Interpolation theory and measures related to operator ideals

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Abstract. Given any operator ideal $\mathfrak{I}$, there are two natural functionals $\gamma_{\mathfrak{I}}(T)$, $\beta_{\mathfrak{I}}(T)$ that one can use to show the deviation of the operator T to the closed surjective hull of $\mathfrak{I}$ and to the closed injective hull of $\mathfrak{I}$, respectively. We describe the behaviour under interpolation of $\gamma_{\mathfrak{I}}$ and $\beta_{\mathfrak{I}}$. The results are part of joint works with A. Martínez, A. Manzano and P. Fernández-Martínez.

Often in analysis we are dealing with an operator that can be considered acting between several Banach spaces. This is, for example, the case of many integral operators that are studied simultaneously in the whole family of L_p-spaces (see [21] and [23]). For this reason it is important to have results which give relationships between properties of a given operator considered in two different spaces. Non-trivial examples of such results are the famous interpolation theorems of Riesz-Thorin (1926/1938) and Marcinkiewicz (1939). Let us recall the statement of Riesz-Thorin theorem.

Let (Ω_i, μ_i), $i = 0, 1$, be measure spaces with σ-finite positive measures μ_i, and let $L_p = L_p(\Omega_i, \mu_i)$ denote the space of all (equivalent classes of) μ_i-measurable functions f on Ω_i, such that

$$\|f\|_p = \left(\int_\Omega |f(x)|^p d\mu_i \right)^{1/p}$$

is finite.

Theorem (Riesz-Thorin theorem). *Assume that* $1 \leq p_i, q_i \leq \infty$ *for* $i = 0, 1$, *and let* T *be a linear operator which maps* $L_{p_i}(\Omega_0, \mu_0)$ *continuously into* $L_{q_i}(\Omega_1, \mu_1)$ *with norm* M_i. *If* $0 < \theta < 1$ *and* $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$, *then* T *maps* $L_p(\Omega_0, \mu_0)$ *continuously into* $L_q(\Omega_1, \mu_1)$ *with norm* $M \leq M_0^{1-\theta} M_1^{\theta}$.

This theorem shows that boundedness can be interpolated between L_p-spaces. In 1960, Krasnosel'skii [22] proved that compactness can be also interpolated. Namely, in the hypotheses of the Riesz-Thorin theorem, if

$T : L_{p_0}(\Omega_0, \mu_0) \longrightarrow L_{q_0}(\Omega_1, \mu_1)$ is not just bounded but also compact, then $T : L_p(\Omega_0, \mu_0) \longrightarrow L_q(\Omega_1, \mu_1)$ is compact too.

Recall that $T \in \mathfrak{L}(A, B)$ is said to be compact if T maps the unit ball of A into a relatively compact set in B.

The proof given by Krasnosel'skii in [22] requires also the assumption $q_0 < \infty$, but this condition is not essential for the result. It can be eliminated by given different arguments.

At the beginning of the sixties, Lions, Peetre, Calderón, Gagliardo, Krein and other authors investigated the validity of these results for general couples of Banach spaces. Two main interpolation methods were developed, the complex method (based on ideas involved in the proof of the Riesz-Thorin theorem) and the real method (connected with the Marcinkiewicz theorem).

In the following years the contributions of these and other authors turned ideas and techniques related to these questions into a new field of study in functional analysis, which is now called interpolation theory and that has found important applications in harmonic analysis, partial differential equations and approximation theory, among other branches of analysis (see the books by Bergh and Löfström [5] and by Triebel [35]).

Let me recall the construction of the real interpolation method.

Let $\overline{A} = (A_0, A_1)$ be a Banach couple, that is, two Banach spaces continuously embedded in a Hausdorff topological vector space $\mathcal{A}$. Then we can form their sum $A_0 + A_1 = \{a \in \mathcal{A} : a = a_0 + a_1,\ a_i \in A_i\}$ and their intersection $A_0 \cap A_1 = \{a \in \mathcal{A} : a \in A_0 \text{ and } a \in A_1\}$. These spaces become Banach spaces when endowed with their natural norms

$$\|a\|_{A_0+A_1} = \inf\{\|a_0\|_{A_0} + \|a_1\|_{A_1} : a = a_0 + a_1,\ a_i \in A_i\},$$
$$\|a\|_{A_0 \cap A_1} = \max\{\|a\|_{A_0}, \|a\|_{A_1}\}.$$

The real interpolation method will allow us to construct intermediate spaces between A_0 and A_1. In other words, spaces that contain continuously $A_0 \cap A_1$ and that are continuously embedded in $A_0 + A_1$. For this aim, we first modify the norm of $A_0 + A_1$ by inserting a scalar parameter $t > 0$. Put

$$K(t, a) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1,\ a_i \in A_i\}.$$

Then, given any $0 < \theta < 1$ and $1 \leq q \leq \infty$, we define $\overline{A}_{\theta,q} = (A_0, A_1)_{\theta,q}$ as the collection of all elements $a \in A_0 + A_1$ having a finite norm

$$\|a\|_{\theta,q} = \begin{cases} \left(\displaystyle\int_0^\infty \left(t^{-\theta} K(t,a)\right)^q \frac{dt}{t} \right)^{1/q} & \text{if } q < \infty \\ \displaystyle\sup_{t>0} \left\{t^{-\theta} K(t,a)\right\} & \text{if } q = \infty. \end{cases}$$

The space $(A_0, A_1)_{\theta,q}$ is called the real interpolation space. It is an intermediate space between A_0 and A_1 and has the following interpolation property for bounded operators:

If (B_0, B_1) is another Banach couple and T is a linear operator from $A_0 + A_1$ into $B_0 + B_1$ whose restrictions $T : A_i \longrightarrow B_i$ are bounded with norm M_i $(i = 0, 1)$, then the restriction of T to $(A_0, A_1)_{\theta,q}$ is a bounded operator $T : (A_0, A_1)_{\theta,q} \longrightarrow (B_0, B_1)_{\theta,q}$ with norm $M \leq M_0^{1-\theta} M_1^{\theta}$.

To make clear the relationship between this interpolation method and our starting point, consider the couple (L_1, L_∞). It turns out that

$$K(t, f) = \int_0^t f^*(s)\, ds, \qquad t > 0$$

where f^* is the non-increasing rearrangement of f on $(0, \infty)$ defined by

$$f^*(t) = \inf\{\delta > 0 : \mu(\{x : |f(x)| > \delta\}) \leq t\}.$$

Hence,

$$(L_1, L_\infty)_{\theta,p} = L_p \quad \text{if} \quad \frac{1}{p} = 1 - \theta,$$

with equivalence of norms. In a more general way, one can check that

$$(L_{p_0}, L_{p_1})_{\theta,p} = L_p \quad \text{if} \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

and if we interpolate with the second parameter q different from p and $p_0 \neq p_1$, then

$$(L_{p_0}, L_{p_1})_{\theta,q} = L_{p,q}.$$

Here $L_{p,q}$ stands for the Lorentz function space defined as the collection of all (equivalent classes of) μ-measurable functions f such that the norm

$$\|f\|_{p,q} = \begin{cases} \left(\int_0^\infty \left(t^{1/p} \int_0^t f^*(s)\, ds \right)^q \frac{dt}{t} \right)^{1/q} & (1 \leq q < \infty) \\ \sup\limits_{0<t<\infty} \left\{ t^{1/p} \int_0^t f^*(s)\, ds \right\} & (q = \infty) \end{cases}$$

is finite. The space $L_{p,\infty}$ is often called the weak L_p-space.

From the early 1960s a number of authors have investigated whether or not Krasnosel'skii's theorem can be extended to abstract couples of Banach spaces. In fact, a question that can be now considered as classical in interpolation theory is to determine if the interpolated operator $T : (A_0, A_1)_{\theta,q} \longrightarrow (B_0, B_1)_{\theta,q}$ inherits a certain property that $T : A_0 \longrightarrow B_0$ has. My talks are devoted to this question. We shall show some properties that pass to the interpolated operators, others that do not pass, paying special attention to quantitative results.

Concerning interpolation of compactness, the first abstract result is due to Lions and Peetre [26] in 1964:

Theorem (Lions-Peetre theorem). *Let B be a Banach space, let $\overline{A} = (A_0, A_1)$ be a Banach couple, $0 < \theta < 1$, $1 \leq q \leq \infty$ and let T be a linear operator.*

a) *Assume that*

$$\begin{aligned} T : A_0 \longrightarrow B \quad & \textit{compactly}, \\ T : A_1 \longrightarrow B \quad & \textit{boundedly}. \end{aligned}$$

Then $T : (A_0, A_1)_{\theta,q} \longrightarrow B$ is compact.

b) *Assume instead that*

$$\begin{aligned} T : B \longrightarrow A_0 \quad & \textit{compactly}, \\ T : B \longrightarrow A_1 \quad & \textit{boundedly}. \end{aligned}$$

Then $T : B \longrightarrow (A_0, A_1)_{\theta,q}$ is compact.

