{L_{\widetilde{A}}(0,1)} \approx \frac{a^{1/n}}{G^{-1}(1/a)},$$

where $G(t) = n't^{n'} \int_0^t \widetilde{A}(s) s^{-n'-1}\, ds$. Passing to inverse functions, we get that (4.2) is equivalent to the existence of some $C_1 > 0$ such that for every $t \in (e, \infty)$,

$$\int_1^t \frac{\widetilde{A}(s)}{s^{n'+1}}\, ds \leq C_1 \log t. \tag{4.5}$$

Thus, to prove the statement of the theorem, we need to construct a function A_1 satisfying (4.3) and, for some $C_2 > 0$,

$$\int_1^t \frac{\widetilde{A}_1(s)}{s^{n'+1}}\,ds \le C_2 \log t, \qquad t \in (e,\infty). \tag{4.6}$$

Suppose that A is fixed and that (4.5) holds. Then, with no loss of generality, we may assume that there is a $C_3 > 0$ such that

$$\widetilde{A}(t) \ge C_3 t^{n'}, \qquad t \in (e,\infty). \tag{4.7}$$

Observe that (4.5) implies that, for some $C_4 > 0$,

$$\widetilde{A}(t) \le C_4 t^{n'} \log t, \qquad t \in (e,\infty). \tag{4.8}$$

Indeed, for $t \in (e,\infty)$, we have

$$\int_1^{2t} \frac{\widetilde{A}(s)}{s^{n'+1}}\,ds \ge \int_t^{2t} \frac{\widetilde{A}(s)}{s^{n'+1}}\,ds \ge \Big(\frac{1-2^{-n'}}{n'}\Big)\widetilde{A}(t)t^{-n'} \tag{4.9}$$

and (4.8) follows from (4.9) and (4.5).

Now, we note that, by (4.7),

$$\frac{\widetilde{A}(s)}{s\log s} \ge c\frac{s^{n'-1}}{\log s} \to \infty \qquad \text{as } s \to \infty. \tag{4.10}$$

Let $\beta > 2^{1-n'}C_4$ be fixed. We associate to each $t \in (e,\infty)$ the set

$$G_t = \Big\{ s \in (0,\infty);\ \frac{\widetilde{A}(s)}{s\log s} \ge \beta(2t)^{n'-1} \Big\},$$

and the number

$$\tau = \tau(t) := \inf G_t. \tag{4.11}$$

By (4.10), $G_t \ne \emptyset$. Further, $\tau > t$, since, by (4.8),

$$\frac{\widetilde{A}(s)}{s\log s} \le C_4 s^{n'-1} \le C_4 t^{n'-1} < \beta(2t)^{n'-1}, \qquad s \in (e,t),$$

and, because the function $\widetilde{A}(s)/(s\log s)$ is continuous in s,

$$\frac{\widetilde{A}(\tau)}{\tau\log\tau} = \beta(2t)^{n'-1}. \tag{4.12}$$

Next, (4.7) implies that every s, satisfying

$$C_3 \frac{s^{n'-1}}{\log s} \geq \beta(2t)^{n'-1},$$

belongs to G_t. Since the inverse function F^{-1} of

$$F(s) := \frac{s^{n'-1}}{\log s}$$

satisfies for large values of t

$$F^{-1}(t) \approx \big(t \log t\big)^{n-1},$$

it follows that there is a $C_5 > 0$ such that

$$\Big(C_5 t\big(\log t\big)^{n-1}, \infty\Big) \subset \Big\{s;\, C_3 \frac{s^{n'-1}}{\log s} \geq \beta(2t)^{n'-1}\Big\} \subset G_t.$$

This, combined with (4.7), implies that

$$\log t < \log \tau \leq C \log t, \qquad t \in (e, \infty). \tag{4.13}$$

We now claim that for every $\lambda > e$ there exists a $t > 0$ such that the corresponding $\tau = \tau(t)$ satisfies $\tau \geq \lambda t$.

Indeed, suppose that the claim is not true, that is, there exists a $\lambda > e$ such that $\tau \leq \lambda t$ for every $t \in (e, \infty)$. Then, by (4.12), (4.13) and the monotonicity of $\widetilde{A}(s)/s$,

$$\beta(2t)^{n'-1} \log t < \frac{\widetilde{A}(\tau)}{\tau} \leq \frac{\widetilde{A}(\lambda t)}{\lambda t},$$

that is,

$$\widetilde{A}(\lambda t) \geq \beta 2^{n'-1} \lambda t^{n'} \log t, \qquad t \in (e, \infty).$$

We therefore obtain, for sufficiently large t,

$$\int_1^t \frac{\widetilde{A}(s)}{s^{n'+1}}\, ds \geq \int_\lambda^t \frac{\widetilde{A}(s)}{s^{n'+1}}\, ds \geq c \int_\lambda^t \frac{\log(s/\lambda)}{s}\, ds \geq c\big(\log t\big)^2,$$

a contradiction with (4.5). This proves our claim. In turn, by convexity of $\widetilde{A}$, we get, for some t_0 large enough,

$$\widetilde{A}(\tau) \geq 2\widetilde{A}(t), \qquad t \geq t_0. \tag{4.14}$$

Next we claim that, for every fixed $M \in \mathbb{N}$, there exists a sequence $t_j \nearrow \infty$ such that

$$\frac{\widetilde{A}(\tau_j)}{\tau_j} \frac{t_j}{\widetilde{A}(Mt_j)} \to \infty, \qquad j \to \infty, \tag{4.15}$$

where τ_j corresponds to t_j in the sense of (4.11). Suppose that our claim is not satisfied, namely, that there is an $M \in \mathbb{N}$ and a $K > 0$ such that

$$\frac{\widetilde{A}(\tau)}{\tau} \frac{t}{\widetilde{A}(Mt)} \leq K, \qquad t \in (e, \infty).$$

Then, for $t > Me$, we have by (4.12) and (4.13)

$$\begin{aligned}\int_1^t \frac{\widetilde{A}(s)}{s^{n'+1}}\, ds \geq \int_M^t \frac{\widetilde{A}(s)}{s^{n'+1}}\, ds &= M^{-n'} \int_1^{t/M} \frac{\widetilde{A}(Ms)}{s^{n'+1}}\, ds \\ &\geq K^{-1} M^{-n'} \int_1^{t/M} \frac{\log s}{s}\, ds \geq c\big(\log t\big)^2,\end{aligned}$$

a contradiction with (4.5). This shows (4.15). Passing to a diagonal sequence if necessary, we obtain

$$\frac{\widetilde{A}(\tau_j)}{\tau_j} \frac{t_j}{\widetilde{A}(jt_j)} \to \infty, \qquad j \to \infty. \tag{4.16}$$

We may assume with no loss of generality that $t_1 \geq t_0$, where t_0 is from (4.14), $t_j > \tau_{j-1}$ and $2t_j < \tau_j$, $j \geq 2$ (again, we can pass to a subsequence when necessary).

Now we are ready to construct A_1. We shall in fact construct $\widetilde{A}_1$. For $j = 2, 3, \ldots$, set

$$\widetilde{A}_1(t) = \begin{cases} \widetilde{A}(t_j) + \dfrac{\widetilde{A}(\tau_j) - \widetilde{A}(t_j)}{\tau_j - t_j}(t - t_j) & \text{if } t_j \leq t \leq \tau_j, \\ \widetilde{A}(t) & \text{otherwise.} \end{cases} \tag{4.17}$$

Obviously, $\widetilde{A}_1(t) \geq \widetilde{A}(t)$ for every $t \in (0, \infty)$, and, moreover, by (4.17), (4.14) and (4.16),

$$\frac{\widetilde{A}_1(2t_j)}{\widetilde{A}(jt_j)} \geq \frac{1}{2} \frac{\widetilde{A}(\tau_j) t_j}{\widetilde{A}(jt_j) \tau_j} \to \infty, \qquad j \to \infty.$$

This shows that $\widetilde{A}_1 \gg \widetilde{A}$, which is equivalent to (4.3).