Note that although the rank of q is $[1, \infty]$, in order to establish the theorem, it suffices to prove a) when $q = \infty$ and b) when $q = 1$. The reason is that for any $1 \leq q \leq \infty$ the following continuous inclusions hold:

$$(A_0, A_1)_{\theta,1} \hookrightarrow (A_0, A_1)_{\theta,q} \hookrightarrow (A_0, A_1)_{\theta,\infty}. \tag{1}$$

A Banach space X intermediate with respect to the couple $\overline{A}$ is said to be of class $\mathcal{C}_K(\theta;\overline{A})$ [resp. $\mathcal{C}_J(\theta;\overline{A})$] if $X \hookrightarrow (A_0, A_1)_{\theta,\infty}$ [resp. $(A_0, A_1)_{\theta,1} \hookrightarrow X$]. It is clear that if the assumptions of a) holds and X is of class $\mathcal{C}_K(\theta;\overline{A})$, then $T : X \longrightarrow B$ is compact. Similarly, under the assumptions of b), if X is of class $\mathcal{C}_J(\theta;\overline{A})$, then $T : B \longrightarrow X$ is compact.

The Lions-Peetre theorem refers to special cases when one of the couples reduces to a single Banach space. However, it is the main tool for proving all known compactness results in interpolation theory.

A quantitative version of Lions-Peetre theorem in terms of entropy numbers can be found in the book by Pietsch [32] (previous results in this direction were obtained by Peetre and by Triebel (see [35]). An analogue of the Lions-Peetre theorem in terms of the measure of non-compactness was established by Edmunds and Teixeira in [34].

It has been also investigated if similar results hold for other properties of the operator T. Let me recall one of these results that refers to properties that generalize the concept of compactness.

An operator $T \in \mathcal{L}(A, B)$ between Banach spaces A and B is said to be strictly singular if the restriction of T to any infinite-dimensional (closed) subspace of A is not an isomorphism into B. The operator T is called strictly cosingular if there is no infinite-codimensional (closed) subspace $F \subset B$, such that $\Phi_F T$ is surjective. Here Φ_F stands for the quotient mapping from B onto B/F.

Every compact operator is strictly singular and strictly cosingular. Moreover, the identity map $I : \ell_1 \longrightarrow \ell_2$ is strictly singular but it is not compact, while $I : c_0 \longrightarrow \ell_\infty$ is strictly cosingular failing also to be compact. Note that $I : c_0 \longrightarrow \ell_\infty$ is not strictly singular either. On the other hand, every surjection from ℓ_1 onto ℓ_2 is strictly singular but it is not cosingular.

The following duality relationship holds between singularity and cosingularity (see [31]): If the dual operator $T^* : B^* \longrightarrow A^*$ is strictly singular (resp. cosingular), then $T : A \longrightarrow B$ is strictly cosingular (resp. singular).

These operators have the following interpolation properties.

Theorem 1. *Let B be a Banach space, let $\overline{A} = (A_0, A_1)$ be a Banach couple, $0 < \theta < 1$, $1 \leq q \leq \infty$ and let T be a linear operator.*

a) *If $T : A_0 \longrightarrow B$ is strictly cosingular and $T : A_1 \longrightarrow B$ is bounded, then $T : (A_0, A_1)_{\theta,q} \longrightarrow B$ is strictly cosingular.*
b) *If $T : B \longrightarrow A_0$ is strictly singular and $T : B \longrightarrow A_1$ is bounded, then $T : B \longrightarrow (A_0, A_1)_{\theta,q}$ is strictly singular.*

Beucher gave in [6] examples showing that strict singularity [resp. cosingularity] cannot be interpolated in the situation a) [resp. b)]. Next we recall one of them.

Example 1. Let $\overline{A} = (L_\infty[0,1], L_1[0,1])$, $B = L_1[0,1]$ and choose T as the identity mapping. It can be checked that $I : L_\infty[0,1] \longrightarrow L_1[0,1]$ is strictly singular, and obviously $I : L_1[0,1] \longrightarrow L_1[0,1]$ is bounded. However, if $1/p = \theta$,

$$I : L_p[0,1] = (L_\infty[0,1], L_1[0,1])_{\theta,p} \longrightarrow L_1[0,1]$$

fails to be strictly singular. The reason is that, according to Khintchine's inequality, the span of Rademacher functions in $L_p[0,1]$ and $L_1[0,1]$ is isomorphic to ℓ_2, so if we denote by E the span of Rademacher functions in $L_p[0,1]$, $I_{|E}$ is an isomorphism into $L_1[0,1]$.

Example 2. In Beucher's example $A_0 + A_1 = L_1[0,1] = B$ and T is equal to the natural embedding J from $\overline{A}_{\theta,p}$ into $A_0 + A_1$.

For any Banach couple $\overline{A} = (A_0, A_1)$, it is possible to give a necessary condition for $J : \overline{A}_{\theta,p} \longrightarrow A_0 + A_1$ to be strictly singular. Namely, any infinite dimensional closed subspace E of $\overline{A}_{\theta,p}$ should contain a subspace isomorphic to ℓ_p.

Indeed, since $J_{|E}$ cannot be an isomorphism, E is not closed in $A_0 + A_1$. Then, according to a result due to Levy [25], E contains a subspace isomorphic to ℓ_p.

Let us investigate the common part of these two theorems. Recall that a class $\mathcal{I}$ of bounded linear operators is said to be an operator ideal if each component $\mathcal{I} \cap \mathcal{L}(A,B) = \mathcal{I}(A,B)$ is a linear subspace of $\mathcal{L}(A,B)$ that contains the finite rank operators and satisfies that $STR \in \mathcal{I}(E,F)$ whenever $R \in \mathcal{L}(E,A)$, $T \in \mathcal{I}(A,B)$ and $S \in \mathcal{L}(B,F)$. The ideal $\mathcal{I}$ is called closed if each component $\mathcal{I}(A,B)$ is closed in $\mathcal{L}(A,B)$.

An ideal $\mathcal{I}$ is called injective if for every isomorphic embedding $J \in \mathcal{L}(B,F)$ and every operator $T \in \mathcal{L}(A,B)$ it follows from $JT \in \mathcal{I}(A,F)$ that $T \in \mathcal{I}(A,B)$. Injectivity means that it does not depend on the size of the target space B whether or not $T \in \mathcal{L}(A,B)$ belongs to $\mathcal{I}$.

The ideal $\mathcal{I}$ is said to be surjective if for every surjection $Q \in \mathcal{L}(E,A)$ and every operator $T \in \mathcal{L}(A,B)$ it follows from $TQ \in \mathcal{I}(E,B)$ that $T \in \mathcal{I}(A,B)$. If $\mathcal{I}$ is surjective then it does not depend on the size of the source space A whether or not $T \in \mathcal{L}(A,B)$ belongs to $\mathcal{I}$.

Compact operators $\mathcal{K}$ is an example of a closed injective and surjective operator ideal. Strictly singular operators $\mathcal{S}$ is an ideal which is closed and injective but it is not surjective, while the ideal $\mathcal{C}$ of strictly cosingular operators is closed and surjective but it is not injective (see [32]).

One may wonder if these properties have any role in the results. In fact, as Heinrich showed in [19], they are sufficient for the conclusion:

Theorem 2. *Let B be a Banach space, let $\overline{A} = (A_0, A_1)$ be a Banach couple, $0 < \theta < 1$, $1 \leq q \leq \infty$ and let T be a linear operator.*

a) *If $\mathcal{I}$ is a surjective closed operator ideal, $T \in \mathcal{I}(A_0, B)$ and $T \in \mathcal{L}(A_1, B)$, then $T \in \mathcal{I}(\overline{A}_{\theta,q}, B)$.*
b) *If $\mathcal{I}$ is an injective closed operator ideal, $T \in \mathcal{I}(B, A_0)$ and $T \in \mathcal{L}(B, A_1)$, then $T \in \mathcal{I}(B, \overline{A}_{\theta,q})$.*

In order to derive a quantitative version of this result, take any operator ideal $\mathcal{I}$ and denote by $\overline{\mathcal{I}}^s$ its closed surjective hull, that is, the smallest closed surjective operator ideal containing $\mathcal{I}$. Put $\overline{\mathcal{I}}^i$ for its closed injective hull.

These ideals can be characterized as follows (see [20]):

i) Let $T \in \mathfrak{L}(A, B)$. The operator T belongs to $\overline{\mathfrak{I}}^s(A, B)$ if and only if for every $\varepsilon > 0$ there is a Banach space E and an operator $R \in \mathfrak{I}(E, B)$ such that

$$T(U_A) \subseteq R(U_E) + \varepsilon U_B.$$

Here U_A stands for the closed unit ball of A.

ii) Let $T \in \mathfrak{L}(A, B)$. The operator T belongs to $\overline{\mathfrak{I}}^i(A, B)$ if and only if for every $\varepsilon > 0$ there is a Banach space F and an operator $R \in \mathfrak{I}(A, F)$ such that

$$\|Tx\|_B \leq \|Rx\|_F + \varepsilon \|x\|_A \,, \qquad x \in A.$$

In the light of these characterizations, it is natural to associate to $\mathfrak{I}$ the following functionals

$$\begin{aligned} \gamma_{\mathfrak{I}}(T) = \gamma_{\mathfrak{I}}(T_{A,B}) &= \inf\{\sigma > 0 \colon T(U_A) \subseteq \sigma U_B + R(U_E), \\ &\qquad R \in \mathfrak{I}(E, B),\ E \text{ any Banach space}\}, \\ \beta_{\mathfrak{I}}(T) = \beta_{\mathfrak{I}}(T_{A,B}) &= \inf\{\sigma > 0 \colon \text{there is a Banach space } F \text{ and } R \in \mathfrak{I}(A, F) \\ &\qquad \text{such that } \|Tx\|_B \leq \sigma\|x\|_A + \|Rx\|_F,\ x \in A\}. \end{aligned}$$

It is clear that

$$\gamma_{\mathfrak{I}}(T) = 0 \text{ if and only if } T \in \overline{\mathfrak{I}}^s,$$

$$\beta_{\mathfrak{I}}(T) = 0 \text{ if and only if } T \in \overline{\mathfrak{I}}^i.$$

So, $\gamma_{\mathfrak{I}}(T)$ shows the deviation of T from the ideal $\overline{\mathfrak{I}}^s$, while $\beta_{\mathfrak{I}}(T)$ gives the deviation from $\overline{\mathfrak{I}}^i$.