It remains to verify (4.6). Fix $j \in \mathbb{N}$ and assume that $t \in [t_j, t_{j+1})$. Then

$$\int_1^t \frac{\widetilde{A}_1(s)}{s^{n'+1}}\,ds \le \int_1^t \frac{\widetilde{A}(s)}{s^{n'+1}}\,ds + \sum_{k=1}^{j} \int_{t_k}^{\tau_k} \left[\widetilde{A}(t_k) + \frac{\widetilde{A}(\tau_k) - \widetilde{A}(t_k)}{\tau_k - t_k}(s - t_k)\right] s^{-n'-1}\,ds, \tag{4.18}$$

and, by (4.12) and (4.13),

$$\begin{aligned} &\int_{t_k}^{\tau_k} \left[\widetilde{A}(t_k) + \frac{\widetilde{A}(\tau_k) - \widetilde{A}(t_k)}{\tau_k - t_k}(s - t_k)\right] s^{-n'-1}\,ds \\ &\le C \int_{t_k}^{\tau_k} \left[\widetilde{A}(t_k) + \frac{\widetilde{A}(\tau_k)}{\tau_k} s\right] s^{-n'-1}\,ds \le C \frac{\widetilde{A}(\tau_k)}{\tau_k t_k^{n'-1}} \\ &\le C \log \tau_k \le C \log t_k. \end{aligned} \tag{4.19}$$

With no loss of generality we may assume that

$$t_j \ge \exp\left(\sum_{k=1}^{j-1} \log t_k\right),$$

whence, by (4.18), (4.19), (4.5), and the fact that $t \ge t_j$,

$$\int_1^t \frac{\widetilde{A}_1(s)}{s^{n'+1}}\,ds \le C \log t + \sum_{k=1}^{j} \log t_k \le C(\log t + \log t_j) \le C \log t.$$

□

Remark 4.4. We have shown that there is no optimal Orlicz range space in (4.4), although, by Theorem 4.1, there is one for the corresponding Sobolev inequality with the fixed domain space $L(\log L)^{1/n'}(\Omega)$. This shows that (2.3) is not equivalent to (1.5) in general.

5 The optimality of Sobolev embeddings in the context of rearrangement-invariant spaces

The surprising results of Theorems 4.3 and 3.3 motivate us to dig a little deeper and to investigate the optimality of Sobolev embeddings in a broader context than in that of, say, Lebesgue, Orlicz or Lorentz-Zygmund spaces. The natural appropriate environment seems to be that of rearrangement-invariant spaces (which moreover includes all the three above-mentioned classes of function spaces). We shall apply Theorem 2.2, as in [Ke1], to associate to a given r.i. norm ϱ_R the essentially smallest r.i. norm ϱ_D for which (1.5) holds. For the sake of brevity, in the case $m = 1$ we restrict ourselves to r.i. norms ϱ_R which satisfy (2.5). When this restriction is removed, we can still obtain certain optimality results (see [EKP] for details).

Theorem 5.1. *Let ϱ_R be an r.i. norm on $\mathfrak{M}_+(0,1)$.*

(i) *Assume that* (2.5) (*with $m = 1$*) *holds. Then the functional*

$$\varrho_D(f) = \varrho_R\left(\int_t^1 f^*(s) s^{-1/n'}\, ds\right), \qquad f \in \mathfrak{M}_+(0,1), \tag{5.1}$$

is equivalent to an r.i. norm. Moreover, it is the optimal (*that is, the smallest*) *r.i. domain norm for ϱ_R in* (1.5) *with $m = 1$.*

(ii) *Let $n \geq 3$ and $2 \leq m \leq n-1$. Then the functional*

$$\varrho_D(f) = \varrho_R\left(\int_t^1 (Pf^*)(s) s^{(m/n)-1}\, ds\right), \qquad f \in \mathfrak{M}_+(0,1), \tag{5.2}$$

is the optimal r.i. domain norm for ϱ_R in (1.5).

The proof of Theorem 5.1 can be derived from Theorem 2.2, Corollary 2.4 and the axioms of an r.i. norm.

Remark 5.2. The condition (2.5) is not necessary for the functional $\varrho_R\big(\int_t^1 f^*(s)s^{-1/n'} ds\big)$ to be equivalent to an r.i. norm; consider for example the Lorentz norm $\varrho_R(f) = \int_0^1 f^*(t)t^{-1/n}\, dt$. Then $\varrho_R\big(\int_t^1 f^*(s)s^{-1/n'}\, ds\big) = \int_0^1 f^*(t)\, dt$, but (2.5) is not satisfied.

Now we turn our attention to the question of the optimal *range* space when the domain space is given.

Theorem 5.3. *Suppose that ϱ_D is an r.i. norm on $\mathfrak{M}_+(0,1)$. Then the functional σ, defined by*

$$\sigma(g) = \left\{ \begin{array}{ll} \varrho_D'(t^{1/n}(Pg^*)(t)) & \text{if } m = 1 \\ \varrho_D'\left(\displaystyle\int_t^1 (Pg^*)(s) s^{(m/n)-1}\, ds\right) & \text{if } n \geq 3 \text{ and } m \geq 2 \end{array} \right\} \tag{5.3}$$

is an r.i. norm on $\mathfrak{M}_+(0,1)$. Put $\varrho_R = \sigma'$. Then ϱ_R is the optimal (*that is, the largest*) *r.i. norm for ϱ_D in* (1.5).

Proof of Theorem 5.3 again follows from Theorem 2.2 and the axioms of an r.i. norm.

6 Optimal pairs of Lorentz-Karamata norms

Our next goal is to combine the results on the optimality of a domain norm with those on the optimality of a range norm and to investigate the optimality of the pair (ϱ_D, ϱ_R), that is, the question of when the ϱ_D associated to ϱ_R in Theorem 5.1 has ϱ_R as its optimal range norm.

The key obstacle is of course the fact that the optimal range norm is given implicitly via its associate norm by (5.3). However, we can apply the construction of Section 5 to a quite large family of classical Lorentz norms (the so-called Lorentz-Karamata norms), which includes almost all examples of norms that have been mentioned so far.

Definition 6.1. A positive function b is said to be *slowly varying* (s.v.) on $(1, \infty)$, in the sense of Karamata, if for each $\varepsilon > 0$, $t^{\varepsilon} b(t)$ is equivalent to an increasing function and $t^{-\varepsilon} b(t)$ is equivalent to a decreasing function on $(1, \infty)$.

Example 6.2. The following functions are slowly varying on $(1, \infty)$:

(i) $b(t) = (e + \log t)^{\alpha} \big(\log(e + \log t) \big)^{\beta}$, $\alpha, \beta \in \mathbb{R}$;
(ii) $b(t) = \exp(\sqrt{\log t})$.

The properties of slowly varying functions are discussed in some detail in [Zy], Chapter 2, p. 184, see also [W] and [Ke2]. We list some that will be needed later.

Lemma 6.3. *Suppose that b is slowly varying on $(1, \infty)$. Then*

(i) *b^r is slowly varying on $(1, \infty)$ for all $r \in \mathbb{R}$;*
(ii) *$\int_{t^{-1}}^{1} s^{-1} b(s^{-1})\, ds$ is slowly varying on $(1, \infty)$ and (see* [Zy], Chapter 2, p. 186)

$$\lim_{t \to \infty} \frac{b(t)}{\int_{t^{-1}}^{1} s^{-1} b(s^{-1})\, ds} = 0; \tag{6.1}$$

(iii) $\lim_{t \to \infty} \dfrac{b(ct)}{b(t)} = 1$ *for all $c > 0$.*

Denote

$$\varrho_p(g) = \|g\|_{L^p(0,1)}, \qquad 0 < p \le \infty.$$

Definition 6.4. Let $0 < p, q \le \infty$, and suppose b is slowly varying on $(1, \infty)$. Let $\phi = \phi_{p,q;b}$ be given by

$$\phi(t) = t^{(1/p)-(1/q)} b(t^{-1}), \qquad 0 < t < 1.$$

Assume that $\varrho_q(t^{-1/q} b(t^{-1})) < \infty$ when $p = \infty$. The *Lorentz-Karamata* (L-K) *(quasi-)norm* $\varrho = \varrho_{p,q;b}$ is given at $f \in \mathfrak{M}_+(0,1)$ by

$$\varrho(f) = \varrho_q(\phi f^*).$$

Standard calculations (see [B]) yield

$$i_\varrho = p.$$

We shall need some information on associate norms of classical Lorentz norms.