The (outer) measure $\gamma_{\mathfrak{I}}$ was introduced by Astala [2], and the (inner) measure $\beta_{\mathfrak{I}}$ by Tylli [36]. These measures satisfy that

$$\max\left\{\gamma_{\mathfrak{I}}(T), \beta_{\mathfrak{I}}(T)\right\} \leq \|T\|,$$

they are subadditive

$$\gamma_{\mathfrak{I}}(S+T) \leq \gamma_{\mathfrak{I}}(S) + \gamma_{\mathfrak{I}}(T), \quad \beta_{\mathfrak{I}}(S+T) \leq \beta_{\mathfrak{I}}(S) + \beta_{\mathfrak{I}}(T)$$

and submultiplicative

$$\gamma_{\mathcal{J}}(ST) \leq \gamma_{\mathcal{J}}(S)\gamma_{\mathcal{J}}(T), \quad \beta_{\mathcal{J}}(ST) \leq \beta_{\mathcal{J}}(S)\beta_{\mathcal{J}}(T).$$

Moreover, if I_A is the identity mapping of A, then

$$\gamma_{\mathcal{J}}(I_A) = \begin{cases} 0 & \text{if} \quad I_A \in \overline{\mathcal{J}}^s \\ 1 & \text{if} \quad I_A \notin \overline{\mathcal{J}}^s \end{cases}, \quad \beta_{\mathcal{J}}(I_A) = \begin{cases} 0 & \text{if} \quad I_A \in \overline{\mathcal{J}}^i \\ 1 & \text{if} \quad I_A \notin \overline{\mathcal{J}}^i \end{cases}.$$

Indeed, if $\gamma_{\mathcal{J}}(I_A) = \lambda > 0$, it is clear that $\lambda \leq \|I_A\| = 1$. On the other hand, using the submultiplicativity, we get

$$\lambda = \gamma_{\mathcal{J}}(I_A) = \gamma_{\mathcal{J}}(I_A I_A) \leq \gamma_{\mathcal{J}}(I_A)\gamma_{\mathcal{J}}(I_A) = \lambda^2.$$

So $\lambda \geq 1$ and therefore $\lambda = 1$. The argument for $\beta_{\mathcal{J}}(I_A)$ is the same.

If the ideal $\overline{\mathcal{J}}^i$ is symmetric, that is, if the dual operator T^* belongs to $\overline{\mathcal{J}}^i(B^*, A^*)$ whenever $T \in \overline{\mathcal{J}}^i(A, B)$, then (see [18], Prop. 1.2)

$$\beta_{\mathcal{J}}(T) = \gamma_{\overline{\mathcal{J}}^i}(T^*).$$

Let us look at some concrete cases.

Example 3. Take $\mathcal{J} = \mathcal{K}$ the ideal of compact operators. It is easy to check that $\gamma_{\mathcal{K}}(T)$ is equal to the (ball) measure of non-compactness of T, i.e.,

$$\gamma_{\mathcal{K}}(T) = \inf \Big\{ \sigma > 0 : \text{ there exist a finite number of elements } b_1, b_2, \ldots, b_k \in B \text{ such that } T(U_A) \subseteq \bigcup_{j=1}^{k} \{b_j + \sigma U_B\} \Big\}.$$

We have

$$\gamma_{\mathcal{K}}(I_A) = 0 \quad \text{if } A \text{ is finite dimensional,}$$
$$\gamma_{\mathcal{K}}(I_A) = 1 \quad \text{if } A \text{ is infinite dimensional.}$$

As we said before, $\mathcal{K} = \overline{\mathcal{K}}^i = \overline{\mathcal{K}}^s$. The ideal $\mathcal{K}$ is also symmetric, so $\beta_{\mathcal{K}}(T) = \gamma_{\mathcal{K}}(T^*)$. It turns out that $\beta_{\mathcal{K}}(T) = \lim_{n\to\infty} c_n(T)$, where $(c_n(T))$ is the sequence of the Gelfand numbers of T, i.e., $\beta_{\mathcal{K}}(T)$ coincides with the infimum of all $\eta > 0$ such that there is a subspace M of A with finite codimension, such that $\|Tx\|_B \leq \eta\|x\|_A$, $x \in M$. Under this form, the measure $\beta_{\mathcal{K}}$ was already considered by Lebow and Schechter in [24], where they establish that $\gamma_{\mathcal{K}}$ and $\beta_{\mathcal{K}}$ are equivalent:

$$\frac{1}{2}\gamma_{\mathcal{K}}(T) \leq \beta_{\mathcal{K}}(T) \leq 2\gamma_{\mathcal{K}}(T).$$

Example 4. Let $\mathcal{I} = \mathcal{W}$ the ideal of weakly compact operators. Recall that $T \in \mathcal{L}(A, B)$ is weakly compact if T carries the closed unit ball of A onto a relatively weakly compact subset of B, or equivalently, for every norm bounded sequence (a_n) of A, (Ta_n) has a weakly convergent subsequence in B. The ideal $\mathcal{W}$ is symmetric, injective, surjective, and closed.

The measure $\gamma_{\mathcal{W}}(T)$ is equal to the measure of weak non-compactness introduced by De Blasi [17]:

$$\begin{aligned}\gamma_{\mathcal{W}}(T) = \inf\{\sigma > 0 : {} & \text{there is a weakly compact set } W \text{ in } B \\ & \text{such that } T(U_A) \subseteq W + \sigma U_B\}.\end{aligned}$$

We have

$$\begin{aligned}\gamma_{\mathcal{W}}(I_A) &= 0 \quad \text{if } A \text{ is reflexive,} \\ \gamma_{\mathcal{W}}(I_A) &= 1 \quad \text{if } A \text{ is nonreflexive.}\end{aligned}$$

Again $\beta_{\mathcal{W}}(T) = \gamma_{\mathcal{W}}(T^*)$ but this time $\gamma_{\mathcal{W}}$ and $\beta_{\mathcal{W}}$ are not equivalent. Namely, there is a Banach space E and a sequence of operators $(T_n) \subseteq \mathcal{L}(E, c_0)$ such that $\beta_{\mathcal{W}}(T_n) = 1$ and $\gamma_{\mathcal{W}}(T_n) \leq 1/n$ (see [3]).

Example 5. Since strictly singular operators $\mathcal{S}$ form a closed injective ideal, we can use $\beta_{\mathcal{S}}(T)$ as a measure of the deviation of T to $\mathcal{S}$. The relevant functional in the case of strictly cosingular operators $\mathcal{C}$ is $\gamma_{\mathcal{C}}$ because $\mathcal{C}$ is a closed surjective ideal.

Example 6. Our last example refers to Rosenthal operators $\mathcal{R}$. Recall that $T \in \mathcal{L}(A, B)$ is said to be a Rosenthal operator if for every bounded sequence $(a_n) \subseteq A$, the sequence (Ta_n) admits a weak Cauchy subsequence. According to Rosenthal's theorem [33], the former condition is equivalent to the fact that no subspace of $T(A)$ is isomorphic to ℓ_1. The identity map I_{c_0} of c_0 is an example of Rosenthal operator; in particular $\mathcal{R} \neq \mathcal{W}$.

The ideal $\mathcal{R}$ is injective, surjective, and closed. Moreover (see [37])

$$\mathcal{R} = \mathcal{S}^s = \mathcal{C}^i.$$

We have now $\gamma_{\mathcal{R}}(I_A) = 0$ if and only if A does not contain any subspace isomorphic to ℓ_1. The same condition is valid for $\beta_{\mathcal{R}}(I_A)$.

The following result is taken from a joint paper with A. Manzano and A. Martínez [11] and refers to interpolation properties of $\gamma_{\mathcal{I}}$ and $\beta_{\mathcal{I}}$ in the Lions-Peetre situation.

Theorem 3. *Let $\mathcal{I}$ be an operator ideal. Assume that B is a Banach space, $\overline{A} = (A_0, A_1)$ is a Banach couple, $0 < \theta < 1$ and $1 \leq q \leq \infty$.*

a) *If $T \in \mathcal{L}(A_0 + A_1, B)$, then*

$$\gamma_{\mathcal{I}}(T_{\overline{A}_{\theta,q},B}) \leq C_{\theta,q}\gamma_{\mathcal{I}}(T_{A_0,B})^{1-\theta}\gamma_{\mathcal{I}}(T_{A_1,B})^{\theta}.$$

b) *If $T \in \mathcal{L}(B, A_0 \cap A_1)$, then*

$$\beta_{\mathcal{I}}(T_{B,\overline{A}_{\theta,q}}) \leq C_{\theta,q}\beta_{\mathcal{I}}(T_{A,B_0})^{1-\theta}\beta_{\mathcal{I}}(T_{A,B_1})^{\theta}.$$

Proof. We only give the details of a). The proof of b) can be found in [11].

It is enough to consider the case $q = \infty$ because as we already pointed out $\overline{A}_{\theta,q} \hookrightarrow \overline{A}_{\theta,\infty}$, being the norm of the embedding less than or equal to $(q\theta(1-\theta))^{1/q}$.

Since $\|a\|_{\theta,\infty} = \sup_{0<t<\infty}\{t^{-\theta}K(t,a)\}$, given any $\varepsilon > 0$, $t > 0$ and $a \in U_{\overline{A}_{\theta,\infty}}$, we can find a decomposition $a = a_0 + a_1$, with $a_i \in A_i$ and $\|a_i\|_{A_i} \leq (1+\varepsilon)t^{\theta-i}$ $(i = 0, 1)$. Thus

$$U_{\overline{A}_{\theta,\infty}} \subseteq (1+\varepsilon)t^{\theta}U_{A_0} + (1+\varepsilon)t^{\theta-1}U_{A_1}.$$

Take now any $\sigma_i > \gamma_{\mathcal{I}}(T_{A_i,B})$. By the definition of $\gamma_{\mathcal{I}}$, there is a Banach space E_i and an operator $S_i \in \mathcal{I}(E_i, B)$ such that

$$T(U_{A_i}) \subseteq \sigma_i U_B + S_i(U_{E_i}) \qquad i = 0, 1.$$

Hence

$$\begin{aligned} T(U_{\overline{A}_{\theta,\infty}}) &\subseteq (1+\varepsilon)t^{\theta}\sigma_0 U_B + (1+\varepsilon)t^{\theta}S_0(U_{E_0}) \\ &\quad + (1+\varepsilon)t^{\theta-1}\sigma_1 U_B + (1+\varepsilon)t^{\theta-1}S_1(U_{E_1}) \\ &\subseteq (1+\varepsilon)(t^{\theta}\sigma_0 + t^{\theta-1}\sigma_1)U_B + S_{t,\varepsilon}(U_E), \end{aligned}$$

where $E = (E_0 \oplus E_1)_{\ell_\infty}$, i.e., $E = \{(x,y) : x \in E_0,\ y \in E_1\}$ normed by $\|(x,y)\|_E = \max\{\|x\|_{E_0}, \|y\|_{E_1}\}$ and $S_{t,\varepsilon}(x,y) = (1+\varepsilon)t^{\theta}S_0 x +$

$(1+\varepsilon)t^{\theta-1}S_1 y$. Since $\mathcal{I}$ is an operator ideal, it is clear that $S_{t,\varepsilon} \in \mathcal{I}(E,B)$. Whence

$$\begin{aligned}\gamma_{\mathcal{I}}(T_{\overline{A}_{\theta,\infty},B}) &\le \inf_{t>0}\left\{t^{\theta}\gamma_{\mathcal{I}}(T_{A_0,B}) + t^{\theta-1}\gamma_{\mathcal{I}}(T_{A_1,B})\right\}\\ &= (1-\theta)^{\theta-1}\theta^{-\theta}\gamma_{\mathcal{I}}(T_{A_0,B})^{1-\theta}\gamma_{\mathcal{I}}(T_{A_1,B})^{\theta}.\end{aligned}$$

□

This theorem gives a quantitative version of Theorem 2 (Heinrich's result), so it comprises Lions-Peetre theorem as well as the result on strict singularity and cosingularity. Writing down the estimate for $\gamma_{\mathcal{I}}$ when $\mathcal{I} = \mathcal{K}$, we get a result of Edmunds and Teixeira [34], Thm. 1/b on the measure of non-compactness. The choice $\mathcal{I} = \mathcal{W}$ allows us do recover an estimate of Aksoy and Maligranda [1], Thm. 2/a on the measure of weak non-compactness.

Example 1 shows that it is not possible to change the role of $\gamma_{\mathcal{I}}$ and $\beta_{\mathcal{I}}$ in the Theorem 3. The example also announces that in the general case of two Banach couples, estimates of similar type are only valid under extra assumptions on the ideal $\mathcal{I}$.

In order to describe the result in the general case, we first introduce some notation.

Let $\ell_1(U_A)$ be the Banach space of all absolutely summable families of scalars (λ_a) indexed by the elements a of U_A. We write $Q_A : \ell_1(U_A) \longrightarrow A$ for the operator defined by $Q_A(\lambda_a) = \sum_{a\in U_A} \lambda_a a$. The operator Q_A is a metric surjection, i.e., $Q_A(\mathring{U}_{\ell_1(U_A)}) = \mathring{U}_A$.

We denote by $J_B : B \longrightarrow \ell_\infty(U_{B^*})$ the isometric embedding $J_B b = (\langle f, b\rangle)_{f\in U_{B^*}}$. Here $\ell_\infty(U_{B^*})$ stands for the Banach space of all bounded families of scalars indexed by the elements of U_{B^*}.

For any $T \in \mathcal{L}(A,B)$ we have

$$\begin{aligned}\gamma_{\mathcal{I}}(J_B T) &\le \|J_B\|\gamma_{\mathcal{I}}(T) \le \gamma_{\mathcal{I}}(T),\\ \beta_{\mathcal{I}}(TQ_A) &\le \beta_{\mathcal{I}}(T)\|Q_A\| \le \beta_{\mathcal{I}}(T).\end{aligned}$$

Moreover, using the extension property of $\ell_\infty(U_{B^*})$ and the lifting property of $\ell_1(U_A)$, one can check that operators J_B and Q_A have the following minimal property (see [2] and [11]):

$$\begin{aligned}\gamma_{\mathcal{I}}(J_B T) &= \min\{\gamma_{\mathcal{I}}(jT) : j : B \longrightarrow F \text{ isometric embedding}\},\\ \beta_{\mathcal{I}}(TQ_A) &= \min\{\beta_{\mathcal{I}}(T\pi) : \pi : E \longrightarrow A \text{ metric surjection}\}.\end{aligned}$$

Given a sequence of Banach spaces (E_m), we denote by $\ell_q(E_m)$ the vector valued ℓ_q-space defined by

$$\ell_q(E_m) = \Big\{x = (x_m) : x_m \in E_m \\ \text{and } \|x\|_{\ell_q(E_m)} = \Big(\sum_{m=-\infty}^{\infty} \|x_m\|_{E_m}^q\Big)^{1/q} < \infty\Big\}.$$

Any operator $T \in \mathfrak{I}\,(\ell_q(E_m), \ell_q(F_m))$ can be imagined as an infinite matrix whose elements are $Q_k T P_n$, where $P_n : E_n \longrightarrow \ell_q(E_m)$ is the embedding given by $P_n x = (\delta_m^n x)$, here δ_m^n is the Kronecker delta, and $Q_k : \ell_q(F_m) \longrightarrow F_k$ is the projection given by $Q_k(y) = y_k$. Note that $Q_k T P_n \in \mathfrak{L}(E_n, F_k)$.

Let $1 < q < \infty$. We say that an operator ideal $\mathfrak{I}$ satisfies the Σ_q-condition if for any sequence of Banach spaces (E_m), (F_m) and any $T \in \mathfrak{L}\,(\ell_q(E_m), \ell_q(F_m))$, it follows from $Q_k T P_n \in \mathfrak{I}(E_n, F_k)$ for any n, k that $T \in \mathfrak{I}\,(\ell_q(E_m), \ell_q(F_m))$.

This condition was introduced by Heinrich in [19], where he also showed that weakly compact operators or Rosenthal operators satisfy it. Another examples that can be also found in [19] are Banach-Saks operators and dual Radon-Nikodym operators. These ideals are injective, surjective and closed, as well.

Note that a necessary condition for $\mathfrak{I}$ to satisfy the Σ_q-condition is that $I_{\ell_q} \in \mathfrak{I}$. Hence compact operators, strictly singular operators or strictly cosingular operators fails the Σ_q-condition.

Theorem 4. *Let $1 < q < \infty$, $0 < \theta < 1$ and let $\mathfrak{I}$ be an operator ideal satisfying the Σ_q-condition. Assume that $\overline{A} = (A_0, A_1)$, $\overline{B} = (B_0, B_1)$ are Banach couples and let $T : A_0 + A_1 \longrightarrow B_0 + B_1$ be a linear operator such that its restrictions $T : A_i \longrightarrow B_i$ are bounded for $i = 0, 1$. Then*

$$\text{a)} \quad \gamma_{\mathfrak{I}}\left(\Big[J_{\overline{B}_{\theta,q}} T\Big]_{\overline{A}_{\theta,q},\ell_\infty(U_{\overline{B}^*_{\theta,q}})}\right) \le C \gamma_{\mathfrak{I}}(T_{A_0,B_0})^{1-\theta} \gamma_{\mathfrak{I}}(T_{A_1,B_1})^{\theta}$$

$$\text{b)} \quad \beta_{\mathfrak{I}}\left(\Big[T Q_{\overline{A}_{\theta,q}}\Big]_{\ell_1(U_{\overline{A}_{\theta,q}}),\overline{B}_{\theta,q}}\right) \le C \beta_{\mathfrak{I}}(T_{A_0,B_0})^{1-\theta} \beta_{\mathfrak{I}}(T_{A_1,B_1})^{\theta}.$$

We refer to [11] for the proof.