Proposition 6.5. *Assume that* $p \in (0, \infty]$ *and* $\phi \in \mathfrak{M}_+(0,1)$. *Let*

$$\varrho(g) = \varrho_{\phi,p}(g) := \varrho_p(\phi g^*).$$

Then

$$\varrho'(g) \approx \left\{ \begin{array}{lll} \varrho_\infty \left(\dfrac{t^{1-(1/p)} (Pg^*)(t)}{(P\phi^p)(t)^{1/p}} \right) & \textit{if} & 0 < p \le 1, \\ \\ \varrho_{p'} \left(\dfrac{\phi^{p-1} Pg^*}{P\phi^p} \right) & \textit{if} & 1 < p < \infty, \\ \\ \varrho_1 \left(\dfrac{g^*}{\phi} \right) & \textit{if} & p = \infty, \ \phi \ \textit{nondecreasing}. \end{array} \right\} \tag{6.2}$$

Proof. By (2.1), $\varrho'(g) = \varrho'_d(g^*)$. In [Sa], Theorem 1, it was shown that if $1 < p < \infty$ (and ϕ is any locally integrable nonnegative function (weight) on $(0,1)$), then

$$\varrho'_d(g) \approx \varrho_{p'} \left(\frac{\phi^{p-1} Pg}{P\phi^p} \right) + \frac{\varrho_1(g)}{\varrho_p(\phi)}, \qquad g \in \mathfrak{M}_+(0,1). \tag{6.3}$$

It remains to observe that when g is decreasing, then the second summand on the right hand side of (6.3) is not greater than a constant multiple of the first one.

The corresponding expression for $0 < p \leq 1$ was obtained in [St], Proposition 1:

$$\varrho_d'(g) \approx \varrho_\infty\left(\frac{t^{1-(1/p)}(Pg)(t)}{(P\phi^p)(t)^{1/p}}\right), \qquad g \in \mathfrak{M}_+(0,1). \tag{6.4}$$

As for $p = \infty$, it is clear that when ϕ is nondecreasing on $(0,1)$, then

$$\varrho_d'(g) = \varrho_1\left(\frac{g}{\phi}\right), \qquad g \in \mathfrak{M}_+(0,1). \tag{6.5}$$

$\square$

Given the index p, $1 < p < \infty$, and the weight ϕ on $(0,1)$, with

$$\int_0^1 \phi(t)^p\,dt < \infty, \tag{6.6}$$

it is not difficult to verify that

$$\varrho(f) = \varrho_p(\phi P f^*), \qquad f \in \mathfrak{M}_+(0,1), \tag{6.7}$$

is an r.i. norm. We will need the following result, which is of independent interest.

Theorem 6.6. *Let $1 < p < \infty$ and suppose the weight ϕ on $(0,1)$ satisfies (6.6). Assume, further, that $\int_0^1 \phi(t)^p t^{-p}\,dt = \infty$ and*

$$\int_0^r \phi(t)^p\,dt \leq Cr^p\left(1 + \int_r^1 \frac{\phi(t)^p}{t^p}\,dt\right), \qquad 0 < r < 1. \tag{6.8}$$

Then the r.i. norm ϱ defined in (6.7) has the associate norm

$$\varrho'(g) \approx \varrho_{p'}(\psi g^*), \qquad g \in \mathfrak{M}_+(0,1),$$

where

$$\psi(s)^{p'} = \frac{d}{ds}\left[\left(1 + \int_s^1 \frac{\phi(y)^p}{y^p}\,dy\right)^{1-p'}\right], \qquad 0 < s < 1. \tag{6.9}$$

Proof. Observe that

$$\psi(t)^{p'} = (p'-1)\left(1+\int_t^1 \frac{\phi(s)^p}{s^p}\,ds\right)^{-p'} \frac{\phi(t)^p}{t^p}.$$

Hence, integrating both sides of (6.9) between 0 and t, we obtain

$$\int_0^t \psi(s)^{p'}\,ds = \left(1+\int_t^1 \frac{\phi(s)^p}{s^p}\,ds\right)^{1-p'}. \tag{6.10}$$

These equations, together with (2.1) and (6.2), yield

$$\varrho'_{\psi,p'}(g) \approx \varrho_p(\phi P g^*) = \varrho(g).$$

Thus, by the duality principle it suffices to show that $\varrho_{\psi,p'}$ is a Banach function norm. According to [Sa], Theorem 4 this will be true if and only if

$$\left(\int_0^r \psi^{p'}\right)^{1/p'} \left(\int_0^r (P\psi^{p'})^{1-p}\right)^{1/p} \le Cr, \qquad 0<r<1.$$

But, by (6.10),

$$\begin{aligned}\int_0^r ((P\psi^{p'})(t))^{1-p}\,dt &= \int_0^r t^{p-1}\left(\int_0^t \psi^{p'}\right)^{1-p} dt\\ &= \frac{r^p}{p}\left(1+\int_r^1 \frac{\phi(t)^p}{t^p}\,dt\right) + \int_0^r t^{p-1}\int_t^r \frac{\phi(s)^p}{s^p}\,ds\,dt \qquad (6.11)\\ &\le Cr^p\left(1+\int_r^1 \frac{\phi(t)^p}{t^p}\,dt\right),\end{aligned}$$

since

$$\int_0^r t^{p-1}\int_t^r \frac{\phi(s)^p}{s^p}\,ds\,dt = \frac{1}{p}\int_0^r \phi(t)^p\,dt \le Cr^p\left(1+\int_r^1 \frac{\phi(t)^p}{t^p}\,dt\right)$$

by (6.8). The result now follows from (6.10) and (6.11). □

Remark 6.7. The norm (6.7) defines a space denoted by $\Gamma^p(\phi^p)$ in [Sa]. An expression equivalent to the associate norm, $\varrho'_{\Gamma^p(\phi^p)}$ was obtained in [GHS], but it is not always easy to compute. Theorem 2.7 thus gives a more tractable way to deal with the associate norm, provided that (6.8) is satisfied. This is the case, for example, when

$$\int_0^r \phi^p \le C\int_r^{2r} \phi^p, \qquad 0<r<1/2,$$

and so, in particular, when ϕ is essentially nondecreasing on $(0,1)$.

More results and references on the associate spaces of classical Lorentz spaces can be found e.g. in [CPSS].

We shall frequently use various weighted estimates for integral operators on $(0,1)$ (possibly restricted to monotone functions). Of the vast literature available on this subject, our standard general reference is [OK] and, when the restricted version is needed, [Sa].

We can now state the main results of this section. First we discuss the case $m=1$.

Theorem 6.8. *Let $p \in (n', \infty]$ and $q \in [1,\infty]$, and suppose that b is a slowly varying function on $(1,\infty)$ such that $\phi(t) = t^{(1/p)-(1/q)} b(t^{-1})$ satisfies $\varrho_q(\phi) < \infty$. Let*

$$\varrho_R(f) = \begin{cases} \varrho_q(\phi f^*) & \text{when } p > q, \\ \varrho_q(\phi P f^*) & \text{when } p \le q, \end{cases}$$

and

$$\varrho_D(f) = \varrho_R\left(\int_t^1 f^*(s) s^{-1/n'} \, ds \right).$$

Then ϱ_R and ϱ_D are optimal r.i. norms in (1.5) (with $m=1$).

Proof. In view of Theorem 5.1, only the optimality of ϱ_R needs to be shown. To this end, we prove

$$\varrho_R'(g) \approx \varrho_D'(t^{1/n}(Pg^*)(t)),$$

and then invoke Theorem 5.3. Now, one always has

$$\varrho_R'(g) \ge \varrho_D'(t^{1/n}(Pg^*)(t)).$$

This is readily seen, via Fubini's theorem and Hölder's inequality, from

$$\begin{aligned} \varrho_D'(t^{1/n}(Pg^*)(t)) &= \sup_{f \not\equiv 0} \frac{\int_0^1 g^*(t) \int_t^1 f(s) s^{1/n'} \, ds \, dt}{\varrho_D(f)} \\ &\le \sup_{f \not\equiv 0} \frac{\varrho_R'(g) \varrho_R\left(\int_t^1 f(s) s^{-1/n'} \, ds \right)}{\varrho_D(f)} \end{aligned}$$

and the definition of $\varrho_D(f)$. Hence, it just remains to show

$$\varrho_D'(t^{1/n}(Pg^*)(t)) \ge c \varrho_R'(g).$$

We shall only sketch the proof in the case when $1 < q < \infty$.