Using operators (T_n) mentioned in Example 4, one can construct an example showing that estimate a) does not hold if we remove the operator $J_{\overline{B}_{\theta,q}}$ (see [11], Remark 3.4).

When $\mathcal{I} = \mathcal{W}$, the ideal of weakly compact operators, Theorem 4 implies that $T : \overline{A}_{\theta,q} \longrightarrow \overline{B}_{\theta,q}$ is weakly compact provided $T : A_0 \longrightarrow B_0$ is weakly compact and $1 < q < \infty$. The restriction on q is essential. For instance, let $1 < p_i < \infty$, $0 < \theta < 1$ and $\dfrac{1}{p} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}$. The couple $\overline{A} = (L_{p_0}[0,1], L_{p_1}[0,1])$ is formed by reflexive spaces, so if we choose T as the identity mapping I, then $I : L_{p_i}[0,1] \longrightarrow L_{p_i}[0,1]$ is weakly compact for $i = 0, 1$. However,

$$(L_{p_0}[0,1], L_{p_1}[0,1])_{\theta,q} = L_{p,q}[0,1]$$

and since $L_{p,1}[0,1]$ (resp. $L_{p,\infty}[0,1]$) contains a subspace isomorphic to ℓ_1 (resp. ℓ_∞), it is not reflexive and so $I : L_{p,1}[0,1] \longrightarrow L_{p,1}[0,1]$ (resp. $I : L_{p,\infty}[0,1] \longrightarrow L_{p,\infty}[0,1]$) cannot be weakly compact.

Let us consider now the problem of finding a necessary and sufficient condition for $T : \overline{A}_{\theta,q} \longrightarrow \overline{B}_{\theta,q}$ to be weakly compact for $1 < q < \infty$.

The root of this problem is the classical result of Davis, Figiel, Johnson and Pelczynski [16] on factorization of weakly compact operators:

Theorem (DFJP Theorem)**.** *Let $T \in \mathcal{L}(A, B)$ be a weakly compact operator. Then there is a reflexive Banach space X and bounded linear operators $R \in \mathcal{L}(A, X)$ and $S \in \mathcal{L}(X, B)$ such that the following diagram commutes*

$$\begin{tikzcd} A \arrow[rr, "T"] \arrow[dr, "R"'] & & B \\ & X \arrow[ur, "S"'] & \end{tikzcd}$$

The proof given in [16] has a clear interpolation flavour. This motivated the investigation on the behaviour of weak compactness under interpolation. And in 1978 Beauzamy [4] proved that if $1 < q < \infty$ a necessary and sufficient condition for $\overline{A}_{\theta,q}$ to be reflexive is that the natural embedding J from $A_0 \cap A_1$ into $A_0 + A_1$ is weakly compact. Later, in 1980, Heinrich [19] gave a new approach to this result that applies also to any injective surjective operator ideal $\mathcal{I}$ satisfying the Σ_q-condition. Other contributions are due to Neidinger [29], [30], Maligranda and Quevedo [27] and Mastylo [28].

In order to establish Heinrich's result, we shall first introduce other two norms on $(A_0, A_1)_{\theta,q}$.

Since

$$\|a\|_{\theta,q} = \left(\int_0^\infty \left(t^{-\theta}K(t,a)\right)^q \frac{dt}{t}\right)^{1/q}$$
$$= \left(\sum_{m=-\infty}^{\infty} \int_{2^{m-1}}^{2^m} \left(t^{-\theta}K(t,a)\right)^q \frac{dt}{t}\right)^{1/q} \sim \left(\sum_{m=-\infty}^{\infty} \left(2^{-\theta m}K(2^m,a)\right)^q\right)^{1/q}$$

it follows that the functional

$$\|a\|_{\theta,q;K} = \left(\sum_{m=-\infty}^{\infty} \left(2^{-\theta m}K(2^m,a)\right)^q\right)^{1/q}$$

is a norm on $\overline{A}_{\theta,q}$ equivalent to $\|\cdot\|_{\theta,q}$.

Note that

$$\|a\|_{\theta,q;K} = \|\left(2^{-\theta m}K(2^m,a)\right)\|_{\ell_q}.$$

In particular, we can now establish the embeddings (1) just taking into account that $\ell_p \hookrightarrow \ell_q$ if $p \le q$.

We denote by $\overline{A}_{\theta,q;K} = (A_0, A_1)_{\theta,q;K}$ the real interpolation space endowed with the norm $\|\cdot\|_{\theta,q;K}$. The interpolation property still holds for the norm $\|\cdot\|_{\theta,q;K}$ but we need the additional constant 2^θ in the norm estimate. Namely

$$\|T\|_{\overline{A}_{\theta,q;K},\overline{B}_{\theta,q;K}} \le 2^\theta \|T\|_{A_0,B_0}^{1-\theta} \|T\|_{A_1,B_1}^{\theta}.$$

Put now

$$J(t,a) = \max\left\{\|a\|_{A_0}, t\|a\|_{A_1}\right\},\ a \in A_0 \cap A_1$$

and denote by $(A_0, A_1)_{\theta,q;J}$ the collection of all elements $a \in A_0 + A_1$ which can be represented by $a = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $A_0 + A_1$) with $(u_m) \subseteq A_0 \cap A_1$ and $\left(\sum_{m=-\infty}^{\infty} \left(2^{-\theta m}J(2^m, u_m)\right)^q\right)^{1/q} < \infty$. The norm of $(A_0, A_1)_{\theta,q;J}$ is given by

$$\|a\|_{\theta,q;J} = \inf\left\{\left(\sum_{m=-\infty}^{\infty} \left(2^{-\theta m}J(2^m,u_m)\right)^q\right)^{1/q} : a = \sum_{m=-\infty}^{\infty} u_m\right\}.$$

It turns out (see [5] or [35]) that $(A_0,A_1)_{\theta,q}=(A_0,A_1)_{\theta,q;J}$ with equivalence of norms. To be precise

$$\|a\|_{\theta,q;K}\le\frac{1}{3-2^{\theta}-2^{1-\theta}}\|a\|_{\theta,q;J}\,,\quad \|a\|_{\theta,q;J}\le 4\|a\|_{\theta,q;K}.$$

In particular,

$$\|T\|_{\overline{A}_{\theta,q;J},\overline{B}_{\theta,q;K}}\le\frac{2^{\theta}}{3-2^{\theta}-2^{1-\theta}}\|T\|^{1-\theta}_{A_0,B_0}\|T\|^{\theta}_{A_1,B_1}. \tag{2}$$

These discrete representations of $(A_0,A_1)_{\theta,q}$ allow us to relate the real interpolation space with vector valued sequence spaces. Indeed, if

$$2^{-\theta m}F_m=\left(A_0+A_1,2^{-\theta m}K(2^m,\cdot)\right)$$

and

$$2^{-\theta m}G_m=\left(A_0\cap A_1,2^{-\theta m}J(2^m,\cdot)\right)$$

then the map $j:(A_0,A_1)_{\theta,q;K}\longrightarrow l_q(2^{-\theta m}F_m)$ defined by $ja=(\dots,a,a,a,\dots)$ is an isometric embedding, and $\pi:\ell_q(2^{-\theta m}G_m)\longrightarrow(A_0,A_1)_{\theta,q;J}$ given by $\pi(u_m)=\sum_{m=-\infty}^{\infty}u_m$ is a metric surjection.

We are now ready to establish Heinrich's results.

Theorem 5. *Let $1<q<\infty$, $0<\theta<1$ and let $\mathfrak{I}$ be an injective surjective operator ideal satisfying the Σ_q-condition. Assume that $\overline{A}=(A_0,A_1)$, $\overline{B}=(B_0,B_1)$ are Banach couples and $T:A_0+A_1\longrightarrow B_0+B_1$ is a linear operator whose restrictions $T:A_i\longrightarrow B_i$ are bounded for $i=0,1$. A necessary and sufficient condition for $T:\overline{A}_{\theta,q}\longrightarrow\overline{B}_{\theta,q}$ to belong to $\mathfrak{I}$ is that $T:A_0\cap A_1\longrightarrow B_0+B_1$ belongs to $\mathfrak{I}$.*

Proof. The necessity is clear because we have the factorization

$$A_0\cap A_1\hookrightarrow\overline{A}_{\theta,q}\xrightarrow{T}\overline{B}_{\theta,q}\hookrightarrow B_0+B_1.$$

Assume then that $T:A_0\cap A_1\longrightarrow B_0+B_1$ belongs to $\mathfrak{I}$. Since $2^{-\theta m}J(2^m,\cdot)$ [resp. $2^{-\theta m}K(2^m,\cdot)$] is an equivalent norm to $\|\cdot\|_{A_0\cap A_1}$ [resp. $\|\cdot\|_{B_0+B_1}$], it follows that $T:2^{-\theta k}G_k\longrightarrow 2^{-\theta n}F_n$ belongs to $\mathfrak{I}$ for any k,n. Hence, by the Σ_q-property of $\mathfrak{I}$, we derive that

$$jT\pi\in\mathfrak{I}\left(\ell_q(2^{-\theta m}G_m),\ell_q(2^{-\theta m}F_m)\right).$$

Now taking into account surjectivity and injectivity of $\mathfrak{I}$, we conclude that $T\in\mathfrak{I}\left(\overline{A}_{\theta,q},\overline{B}_{\theta,q}\right)$. $\square$

Next we derive the general factorization result.