By known weighted inequalities for Hardy integral operators we have

$$\varrho_R(f) \approx \varrho_q(\phi f^*).$$

Thus, by (6.2),

$$\varrho_R'(g) \approx \varrho_{q'}\Big(\frac{\phi^{q-1}}{P\phi^q} Pg^*\Big) \approx \varrho_{q'}\Big(\frac{Pg^*}{\phi}\Big). \tag{6.12}$$

Again,

$$\varrho_D(f) \approx \varrho_q(t^{1/n}\phi(t)f(t)).$$

So, by (6.12),

$$\varrho_D'(t^{1/n}(Pg^*)(t)) \geq c\varrho_{q'}\left(\frac{t^{-1/n}t^{1/n}(Pg^*)(t)}{\phi(t)}\right) \geq c\varrho_R'(g).$$

□

In the case $m \geq 2$ we have the following result:

Theorem 6.9. *Let $n \geq 3$ and $2 \leq m \leq n-1$. Let $1 \leq p, q \leq \infty$, and let b, ϕ and ϱ_R be as in* Theorem 6.8. *Given $f \in \mathfrak{M}_+(0,1)$, define*

$$\varrho_D(f) = \varrho_R\bigg(\int_t^1 Pf^*(s)s^{(m/n)-1}\,ds\bigg).$$

Then ϱ_R and ϱ_D are optimal r.i. norms in (1.5) *if either $p > n/(n-m)$ or $p = n/(n-m)$, $q = \infty$ and b is bounded away from zero. In all the other cases of p, q and b, ϱ_D is optimal but ϱ_R is not.*

Proof. Once again, just the optimality of ϱ_R is in question. We examine this by cases.

Case 1: $n/(n-m) < p \leq \infty$

Assume that $1 < q < \infty$. It is easy to see that it suffices to get ϱ_R optimal in (1.5) for *some* r.i. norm. In view of Theorem 5.3, we need only to find an r.i. norm ϱ such that

$$\varrho_R'(g) \approx \varrho'\bigg(\int_t^1 (Pg^*)(s)s^{(m/n)-1}\,ds\bigg). \tag{6.13}$$

Since $p' < n/m$, we have $\varrho_R(f) \approx \varrho_q(\phi f^*)$, so, from (6.2),

$$\varrho'_R(g) \approx \varrho_{q'}(\psi P g^*), \tag{6.14}$$

where

$$\psi(t) = \begin{cases} \phi(t)^{-1}, & \text{when } p < \infty, \\ \dfrac{t^{1/q} b(t^{-1})^{q-1}}{\int_0^t s^{-1} b(s^{-1})^q \, ds}, & \text{when } p = \infty. \end{cases}$$

We claim that a suitable choice for ϱ is the one with associate norm

$$\varrho'(g) = \varrho_{q'}(t^{-m/n}\psi(t)(Pg^*)(t)).$$

A weighted inequality for the operator P yields

$$\varrho'(g) \approx \varrho_{q'}(t^{-m/n}\psi(t) g^*(t)). \tag{6.15}$$

Now,

$$\begin{aligned} &\varrho'\left(\int_t^1 (Pg^*)(s) s^{(m/n)-1} \, ds\right) \\ &\qquad \geq \varrho_{q'}\left(\chi_{(0,1/2)}(t) t^{-m/n} \psi(t) \int_t^{2t} (Pg^*)(s) s^{(m/n)-1} \, ds\right) \\ &\qquad \geq c\varrho_{q'}\Big(\chi_{(0,1/2)}(t)\psi(2t)(Pg^*)(2t)\Big) \geq c\varrho'_R(g). \end{aligned}$$

Similarly, using (6.15) and (6.14),

$$\begin{aligned} \varrho'\left(\int_t^1 (Pg^*)(s) s^{(m/n)-1} \, ds\right) &\approx \varrho_{q'}\left(t^{-m/n}\psi(t) \int_t^1 (Pg^*)(s) s^{(m/n)-1} \, ds\right) \\ &\leq C\varrho_{q'}(\psi P g^*) \leq C\varrho'_R(g), \end{aligned}$$

whence (6.13) is verified. The proof in other cases of q is similar.

Case 2: $p < n/(n-m)$ or $p = n/(n-m)$ and $\varrho_q(t^{-1/q} b(t^{-1})) < \infty$

In this case, $\varrho_D(f) \approx \varrho_1(f)$. Indeed,

$$\begin{aligned} \varrho_D(f) &= \varrho_q\left(\phi(t) \int_t^1 (Pf^*)(s) s^{(m/n)-1} \, ds\right) \\ &\leq \varrho_q\left(\phi(t)\varrho_1(f^*) \int_t^1 s^{(m/n)-2} \, ds\right) \leq C\varrho_1(f), \end{aligned}$$

and the converse inequality follows from the axiom (A_6) of the r.i. norm, applied to $\varrho = \varrho_D$.

The optimal r.i. range norm for ϱ_1 is the classical Lorentz norm

$$\sigma(f) = \varrho_\infty(t^{1-(m/n)} f^*(t)).$$

This follows from Theorem 5.3, since

$$\begin{aligned}\varrho_1'\left(\int_t^1 (Pg^*)(s)s^{(m/n)-1}\,ds\right) &= \varrho_\infty\left(\int_t^1 (Pg^*)(s)s^{(m/n)-1}\,ds\right)\\ &= \varrho_1(Pg^*(t)t^{(m/n)-1}) \approx \varrho_1(t^{(m/n)-1}g^*(t)) \approx \sigma'(g).\end{aligned}$$

We conclude that ϱ_R is optimal if and inly if $p' = n/m$, $q = \infty$, and b is equivalent to a constant.

Case 3: $p' = n/m$ and $\varrho_q(t^{-1/q}b(t^{-1})) = \infty$

By Theorem 5.3, the optimality of ϱ_R in (1.5) is equivalent to

$$\varrho_R'(g) \approx \varrho_D'\left(\int_t^1 (Pg^*)(s)s^{(m/n)-1}\,ds\right). \tag{6.16}$$

We start with $1 < q < \infty$. By weighted inequalities for P and its dual,

$$\varrho_D(f) \approx \varrho_q\left(\phi(t)\int_t^1 (Pf^*)(s)s^{(m/n)-1}\,ds\right) \approx \varrho_q(t^{m/n}\phi(t)(Pf^*)(t)),$$

so, by Theorem 6.6,

$$\varrho_D'(g) \approx \varrho_{q'}(\psi g^*), \qquad \psi(t) = \frac{t^{-1/q'}b(t^{-1})^{q-1}}{1+\int_t^1 s^{-1}b(s^{-1})^q\,ds}.$$

Direct calculation yields

$$\varrho_D'\left(\int_t^1 (P\chi_{(0,a)})(s)s^{(m/n)-1}\,ds\right) \approx a^{m/n}\left(1+\int_a^1 s^{-1}b(s^{-1})^q\,ds\right)^{-1/q},$$

and

$$\varrho_R'(\chi_{(0,a)}) \approx \frac{a}{\varrho_R(\chi_{(0,a)})} \approx \frac{a}{a^{1/p}b(a^{-1})} = a^{m/n}b(a^{-1})^{-1}.$$

Thus,

$$\frac{\varrho_D'\left(\int_t^1 (P\chi_{(0,a)})(s)s^{(m/n)-1}\,ds\right)}{\varrho_R'(\chi_{(0,a)})} = \left[\frac{b(a^{-1})^q}{1+\int_a^1 s^{-1}b(s^{-1})^q\,ds}\right]^{1/q} \to 0$$

as $a \to 0_+$ by (6.1). As this is not compatible with (6.16), ϱ_R cannot be optimal.

Let $q = 1$. Then, by Fubini's theorem,

$$\varrho_R(f) = \varrho_1\left(f^*(t)\int_t^1 s^{(1/p)-2}b(s^{-1})\,ds\right) \approx \varrho_1(\phi f^*),$$

since b is slowly varying. Denoting

$$\beta(t) = \int_{t^{-1}}^1 s^{-1}b(s^{-1})\,ds, \qquad 1 < t < \infty,$$

and using Fubini's theorem and the slow variance of b, we obtain

$$\begin{aligned}\varrho_D(f) &\approx \varrho_1\left(\phi(t)\int_t^1 (Pf^*)(s)s^{(m/n)-1}\,ds\right)\\ &\approx \varrho_1(b(t^{-1})(Pf^*)(t)) \approx \varrho_1(\beta(t^{-1})f^*(t)),\end{aligned}$$

Therefore, from (6.4) and the slow variance of b (hence of β)

$$\varrho_R'(g) \approx \varrho_\infty\Big(\frac{Pg^*}{P\phi}\Big) \approx \varrho_\infty\Big(\frac{g^*}{\phi}\Big),$$

and

$$\varrho_D'(g) \approx \varrho_\infty\left(\frac{\int_0^t g^*(s)\,ds}{\int_0^t \beta(s^{-1})\,ds}\right) \approx \varrho_\infty\Big(\frac{g^*(t)}{\beta(t^{-1})}\Big).$$

In particular,

$$\begin{aligned}\frac{\varrho_D'\left(\int_t^1 (P\chi_{(0,a)})(s)s^{(m/n)-1}\,ds\right)}{\varrho_R'(\chi_{(0,a)})} &\approx \frac{a^{m/n}\left(1+\int_a^1 s^{-1}b(s^{-1})\,ds\right)^{-1}}{a^{m/n}b(a^{-1})^{-1}}\\ &\approx \frac{b(a^{-1})}{1+\int_a^1 s^{-1}b(s^{-1})\,ds} \to 0 \qquad \text{as } a \to 0_+\,.\end{aligned}$$

As before, this rules out (6.16), whence ϱ_R is not optimal.

Finally, let $q = \infty$. Then

$$\varrho_R(f) = \varrho_\infty(\phi P f^*) \approx \varrho_\infty(\phi f^*),$$

so, by (6.5)

$$\varrho_R'(g) \approx \varrho_1\Big(\frac{g^*}{\phi}\Big);$$

moreover,

$$\varrho_D(f) \approx \varrho_\infty\left(\phi(t)\int_t^1 (Pf^*)(s)s^{(m/n)-1}\,ds\right) \approx \varrho_\infty(tb(t^{-1})(Pf^*)(t)).$$

If $b(t) \equiv 1$, then one readily verifies that

$$\varrho_D'\left(\int_t^1 (Pg^*)(s)s^{(m/n)-1}\,ds\right) \approx \varrho_R'(g)$$

whence ϱ_R is optimal. In other cases we may assume that b is continuous and (replacing $b(t)$ by $\beta(t) = \inf_{s>t} b(s)$, if necessary) that it increases to infinity. If, in addition, we require that $\dfrac{d}{dt}b(t^{-1})^{-1}$ is nonincreasing, then

$$\varrho_D'(g) = \varrho_1\Big(\frac{d}{dt}b(t^{-1})^{-1}g^*(t)\Big),$$

and hence, applying Fubini's theorem twice,

$$\begin{aligned}\varrho_D'\left(\int_t^1 (Pg^*)(s)s^{(m/n)-1}\,ds\right) &\approx \varrho_1\left(\left[\frac{d}{dt}b(t^{-1})^{-1}\right]\int_t^1 (Pg^*)(s)s^{(m/n)-1}\,ds\right)\\ &= \varrho_1(t^{(m/n)-1}b(t^{-1})^{-1}(Pg^*)(t)) \approx \varrho_1(t^{(m/n)-1}b(t^{-1})^{-1}g^*(t)) \approx \varrho_R'(g).\end{aligned}$$

Thus, ϱ_R is optimal, again. We conclude that ϱ_R is optimal if and only if $q = \infty$ and b increases to infinity. □

7 Examples

We present here examples of norms, ϱ_D and ϱ_R, which, in view of Theorems 6.8, 6.9 and 5.1, are optimal in (1.5). Throughout this section, b is a slowly-varying function on $(1, \infty)$.

Example 7.1 (nonlimiting case). Let $1 < p < n/m$ and let

$$\varrho_D(f) = \varrho_p(f)$$

and

$$\varrho_R(f) = \varrho_p\Big(t^{-m/n} f^*(t)\Big).$$

Then (ϱ_R, ϱ_D) is an optimal pair of r.i. norms in (1.5).

Indeed, this follows for $m > 1$ from Theorem 6.9 on observing that

$$\varrho_p\left(t^{-m/n} \int_t^1 (Pf^*)(s) s^{(m/n)-1}\, ds\right) \approx \varrho_p(Pf^*) \approx \varrho_p(f).$$

When $m = 1$, we have to combine the arguments of Theorems 6.8 and 5.1 (i) and note that (2.5) is satisfied because $i_{\varrho_R} = np/(n-p) > n'$.

In particular, this improves (1.1) in the following sense: the range space $L^{p^*}(\Omega)$ in (1.1) can be replaced by the (smaller) Lorentz space $L_{p^*,p}(\Omega)$. That is a well-known fact, but here we may moreover conclude that $L_{p^*,p}(\Omega)$ is the smallest possible such rearrangement-invariant space.

Example 7.2 (limiting case). Suppose that $\varrho_D(f) = \varrho_{n/m}(f)$ and that ϱ_R is the norm of the Lorentz-Zygmund space $L_{\infty,n/m,-1}$:

$$\varrho_R(f) = \varrho_{n/m}\Big(t^{-m/n}\big(\log(e/t)\big)^{-1} f^*(t)\Big). \tag{7.1}$$

Then ϱ_R and ϱ_D are r.i. norms satisfying (1.5); moreover, by Theorems 5.3 and 6.6, ϱ_R is optimal, though ϱ_D is not (see Corollary 3.4). The ϱ_R given by (7.1) and the ϱ_D, defined by

$$\begin{aligned}\varrho_D(f) &= \varrho_{n/m}\Big(t^{-m/n}\big(\log(e/t)\big)^{-1}\int_t^1 (Pf^*)(s) s^{(m/n)-1}\, ds\Big)\\ &\approx \varrho_{n/m}\Big(t^{-m/n}\big(\log(e/t)\big)^{-1}\int_t^1 f^*(s) s^{(m/n)-1}\, ds\Big)\end{aligned}$$

are optimal in (1.5).

For $m > 1$, this follows from Theorem 6.9 with $p = \infty$, $q = m/n$ and $b(t) = (\log t)^{-1}$. When $m = 1$, we have $i_{\varrho_R} = \infty$, whence, by Theorem 2.5, (2.5) is satisfied. The assertion then follows, as above, from the combination of arguments of Theorem 5.1 (i) and Theorem 6.8.

Example 7.3 (limiting case—a general version). The above example can be formulated for a general slowly-varying function as follows: The r.i. norms

$$\begin{aligned}\varrho_D(f) &= \varrho_{n/m}\left(t^{-m/n}b(t^{-1})\int_t^1 (Pf^*)(s)s^{(m/n)-1}\,ds\right)\\ &\approx \varrho_{n/m}\left(t^{-m/n}b(t^{-1})\int_t^1 f^*(s)s^{(m/n)-1}\,ds\right)\end{aligned}$$

and

$$\varrho_R(f) = \varrho_{n/m}\Big(t^{-m/n}b(t^{-1})f^*(t)\Big)$$

are optimal in (1.5).

When $b(t) = \big[1+\log\big(1+\log t\big)\big]^{-1}$, this result extends and gives the best possible refinement of the double-exponential analogue of (1.3), proved in [EGO].

Example 7.4 (the optimal domain space for $L^\infty(\Omega)$). Take

$$\begin{aligned}\varrho_D(f) &= \varrho_\infty\left(b(t^{-1})\int_t^1 (Pf^*)(s)s^{(m/n)-1}\,ds\right)\\ &\approx \varrho_\infty\left(b(t^{-1})\int_t^1 f^*(s)s^{(m/n)-1}\,ds\right)\end{aligned}$$

and

$$\varrho_R(f) = \varrho_\infty\Big(b(t^{-1})(Pf^*)(t)\Big) \approx \varrho_\infty\Big(b(t^{-1})f^*(t)\Big).$$

Then (ϱ_D, ϱ_R) is an optimal pair of r.i. norms in (1.5). Given $b(t) \equiv 1$ and $m = 1$, this yields the pair $\varrho_D(f) = \varrho_1\big(t^{-1/n'}f^*(t)\big)$ and $\varrho_R(f) = \varrho_\infty(f)$. In other words, when $L^\infty(\Omega)$ is the target space, then the optimal rearrangement-invariant domain space is the Lorentz space $L_{n,1}(\Omega)$. This fact was obtained also in [CPi], Theorem 5.3, by different means (see Theorem 10.2 below).