Theorem 6. *Let $\mathfrak{I}$ be an injective surjective operator ideal satisfying the Σ_q-condition and let $T \in \mathfrak{I}(A, B)$. Then there is a Banach space X such that $I_X \in \mathfrak{I}$ and operators $R \in \mathfrak{L}(A, X)$ and $S \in \mathfrak{L}(X, B)$ so that the following diagram commutes*

$$\begin{tikzcd} A \arrow[rr, "T"] \arrow[dr, "R"'] & & B \\ & X \arrow[ur, "S"'] & \end{tikzcd}$$

Proof. Factorize first T as

$$\begin{tikzcd} A \arrow[r, "T"] \arrow[d, "\Phi"'] & B \\ A/\operatorname{Ker}(T) \arrow[r, "j_0"'] & B \arrow[u, "I_B"'] \end{tikzcd}$$

Here $\Phi(x) = [x]$ is the quotient mapping and $j_0[x] = Tx$. Put $A_0 = A/\operatorname{Ker}(T)$ and $A_1 = B$. Then $j_0 : A_0 \longrightarrow A_1$ is a continuous embedding, and therefore (A_0, A_1) is a Banach couple. It is clear that $A_0 \cap A_1 = A_0$ and $A_0 + A_1 = A_1$. Moreover, since Φ is a quotient mapping and $\mathfrak{I}$ is surjective, we get that $j_0 : A_0 \cap A_1 \longrightarrow A_0 + A_1$ belongs to $\mathfrak{I}$. Take any $0 < \theta < 1$ and let $X = (A_0, A_1)_{\theta,q}$. By Theorem 5, the identity mapping I_X of X belongs to $\mathfrak{I}$, and operators $R = j_0\Phi \in \mathfrak{L}(A, X)$, $S = I \in \mathfrak{L}(X, B)$ give the wanted factorization. $\square$

In the special case $\mathfrak{I} = \mathfrak{W}$ we recover the theorem of Davis, Figiel, Johnson and Pelczynski.

Next we shall describe a quantitative version of Theorem 5. Instead of estimating the measure of the interpolated operator against the measures of the restrictions $T : A_i \longrightarrow B_i$, we are looking for something different involving the measure of $T : A_0 \cap A_1 \longrightarrow B_0 + B_1$.

The first attempt in this direction was done by Aksoy and Maligranda [1]. They considered the case of the measure of weak non-compactness and $T : A_0 + A_1 \longrightarrow B$, trying to establish the inequality

$$\gamma_{\mathcal{W}}\left(T_{\overline{A}_{\theta,q},B}\right) \leq C_{\theta,q}\gamma_{\mathcal{W}}(T_{A_0\cap A_1,B})^{\theta}\|T\|_{\overline{A},B}^{1-\theta}, \tag{3}$$

where $\|T\|_{\overline{A},B} = \max\{\|T\|_{A_0,B}, \|T\|_{A_1,B}\}$. But this can only be true for $0 < \theta \leq 1/2$ as the following example shows.

Example 7. Take any $0 < \lambda < 1$ and put $\lambda c_0 = (c_0, \lambda\|\cdot\|_{c_0})$. Choose $\overline{A} = (c_0, \lambda c_0)$, $B = c_0$ and $T = I$ as the identity operator. Then

$$\|T\|_{\overline{A},B} = \max\{1, \lambda^{-1}\} = \lambda^{-1},$$
$$\gamma_{\mathcal{W}}\left(T_{\Delta(\overline{A}),B}\right) = \gamma_{\mathcal{W}}\left(I_{c_0}\right) = 1.$$

On the other hand,

$$(c_0, \lambda c_0)_{\theta,q} = \lambda^{\theta} c_0$$

with equivalence of norms, being the constants independent of λ. Hence

$$\gamma_{\mathcal{W}}\left(T_{\overline{A}_{\theta,q},B}\right) = \gamma_{\mathcal{W}}\left(I_{\lambda^{\theta}c_0,c_0}\right) = \lambda^{-\theta}.$$

If (3) holds, then

$$\lambda^{-\theta} \leq C_{\theta,q}\lambda^{-(1-\theta)}, \qquad 0 < \lambda < 1$$

or equivalently

$$\sup_{0<\lambda<1}\left\{\lambda^{1-2\theta}\right\} < \infty$$

which only happens when $0 < \theta \leq 1/2$.

As we show next, inequality (3) is valid for the whole rank of θ if we replace θ by $\Theta = \min\{\theta, 1-\theta\}$. Moreover, the argument we give works for any operator ideal $\mathfrak{I}$. The result as well as the former example are taken from a joint paper with A. Martínez [12].

Theorem 7. *Let $\mathfrak{I}$ be an operator ideal. Assume that B is a Banach space, $\overline{A} = (A_0, A_1)$ is a Banach couple, $1 \leq q \leq \infty$, $0 < \theta < 1$ and let $\Theta = \min\{\theta, 1-\theta\}$.*

a) *If* $T \in \mathcal{L}(A_0 + A_1, B)$, *then*

$$\gamma_{\mathcal{J}}(T_{\overline{A}_{\theta,q},B}) \leq C_{\theta,q}\gamma_{\mathcal{J}}(T_{A_0\cap A_1,B})^{\Theta}\|T\|^{1-\Theta}_{\overline{A},B}.$$

b) *If* $T \in \mathcal{L}(B, A_0 \cap A_1)$, *then*

$$\beta_{\mathcal{J}}(T_{B,\overline{A}_{\theta,q}}) \leq C_{\theta,q}\beta_{\mathcal{J}}(T_{B,A_0+A_1})^{\Theta}\|T\|^{1-\Theta}_{B,\overline{A}}.$$

Proof. We only sketch the main ideas of the proof of a). Statement b) follows from a) by duality arguments. Full details can be found in [12].

It suffices to consider the case $q = \infty$. Let $\eta \leq 1$, and take t, s so that $t^{\theta} > 1/\eta$, $s^{1-\theta} > 1/\eta$. A similar reasoning to that in [28], Thm. 8 (see also [1], Thm. 2/b), gives that for any $\varepsilon > 0$

$$U_{\overline{A}_{\theta,\infty}} \subseteq 2(1+\varepsilon)\eta\max\{s,t\}U_{A_0\cap A_1} + 2(1+\varepsilon)\eta U_{A_0+A_1}.$$

Let $\sigma > \gamma_{\mathcal{J}}(T_{A_0\cap A_1,B})$. By the definition of $\gamma_{\mathcal{J}}$, there is a Banach space E and an operator $R \in \mathcal{J}(E,B)$ such that

$$T(U_{A_0\cap A_1}) \subseteq \sigma U_B + R(U_E).$$

The operator $S = 2(1+\varepsilon)\eta\max\{s,t\}R$ belongs to $\mathcal{J}(E,B)$ and satisfies

$$T(U_{\overline{A}_{\theta,\infty}}) \subseteq [2(1+\varepsilon)\eta\sigma\max\{s,t\} + 2(1+\varepsilon)\eta\|T\|_{\overline{A},B}]U_B + S(U_E).$$

Whence

$$\gamma_{\mathcal{J}}\left(T_{\overline{A}_{\theta,\infty},B}\right) \leq 2(1+\varepsilon)\eta\sigma\max\{s,t\} + 2(1+\varepsilon)\eta\|T\|_{\overline{A},B}$$

and so

$$\gamma_{\mathcal{J}}\left(T_{\overline{A}_{\theta,\infty},B}\right) \leq 2\eta\gamma_{\mathcal{J}}(T_{A_0\cap A_1,B})\max\{s,t\} + 2\eta\|T\|_{\overline{A},B}.$$

Now choosing $\eta = \left(\gamma_{\mathcal{J}}(T_{A_0\cap A_1,B})/\|T\|_{\overline{A},B}\right)^{\Theta}$ and taking the infimum over all $t > (1/\eta)^{1/\theta}$ and $s > (1/\eta)^{1/(1-\theta)}$, the result follows. $\square$

For $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$, Banach couples, we shall write $T : \overline{A} \longrightarrow \overline{B}$ to mean that T is a linear operator from $A_0 + A_1$ into $B_0 + B_1$, whose restrictions to each A_i define a bounded operator from A_i into B_i $(i = 0, 1)$. We put

$$\|T\|_{\overline{A},\overline{B}} = \max\{\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}\}$$

In the general case the estimate has been obtained in another joint paper with A. Martínez [13]. In order to describe it, we denote by T_m the restriction of T from $\left(A_0 \cap A_1, 2^{-\theta m}J(2^m,\cdot)\right)$ into $\left(B_0 + B_1, 2^{-\theta m}K(2^m,\cdot)\right)$.