8 The optimal rearrangement-invariant domain space in (1.4)

A particular case of Example 7.2 finishes the analysis of the optimality of the limiting case of Sobolev inequality initiated by the domain space $L^n(\Omega)$. We can reformulate the result in terms of function spaces in the following way: For the domain space $L^n(\Omega)$, the smallest possible rearrangement-invariant range space is the Lorentz-Zygmund space $L_{\infty,n;-1}(\Omega)$. However, then $L^n(\Omega)$ is not optimal (the largest possible) rearrangement-invariant domain space for the Sobolev embedding into $L_{\infty,n;-1}(\Omega)$. This is already known to us from Corollary 3.4. By Theorem 5.1 (i), the optimal rearrangement-invariant domain space, denoted by $X = X(\Omega)$, say, is normed by

$$\|f\|_X = \left\| \int_t^1 s^{-1/n'} (Pf^*)(s)\, ds \right\|_{L_{\infty,n;-1}(0,1)}. \tag{8.1}$$

We shall see that the space X is still essentially larger than $\big(L^n(\Omega) + L_{n,1;-1/n'}(\Omega)\big)$, obtained by Evans, Opic and Pick (Corollary 3.4 above). It turns out that X is a new type of a very important function space. Our goal in this section is to carry out a detailed study of X, in particular to describe its relations to familiar function spaces.

First we recall the important fact that there exists exactly one Orlicz space per a fundamental function.

Remark 8.1. Given a quasiconcave function φ on $[0,1]$, then there exists exactly one (up to equivalence of norms) Orlicz space $L_A(\mathcal{R})$ whose fundamental function is φ. This space is determined by the Young function A, satisfying

$$A(t) \approx \frac{1}{\varphi^{-1}(1/t)}, \qquad t \in (1,\infty).$$

Then, of course, by (3.1)

$$\Lambda_\varphi(\mathcal{R}) \hookrightarrow L_A(\mathcal{R}) \hookrightarrow M_\varphi(\mathcal{R}),$$

and $L_A(\mathcal{R})$ may coincide with either (or both) of the endpoint spaces.

The main result of this section is the following theorem.

Theorem 8.2. *Let the space X be defined by* (8.1). *Then*

(i) *the fundamental function φ_X of X satisfies*

$$\varphi_X(t) \approx t^{1/n}\big(\log(e/t)\big)^{-1/n'}, \qquad t \in (0,1); \tag{8.2}$$

(ii) *the following relations hold:*

$$\big(L^n(\Omega) + L_{n,1;-1/n'}(\Omega)\big) \hookrightarrow X, \tag{8.3}$$

$$X \subset \Big(L_{n,n;-1/n'}(\Omega) \cap \bigcap_{\alpha>1} L_{n,1;-\alpha/n'}(\Omega) \Big); \tag{8.4}$$

and both the embedding (8.3) *and the inclusion* (8.4) *are strict;*

(iii) *X is incomparable to every space from the scale of Lorentz-Zygmund spaces*

$$\big\{ L_{n,r;-1/n'}(\Omega) \big\}, \qquad r \in (1,n);$$

(iv) *X is incomparable to every space from the scale of Orlicz spaces*

$$\big\{ L_{n,n;-\alpha/n'}(\Omega) \big\}, \qquad \alpha \in (0,1).$$

Proof. To show (i) is an easy exercise. We shall prove (ii) in several steps.

First, since $L_{n,1;-1/n'}(\Omega)$ is the endpoint Lorentz space corresponding to the fundamental function φ_X from (8.2), it follows from (i) and (3.1) that

$$L_{n,1;-1/n'}(\Omega) \hookrightarrow X.$$

We next observe that $L^n(\Omega)$ is admissible as a domain space in the Sobolev embedding with the range space $L_{\infty,n;-1}(\Omega)$, and X is the largest such space. Hence it must be

$$L^n(\Omega) \hookrightarrow X,$$

showing (8.3). We shall verify that this embedding is strict.

Using [BS], Chapter 3, Exercise 5, we obtain a formula for the associate space of $(L^n(\Omega) + L_{n,1;-1/n'}(\Omega))$, namely

$$\big(L^n(\Omega) + L_{n,1;-1/n'}(\Omega)\big)' = \big(L^{n'}(\Omega) \cap L_{n',\infty;1/n'}(\Omega)\big).$$

Thus, it will suffice to find $f \in X$ and $g \in \Big(L^{n'}(\Omega) \cap L_{n',\infty;1/n'}(\Omega)\Big)$ such that $\int_0^1 f^* g^* = \infty$. A little calculation shows that such functions are, for example, those having rearrangements

$$f^*(t) = \sum_{k=1}^{\infty} a_k \chi_{I_k}(t), \qquad g^*(t) = \sum_{k=1}^{\infty} b_k \chi_{I_k}(t),$$

where I_k are the intervals $\big[\exp(-2^{k+1}), \exp(-2^k)\big)$, $k \in \mathbb{N}$, and the sequences $\{a_k\}$, $\{b_k\}$ are defined by

$$a_k = k^{-1} 2^{k/n'} \exp\Big(\frac{2^k}{n}\Big), \qquad b_k = 2^{-k/n'} \exp\Big(\frac{2^k}{n'}\Big).$$

There are several ways of showing that

$$X \hookrightarrow L_{n,n;-1/n'}(\Omega). \tag{8.5}$$

For example, one can use weighted estimates on Hardy-type operators restricted to monotone functions. We present a simple direct proof.

First,

$$\|f\|_X \approx \left\| \int_t^1 s^{-1/n'} f^*(s)\, ds \right\|_{L_{\infty,n;-1}(0,1)}.$$

It will be useful to observe that

$$\|f\|_X^n \approx \int_0^1 t^{-1} \left(\log(e/t)\right)^{-n} \left(\int_t^1 s^{-1/n'} f^*(s)\, ds \right)^n dt, \tag{8.6}$$

and

$$\begin{aligned} \|f\|_{n,n;-1/n'}^n &= \int_0^1 \left(\log(e/t)\right)^{1-n} f^*(t)^n \, dt \\ &\approx \int_0^1 t^{-1} \left(\log(e/t)\right)^{-n} \int_t^1 f^*(s)^n \, ds \, dt \end{aligned} \tag{8.7}$$

We claim that, for $t \in (0, 1/2)$,

$$\int_0^1 f^*(y)^n \min(t, y)\, dy \le C \int_0^t \left(\int_s^1 f^*(y) y^{-1/n'} \, dy \right)^n ds. \tag{8.8}$$

Indeed,

$$\begin{aligned} \int_0^t \left(\int_s^1 f^*(y) y^{-1/n'} \, dy \right)^n ds &\ge \int_0^t \left(\int_s^{2s} f^*(y) y^{-1/n'} \, dy \right)^n ds \\ &\ge c \int_0^t f^*(2s)^n s \, ds \ge c \int_0^t f^*(s)^n s \, ds, \end{aligned} \tag{8.9}$$

and also

$$\begin{aligned} \int_0^t \left(\int_s^1 f^*(y) y^{-1/n'} \, dy \right)^n ds &\ge \int_0^{t/2} \left(\int_s^1 f^*(y) y^{-1/n'} \, dy \right)^n ds \\ &\ge ct \left(\int_{t/2}^1 f^*(y) y^{-1/n'} \, dy \right)^n. \end{aligned} \tag{8.10}$$

Now, an argument similar to that of [Ca], Theorem 7, shows that

$$\left(\int_t^1 f^*(y)^n\,dy\right)^{1/n} \le C\int_{t/2}^1 f^*(y)y^{-1/n'}\,dy, \qquad t\in(0,1).$$

Combined with (8.10) and (8.9), this yields (8.8).

Now, (8.8) can be rewritten, using Fubini's theorem, as

$$\int_0^t\int_s^1 f^*(y)^n\,dy\,ds \le C\int_0^t\left(\int_s^1 f^*(y)y^{-1/n'}\,dy\right)^n ds.$$

Hence, by Hardy's lemma ([BS], Chapter 2, Proposition 3.6),

$$\begin{aligned}\int_0^1 t^{-1}\big(\log(e/t)\big)^{-n}&\int_t^1 f^*(s)^n\,ds\,dt\\ &\le C\int_0^1 t^{-1}\big(\log(e/t)\big)^{-n}\left(\int_t^1 f^*(y)y^{-1/n'}\,dy\right)^n dt,\end{aligned}\tag{8.11}$$

and (8.5) follows from (8.6), (8.7) and (8.11).

It is easy to show that

$$X\ne L_{n,n;-1/n'}(\Omega),$$

as this would contradict Theorem 4.1.

In order to prove the embedding

$$X\hookrightarrow L_{n,1;-\alpha/n'}(\Omega), \qquad \alpha>1, \tag{8.12}$$

we define

$$u(t)=\frac{1}{t}\big(\log(e/t)\big)^{-(\alpha/n')-1}, \qquad w(t)=\frac{1}{t}\big(\log(e/t)\big)^{-n}.$$

Note that

$$\|f\|_{n,1;-\alpha/n'}=\int_0^1 u(t)\int_t^1 f^*(y)y^{-1/n'}\,dy\,dt$$

and

$$\|f\|_X=\left(\int_0^1 w(t)\left(\int_t^1 f^*(y)y^{-1/n'}\,dy\right)^n dt\right)^{1/n},$$

hence (8.12) follows from the weighted embedding

$$\int_0^1 g(t)u(t)\,dt \le C\left(\int_0^1 g(t)^n w(t)\,dt\right)^{1/n}, \qquad g \in \mathfrak{M}_+(0,1),$$

which is (cf. [Ka], [Av]) equivalent to

$$\int_0^1 \left(\frac{u(t)}{w(t)}\right)^{n'} w(t)\,dt < \infty. \tag{8.13}$$

It is a matter of a simple calculation to verify (8.13).

To prove that

$$X \ne \bigcap_{\alpha>1} L_{n,1;-\alpha/n'}(\Omega),$$

it is enough to take any function

$$g \in \bigcap_{\alpha>1} L_{n,1;-\alpha/n'}(\Omega) \setminus L_{n,n;-1/n'}(\Omega)$$

(this set difference is not empty, cf. [EOP]).

This finishes the proof of (ii).

(iii) It is easy to verify that the function

$$g(t) = t^{-1/n}\big(\log(e/t)\big)^{-1/n}$$

satisfies

$$g \in L_{n,r;-1/n'}(\Omega) \setminus X, \qquad 1 < r < n.$$

Conversely, every function h_α, whose rearrangement is

$$h_\alpha^*(t) = \sum_{k=1}^{\infty} \chi_{[\exp(-2^{k+1}),\exp(-2^k))}(t) k^{-\alpha} 2^{k/n'} \exp\left(\frac{2^k}{n}\right),$$

satisfies

$$h_\alpha \in X \setminus L_{n,r;-1/n'}(\Omega) \qquad \text{when} \qquad \frac{1}{n} < \alpha < \frac{1}{r}.$$

(iv) By Theorem 3.2 (ii),

$$L_{n,n;-\alpha/n'} \hookrightarrow L_{n,r;-1/n'} \qquad \text{if} \quad 0<\alpha<1 \quad \text{and} \quad \frac{n'}{(n'-\alpha)} < r < n.$$

Therefore, we conclude that, for $\alpha \in (0,1)$,

$$X \not\hookrightarrow L_{n,n;-\alpha/n'}(\Omega),$$

as the embedding would imply $X \hookrightarrow L_{n,r;-1/n'}(\Omega)$ and thus contradict (iii). Conversely, assume that, for some $\alpha \in (0,1)$, the embedding

$$L_{n,n;-\alpha/n'}(\Omega) \hookrightarrow X$$

holds. Then (recall that $L_{n,n;-\alpha/n'}(\Omega)$ coincides with the Orlicz space $L_A(\Omega)$ where $A(t) \approx t^n(\log t)^{\alpha(1-n)}$) we get a contradiction with Theorem 4.1. □

Remark 8.3. It has been recently brought to our attention that spaces of type X appear naturally also in certain limiting interpolation problems (see [CPu], [Pu]).

9 The optimality of (1.3) in the context of Orlicz spaces—Part 2

Let us now return to the question of the optimality of (1.3) in the context of Orlicz spaces. Theorem 4.1 of A. Cianchi shows that, given a fixed Orlicz domain space, there always exists the optimal Orlicz range space. On the other hand, the situation described in Theorem 4.3 shows that for a given Orlicz *range* space, the optimal Orlicz domain space need not necessarily exist. Still, this situation is not universal: consider the simplest possible example of Orlicz range space, i.e. a Lebesgue space $L^q(\Omega)$, $n' < q < \infty$. Then it can be shown that the optimal Orlicz domain space is the Lebesgue space $L^r(\Omega)$ with

$$r = \frac{qn}{q+n} < q. \tag{9.1}$$

A natural question now occurs: what governs the difference between the case represented by the range space $\exp L^{n'}(\Omega)$ (for which there is no optimal Orlicz domain space) and the case represented by the range space $L^q(\Omega)$, $q \in (n', \infty)$ (for which the optimal Orlicz domain space is readily found)?

Certain insight into this problem is achieved when the "optimal fundamental function" is calculated and the corresponding Orlicz space is considered. We shall give the details below, but perhaps it might be helpful to the reader if we outline the idea first:

(i) start with a given Orlicz range space L_A;
(ii) find the corresponding optimal rearrangement-invariant domain space X;
(iii) calculate its fundamental function $\varphi = \varphi_X$;
(iv) find the (unique) Orlicz space whose fundamental function is equivalent to φ, denote this space by L_B;
(v) find out whether or not L_B is a candidate as a domain space for the Sobolev embedding into L_A;
(vi) if the answer to the question in (v) is affirmative, then L_B is the optimal Orlicz domain space for L_A in (1.5).

Now, when $L_A = L^q$, $q \in (n', \infty)$, then X is the Lorentz space $L_{r,q}$ with the r from (9.1), $\varphi(t) = t^{1/r}$, and therefore $L_B = L^r$, which is a good candidate for the Sobolev embedding into L^q. Thus, it is the optimal Orlicz domain space for such embedding. On the other hand, when $L_A = \exp L^{n'}$, then X is the space given by (8.1), the optimal fundamental function φ is by (8.2) equivalent to $t^{1/n}(\log(e/t))^{-1/n'}$, and $L_B = L_{n,n;-1/n'}$, which is the Orlicz space generated by the Young function B satisfying $B(t) \approx t^n\big(\log t\big)^{1-n}$ for large t. It follows from Theorem 4.1 that L_B is too big to be a candidate for the Sobolev embedding into $\exp L^{n'}$.

This relatively simple observation can be extended to a general principle which provides us with a sufficient condition for the existence of an optimal Orlicz domain space.

We shall first determine the optimal fundamental functions. The proof of the following proposition is an easy exercise.

Proposition 9.1. (i) *Let ϱ_R be a rearrangement-invariant norm on $\mathfrak{M}_+(0,1)$. In case $m = 1$ assume further that* (2.5) *holds. Let ϱ_D be given by* (5.2). *Then*

$$\varrho_D(\chi_{(0,a)}) \approx a\varrho_R\Big(\chi_{(a,1)}(t)t^{(m/n)-1}\Big), \qquad a \in (0, 1/2).$$

(ii) *Let ϱ_D be a rearrangement-invariant norm. Let σ be given by* (5.3) (*with $\varrho = \varrho_D$*) *and set $\varrho_R = \sigma'$. Then*

$$\varrho_R(\chi_{(0,a)}) \approx \frac{1}{\varrho'_D\Big(\chi_{(a,1)}(t)t^{(m/n)-1}\Big)}, \qquad a \in (0, 1/2).$$

Of course, the above-outlined procedure works as well for any given r.i. range space which is not necessarily an Orlicz space. Here is a general sufficient condition for the existence of the optimal Orlicz domain space when an arbitrary r.i. range space is given.

Theorem 9.2. *Let ϱ_R be a rearrangement-invariant norm on $\mathfrak{M}_+(0,1)$ such that (2.5) holds when $m=1$. Let ϱ_D be given by (5.2). Assume that there exists a quasiconcave function φ on $[0,1]$, satisfying*

$$\varphi(t) \approx t\varrho_R\Big(\chi_{(t,1)}(s)s^{(m/n)-1}\Big), \qquad t \in (0,1/2). \tag{9.2}$$

Let A be a Young function such that

$$A(t) \approx \frac{1}{\varphi^{-1}(1/t)}, \qquad t \in (1,\infty).$$

Assume that there is a $C>0$ such that, for every $u \in L_A(\Omega)$,

$$\varrho_D(u^*) \le C\|u\|_{L_A(\Omega)}. \tag{9.3}$$

Then for every $u \in C_0^1(\Omega)$

$$\varrho_R(u^*) \le C\|\nabla^m u\|_{L_A(\Omega)},$$

and $L_A(\Omega)$ is the optimal (largest) such Orlicz space.