Theorem 8. *Let $1 < q < \infty$, $0 < \theta < 1$, put $\Theta = \min\{\theta, 1-\theta\}$, and let $\mathcal{I}$ be an operator ideal satisfying the Σ_q-condition. Assume that $T : \overline{A} \longrightarrow \overline{B}$ and define T_m as before. Then*

a) $$\gamma_{\mathcal{I}}\left(\left[J_{\overline{B}_{\theta,q}}T\right]_{\overline{A}_{\theta,q},\ell_\infty(U_{\overline{B}^*_{\theta,q}})}\right) \leq C \sup\left\{\gamma_{\mathcal{I}}(T_m)^{\Theta} : m \in \mathbb{Z}\right\} \|T\|_{\overline{A},\overline{B}}^{1-\Theta},$$

b) $$\beta_{\mathcal{I}}\left(\left[TQ_{\overline{A}_{\theta,q}}\right]_{\ell_1(U_{\overline{A}_{\theta,q}}),\overline{B}_{\theta,q}}\right) \leq C \sup\left\{\beta_{\mathcal{I}}(T_m)^{\Theta} : m \in \mathbb{Z}\right\} \|T\|_{\overline{A},\overline{B}}^{1-\Theta}.$$

See [13] for the proof.

Comparing Theorems 7 and 8, it is natural to wonder if in the general case where $B_0 \neq B_1$, all assumptions of Theorem 8 are necessary for the conclusion. This is just the case (see [13]). We shall only review an example showing that a finite number of $\gamma_{\mathcal{I}}(T_m)$ is not enough to dominate $\gamma_{\mathcal{I}}\big(J_{\overline{B}_{\theta,q}}T\big)$.

Example 8. Let $0 < \lambda < 1$, $\mathcal{I} = \mathcal{W}$, choose $\overline{A} = \overline{B} = (c_0, \lambda c_0)$ and let $T = I$ be the identity operator. Then

$$\big(A_0 \cap A_1, 2^{-\theta m} J(2^m, \cdot)\big) = 2^{-\theta m} \max\{1, \lambda 2^m\} c_0$$
$$\big(B_0 + B_1, 2^{-\theta m} K(2^m, \cdot)\big) = 2^{-\theta m} \min\{1, \lambda 2^m\} c_0.$$

Thus

$$\gamma_{\mathcal{W}}(T_m) = \min\left\{\lambda 2^m, \frac{1}{\lambda 2^m}\right\}.$$

Clearly

$$\|T\|_{\overline{A},\overline{B}} = 1.$$

Moreover, taking into account that $(c_0, \lambda c_0)_{\theta,q} = \lambda^\theta c_0$ with equivalence of norms with constants independent of λ, and using the extension property of ℓ_∞-spaces, one can check that

$$\gamma_{\mathcal{W}}\left(J_{\overline{B}_{\theta,q}}T\right) = \gamma_{\mathcal{W}}(I_{\lambda^\theta c_0, \lambda^\theta \ell_\infty}) = 1.$$

Consequently, if the inequality a) in Theorem 8 would be true with only a finite number of $\gamma_{\mathcal{I}}(T_m)$, say $\left\{\gamma_{\mathcal{I}}(T_m)\right\}_{-N \leq m \leq N}$, it would follow

$$1 \leq C(\lambda 2^N)^{\Theta} \quad \text{for every } \lambda \in \left(0, \frac{1}{2^N}\right)$$

which is impossible.

Compact operators do not satisfy Σ_q-condition, so Theorems 4 and 8 do not apply to the ideal $\mathcal{K}$. In fact, it is easy to show by means of examples that $T : A_0 \cap A_1 \longrightarrow B_0 + B_1$ may be compact and $T : \overline{A}_{\theta,q} \longrightarrow \overline{B}_{\theta,q}$ may not be:

Example 9. Take $\overline{A} = \overline{B} = (\ell_q, \ell_q(n))$, where $\ell_q(n) = \{\xi_n : (n\xi_n) \in \ell_q\}$, and choose T as the identity operator. Then

$$A_0 \cap A_1 = \ell_q(n) \,, \qquad B_0 + B_1 = \ell_q$$

and $T : A_0 \cap A_1 \longrightarrow B_0 + B_1$ is compact because it is the limit of a sequence of finite rank operators. However $(A_0, A_1)_{\theta,q} = (B_0, B_1)_{\theta,q} = \ell_q(n^\theta)$ and the identity operator of this infinite-dimensional space is not compact.

In 1992, culminating the efforts of several authors (see the papers [7], [8], [14]), Cwikel proved in [15] that a sufficient condition for compactness of $T : \overline{A}_{\theta,q} \longrightarrow \overline{B}_{\theta,q}$ $(1 \leq q \leq \infty)$ is that one of the restrictions of T, say $T : A_0 \longrightarrow B_0$, is compact. I will finish this talk by describing a quantitative version of this result in terms of the measure of non-compactness. The result is taken from a joint paper with P. Fernández-Martínez and A. Martínez [9].

Theorem 9. *Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be Banach couples and let $T : \overline{A} \longrightarrow \overline{B}$. Then, for any $1 \leq q \leq \infty$ and $0 < \theta < 1$,*

$$\gamma_{\mathcal{K}}\left(T_{\overline{A}_{\theta,q;J}, \overline{B}_{\theta,q;K}}\right) \leq 16\,\delta \gamma_{\mathcal{K}}(T_{A_0,B_0})^{1-\theta} \gamma_{\mathcal{K}}(T_{A_1,B_1})^{\theta}$$

where $\delta = 2^\theta/(3 - 2^\theta - 2^{1-\theta})$.

Proof. I will only sketch the main ideas. Full details can be found in [9].

Let again $G_m = (A_0 \cap A_1, J(2^m, .))$ and $F_m = (B_0 + B_1, K(2^m, .))$. As we have already said, the operator $\pi : \ell_q(2^{-\theta m} G_m) \longrightarrow \overline{A}_{\theta,q;J}$ defined by $\pi(u_m) = \sum_{m=-\infty}^{\infty} u_m$ is a metric surjection. Note that

$$\pi : \ell_1(G_m) \longrightarrow A_0 \quad \text{and} \quad \pi : \ell_1(2^{-m} G_m) \longrightarrow A_1$$

are also bounded with norm ≤ 1.

Let j be the operator associating to any $b \in B_0 + B_1$ the constant sequence $jb = (\ldots, b, b, b, \ldots)$. Then $j : \overline{B}_{\theta,q;K} \longrightarrow l_q(2^{-\theta m} F_m)$ is a metric injection, and operators $j : B_0 \longrightarrow l_\infty(F_m)$, $j : B_1 \longrightarrow l_\infty(2^{-m} F_m)$ are bounded with norms ≤ 1.

We have the following diagram of bounded operators

$$\ell_1(G_m) \xrightarrow{\ \pi\ } A_0 \xrightarrow{\ T\ } B_0 \xrightarrow{j} \ell_\infty(F_m)$$

$$\ell_1(2^{-m}G_m) \xrightarrow{\ \pi\ } A_1 \xrightarrow{\ T\ } B_1 \xrightarrow{\ j\ } \ell_\infty(2^{-\theta m}F_m)$$

$$\ell_q\left(2^{-\theta m}G_m\right) \xrightarrow{\ \pi\ } (A_0,A_1)_{\theta,q} \xrightarrow{\ T\ } (B_0,B_1)_{\theta,q} \xrightarrow{\ j\ } \ell_q\left(2^{-\theta m}F_m\right).$$

Put $\overline{T} = jT\pi$. We have that

$$\begin{aligned}\gamma_{\mathcal{K}}&\big(T:(A_0,A_1)_{\theta,q;J} \longrightarrow (B_0,B_1)_{\theta,q;K}\big)\\ &\le 2\gamma_{\mathcal{K}}\big(jT:(A_0,A_1)_{\theta,q;J} \longrightarrow \ell_q(2^{-\theta m}F_m)\big)\\ &= 2\gamma_{\mathcal{K}}\big(\overline{T}:\ell_q(2^{-\theta m}G_m) \longrightarrow \ell_q(2^{-\theta m}F_m)\big).\end{aligned}$$

So, in order to establish the theorem, it suffices to show that

$$\begin{aligned}\gamma_{\mathcal{K}}&\left(\overline{T}:\ell_q(2^{-\theta m}G_m) \longrightarrow \ell_q(2^{-\theta m}F_m)\right)\\ &\le 8\delta\gamma_{\mathcal{K}}(T:A_0 \longrightarrow B_0)^{1-\theta}\gamma_{\mathcal{K}}(T:A_1 \longrightarrow B_1)^{\theta}.\end{aligned} \tag{4}$$

The advantage of working with $\overline{T}$ instead of T is that we can use the following families of projections on the couples of vector valued sequences: For each positive integer $n \in \mathbb{N}$ define operators P_n, Q_n^+, Q_n^- by

$$\begin{aligned}P_n(u_m) &= (\dots,0,0,u_{-n},u_{-n+1},\dots,u_{n-1},u_n,0,0,\dots)\\ Q_n^+(u_m) &= (\dots,0,0,u_{n+1},u_{n+2},\dots)\\ Q_n^-(u_m) &= (\dots,u_{-n-2},u_{-n-1},0,0,\dots).\end{aligned}$$

The following three conditions holds:

I) The identity mapping on $\ell_1(G_m)+\ell_1(2^{-m}G_m)$ can be decomposed as $I = P_n + Q_n^+ + Q_n^-$ for $n = 1,2,\dots$.