Proof. This readily follows from the fact that there is only one Orlicz space per a fundamental function (cf. Remark 8.1). More precisely, suppose $L_B(\Omega)$ is another such space with $A \gg B$. Then, since ϱ_D is the optimal r.i. domain space, we get for every $u \in L_A(\Omega)$

$$\varrho_D(u^*) \le C\|u\|_{L_B(\Omega)} \le \|u\|_{L_A(\Omega)}.$$

By assumptions of the theorem, ϱ_D and $\|\cdot\|_A$ have the same fundamental function, equivalent to φ. Thus, necessarily, the fundamental function of $\|\cdot\|_B$ is also equivalent to φ, which contradicts $A \gg B$. □

For the case when ϱ_R is an Orlicz norm, we have the following result, which follows immediately from Theorem 9.2, (9.2) and Lemma 4.2.

Theorem 9.3. *Let B be a Young function. If $m=1$, we assume that the norm ϱ, given by $\varrho(f)=\|f\|_{L_B(0,1)}$, satisfies (2.5). Suppose that there is a quasiconcave function φ on $[0,1]$, satisfying*

$$\varphi(t) \approx \frac{t^{m/n}}{\psi^{-1}(t)}, \qquad t \in (0,1/2),$$

where ψ^{-1} is the inverse function of

$$\psi(t) = t^{n/(n-m)} \int_0^t \frac{B(s)}{s^{(n/(n-m))+1}}\, ds.$$

Let A be a Young function such that

$$A(t) \approx \frac{1}{\varphi^{-1}(1/t)}, \qquad t \in (1, \infty).$$

Assume that there is a $C > 0$ such that, for every $f \in L_A(0,1)$,

$$\left\| \int_t^1 (Pf^*)(s) s^{(m/n)-1}\, ds \right\|_{L_B(0,1)} \le C \|f\|_{L_A(0,1)}.$$

Then $L_A(\Omega)$ is the optimal (largest) Orlicz domain space such that

$$\|u\|_{L_B(\Omega)} \le C \|\nabla^m u\|_{L_A(\Omega)}.$$

Remark 9.4. It is not known to us whether or not the condition (9.3) is also necessary for the existence of the optimal Orlicz domain space.

10 The optimal domain spaces for Sobolev embeddings into L^∞, BMO and VMO.

We conclude with a brief survey of results from [CPi] on the optimality of domain space for a Sobolev embedding of order 1 when the range space is either BMO or VMO. The space BMO of functions having *bounded mean oscillation*, introduced by John and Nirenberg [JN], has proved to be particularly useful in various areas of analysis, especially harmonic analysis and interpolation theory. For example, it serves as an appropriate substitute for L^∞ when L^∞ does not work. In this section we establish necessary and sufficient conditions for the membership of a function to BMO, VMO, or to L^∞ in terms of the summability properties of its gradient.

Definition 10.1. Let Q be a cube in $\mathbb{R}^n$ such that $|Q| = 1$. The space BMO(Q) is the class of real-valued integrable functions on Q such that

$$\|f\|_{*,Q} = \sup_{Q' \subset Q} \frac{1}{|Q'|} \int_{Q'} |f(x) - f_{Q'}|\, dx < \infty,$$

where the supremum is extended over all subcubes Q' of Q, and $f_{Q'} = |Q'|^{-1} \int_{Q'} f$. Let us recall that BMO is not a Banach space, although it can be turned into one by introducing the norm

$$\|f\|_{\mathrm{BMO}(Q)} = \|f\|_{*,Q} + \|f\|_{L^1(Q)}.$$

We say that a function $f : Q \to \mathbb{R}$ belongs to VMO(Q), the space of functions with *vanishing mean oscillation*, if $\lim_{t\to 0_+} \theta_f(t) = 0$, where

$$\theta_f(t) = \sup_{|Q'|\le t} \frac{1}{|Q'|} \int_{Q'} |f(x) - f_{Q'}|\, dx, \qquad t \in (0,1).$$

Throughout this section, if $X = X(Q)$ is an r.i. space on $\mathfrak{M}_+(Q)$, then we denote by ϱ_X its representation r.i. norm on $(0,1)$, that is, $\|u\|_X := \varrho_X(u^*)$ for every $u \in \mathfrak{M}_+(Q)$ (see [BS] for details).

We start with recalling the result on $L^\infty(Q)$ (cf. Example 7.4 above), formulated in a slightly modified way.

Theorem 10.2. *Let $X(Q)$ be an r.i. space on $\mathfrak{M}_+(Q)$. Then the following statements are equivalent:*

(i) $\|u\|_{L^\infty(Q)} \le C\varrho_X(|\nabla u|^*), \quad u \in C_0^1(Q)$;
(ii) $\varrho_X'\big(t^{-1/n'}\chi_{(0,1)}(t)\big) < \infty$;
(iii) $X(Q) \hookrightarrow L_{n,1}(Q)$.

In other words, the space $L_{n,1}(Q)$ is the largest r.i. space $X(Q)$ that renders (i) *true.*

Main results of this section are the following two theorems. The techniques of proofs are based on the notion of *signed nonincreasing rearrangement*. Details can be found in [CPi].

Theorem 10.3. *Let $X(Q)$ be an r.i. space on $\mathfrak{M}_+(Q)$. Then the following statements are equivalent:*

(i) $\|u\|_{*,Q} \le C\varrho_X(|\nabla u|^*), \qquad u \in C_0^1(Q)$;
(ii) $\sup_{0<t<1} t^{-1}\varrho_X'\big(s^{1/n}\chi_{(0,t)}(s)\big) < \infty$;
(iii) $X(Q) \hookrightarrow L_{n,\infty}(Q)$.

In other words, the space $L_{n,\infty}(Q)$ is the largest r.i. space $X(Q)$ that renders (i) *true.*

Theorem 10.4. *Let $X(Q)$ be an r.i. space on $\mathfrak{M}_+(Q)$. Then the following statements are equivalent:*

(i) $\lim_{t\to 0_+} \sup_{\varrho_X(|\nabla u|^*)\le 1} \theta_u(t) = 0$;
(ii) $\lim_{t\to 0_+} t^{-1}\varrho_X'\big(s^{1/n}\chi_{(0,t)}(s)\big) = 0$;
(iii) $X(Q) \subset (L_{n,\infty})_a(Q)$, *where $(L_{n,\infty})_a(Q)$ is the set of all functions having absolutely continuous norm in $L_{n,\infty}(Q)$.*

In particular, for Orlicz spaces we get the following results:

Corollary 10.5. *Let A be a Young function.*

(i) *The embedding*

$$\|u\|_{*,Q} \leq C\|\nabla u\|_{L_A(Q)}, \qquad u \in C_0^1(Q), \tag{10.1}$$

holds if and only if there is a $C > 0$ such that for all large t

$$\int_0^t \widetilde{A}(s)\,ds \leq Ct^{n'+1}.$$

(ii) *The embedding*

$$\|u\|_{L^\infty(Q)} \leq C\|\nabla u\|_{L_A(Q)}, \qquad u \in C_0^1(Q), \tag{10.2}$$

holds if and only if

$$\int_1^\infty \widetilde{A}(s)s^{-n'-1}\,ds < \infty.$$

To conclude, let us investigate the existence of an optimal Orlicz domain space.

Theorem 10.6. (i) *The space $L_n(Q)$ is the largest Orlicz space $L_A(Q)$ such that* (10.1) *holds.*

(ii) *There does not exist any largest Orlicz space $L_A(Q)$ such that* (10.2) *holds.*

The assertion (i) recovers a result obtained earlier by Fiorenza in [F] by different means. It can be also shown by a method analogous to that of the proof of Theorem 9.2.

Acknowledgements: This research was partly supported by the grant no. GR/L79236 from the EPSRC, NATO Collaborative Research Grant OUTR.CRG 970071, Royal Society grant `rc/jp/nov`, and grants no. 201/94/1066 and 201/97/0744 of the Grant Agency of the Czech Republic.

The author wishes to thank B. Opic and A. Kufner for a careful reading of the text and for many valuable comments.

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