II) Operators P_n, Q_n^+ and Q_n^- are uniformly bounded in the couple, i.e.,

$$\|P_n:\ell_1(G_m) \longrightarrow \ell_1(G_m)\| = \|P_n:\ell_1(2^{-m}G_m) \longrightarrow \ell_1(2^{-m}G_m)\| = 1$$

and similarly for Q_n^+ and Q_n^-.

III) $\|Q_n^+ : \ell_1(G_m) \longrightarrow \ell_1(2^{-m}G_m)\|$
$$= 2^{-(n+1)} = \|Q_n^- : \ell_1(2^{-m}G_m) \longrightarrow \ell_1(G_m)\|.$$

The same families of projections can be defined on the couple

$$\left(\ell_\infty(F_m)\, ,\, \ell_\infty(2^{-m}F_m)\right).$$

Call them R_n , S_n^+ , S_n^-. They satisfy the corresponding versions of (I), (II) and (III).

Next we shall decompose the interpolated operator by means of these projections. We shall need a more refined splitting than the one use in [10] to develop a new approach to Cwikel's result.

It follows from

$$\begin{aligned}\overline{T} = \overline{T}(P_n + Q_n^+ + Q_n^-) =& \overline{T}P_n + (R_n + S_n^+ + S_n^-)\overline{T}(Q_n^+ + Q_n^-)\\ =& \overline{T}P_n + R_n\overline{T}(Q_n^+ + Q_n^-) + S_n^+\overline{T}Q_n^- + S_n^-\overline{T}Q_n^+ + S_n^+\overline{T}Q_n^+ + S_n^-\overline{T}Q_n^-\end{aligned}$$

that

$$\begin{aligned}\gamma_{\mathcal{K}}(\overline{T}) \le \gamma_{\mathcal{K}}(\overline{T}P_n) + \gamma_{\mathcal{K}}(R_n\overline{T}(Q_n^+ + Q_n^-)) + \|S_n^+\overline{T}Q_n^-\|\\ + \|S_n^-\overline{T}Q_n^-\| + \|S_n^+\overline{T}Q_n^+\| + \|S_n^-\overline{T}Q_n^-\|.\end{aligned}$$

Here all operators act from $\ell_q\left(2^{-\theta m}G_m\right)$ into $\ell_q\left(2^{-\theta m}F_m\right)$.

It is not hard to check that the inclusions

$$\begin{aligned}\ell_q\left(2^{-\theta m}G_m\right) \hookrightarrow \left(\ell_1(G_m), \ell_1(2^{-m}G_m)\right)_{\theta,q;J}\,,\\ \left(\ell_\infty(F_m), \ell_\infty(2^{-m}F_m)\right)_{\theta,q;K} \hookrightarrow \ell_q\left(2^{-\theta m}F_m\right)\end{aligned}$$

have norm less than or equal to 1. This remark and interpolation property (2) will be useful in our estimates.

Let us consider first the $(+,-)$-term. We have

$$\begin{aligned}\|S_n^+\overline{T}Q_n^-\| &\le \delta\|S_n^+\overline{T}Q_n^-\|^{1-\theta}_{\ell_1(G_m),\ell_\infty(F_m)}\|S_n^+\overline{T}Q_n^-\|^{\theta}_{\ell_1(2^{-m}G_m),\ell_\infty(2^{-m}F_m)}\\ &\le \delta\|T\|^{1-\theta}_{A_0,B_0}\|Q_n^-\|^{\theta}_{\ell_1(2^{-m}G_m),\ell_1(G_m)}\\ &\quad\times\|\overline{T}\|^{\theta}_{\ell_1(G_m),\ell_\infty(F_m)}\|S_n^+\|^{\theta}_{\ell_\infty(F_m),\ell_\infty(2^{-m}F_m)}\\ &\le \delta 2^{-2(n+1)\theta}\|T\|_{A_0,B_0} \to 0 \text{ as } n \to \infty.\end{aligned}$$

In the same way

$$\|S_n^-\overline{T}Q_n^+\| \to 0 \text{ as } n \to \infty.$$

Concerning the $(+,+)$-term, we shall show that for any $\varepsilon > 0$, choosing n big enough, we have

$$\|S_n^+\overline{T}Q_n^+\| \leq 2\delta\gamma_{\mathcal{K}}(T : A_0 \longrightarrow B_0)^{1-\theta}\gamma_{\mathcal{K}}(T : A_1 \longrightarrow B_1)^{\theta} + \varepsilon. \qquad (5)$$

Indeed, according to (2) and (II),

$$\begin{aligned}\|S_n^+\overline{T}Q_n^+\| &\leq \delta\|S_n^+\overline{T}Q_n^+\|^{1-\theta}_{\ell_1(G_m),\ell_\infty(F_m)}\|S_n^+\overline{T}Q_n^+\|^{\theta}_{\ell_1(2^{-m}G_m),\ell_\infty(2^{-m}F_m)}\\ &\leq \delta\|\overline{T}Q_n^+\|^{1-\theta}_{\ell_1(G_m),\ell_\infty(F_m)}\|S_n^+\overline{T}Q_n^+\|^{\theta}_{\ell_1(2^{-m}G_m),\ell_\infty(2^{-m}F_m)}.\end{aligned}$$

The sequence $(\|\overline{T}Q_n^+\|)$ is decreasing since $Q_n^+Q_{n+1}^+ = Q_{n+1}^+$. Hence there is $\lambda \geq 0$ such that $\|\overline{T}Q_n^+\| \to \lambda$ as $n \to \infty$. Choose $(u_n) \subseteq U_{\ell_1(G_m)}$ so that

$$\|\overline{T}Q_n^+u_n\|_{\ell_\infty(F_m)} \to \lambda \text{ as } n \to \infty.$$

By the definition of $\gamma_{\mathcal{K}}$, given any $\sigma_0 > \gamma_{\mathcal{K}}(T : A_0 \longrightarrow B_0)$ there are finitely many vectors $b_1, \ldots, b_s$ in B_0 such that

$$TU_{A_0} \subseteq \bigcup_{r=1}^{s}\{b_r + \sigma_0 U_{B_0}\}.$$

Then there is some r, say $r = 1$, and some subsequence (n') of $\mathbb{N}$ with

$$T\pi Q_{n'}^+u_{n'} \in \{b_1 + \sigma_0 U_{B_0}\} \text{ for any } n'.$$

This allows us to estimate $K(2^m, b_1)$ because, using (III), we get

$$\begin{aligned}K(2^m, b_1) &\leq \|b_1 - T\pi Q_{n'}^+u_{n'}\|_{B_0} + 2^m\|T\pi Q_{n'}^+u_{n'}\|_{B_1}\\ &\leq \sigma_0 + 2^{m-n'}\|T\|_1 \to \sigma_0 \text{ as } n' \to \infty.\end{aligned}$$

Whence

$$\begin{aligned}\lambda &= \lim_{n'\to\infty}\|\overline{T}Q_{n'}^+u_{n'}\|_{\ell_\infty(F_m)}\\ &\leq \sup_{n'}\left\{\|T\pi Q_{n'}^+u_{n'} - b_1\|_{B_0} + \sup_{m\in\mathbb{Z}}\{K(2^m, b_1)\}\right\} \leq 2\sigma_0 + \varepsilon.\end{aligned}$$

This implies that there is $N_1 \in \mathbb{N}$ such that for $n \geq N_1$,

$$\|\overline{T}Q_n^+\|^{1-\theta}_{\ell_1(G_m),\ell_\infty(F_m)} \leq \big(2\gamma_{\mathcal{K}}(T : A_0 \longrightarrow B_0)\big)^{1-\theta} + \varepsilon.$$

Let now pass to $\|S_n^+\overline{T}Q_n^+\|$. First note that the set D of all sequences having only a finite number of non-zero coordinates is dense in $\ell_1(2^{-m}G_m)$. Given any $\sigma_1 > \gamma_{\mathcal{K}}(T : A_1 \longrightarrow B_1)$, we can find a finite set $\{v_r\}_{r=1}^s \subseteq D$ so that

$$\min_{1\leq r\leq s}\left\{\|\overline{T}v - \overline{T}v_r\|_{\ell_\infty(2^{-m}F_m)}\right\} \leq 2\sigma_1 \,, \qquad v \in U_{\ell_1(2^{-m}G_m)}.$$

Then $\left\{\overline{T}v_r\right\}_{r=1}^s \subseteq \ell_\infty(F_m) \cap \ell_\infty(2^{-m}F_m)$, and by (III)

$$\|S_n^+\overline{T}v_r\|_{\ell_\infty(2^{-m}F_m)} \leq \varepsilon \,, \qquad r = 1, \ldots, s$$

for any $n \geq N_2 \in \mathbb{N}$. Given $v \in U_{\ell_1(2^{-m}G_m)}$ we have

$$\begin{aligned}
&\|S_n^+\overline{T}Q_n^+v\|_{\ell_\infty(2^{-m}F_m)} \\
&\quad\leq \min_{1\leq r\leq s}\left\{\|S_n^+\overline{T}Q_n^+v - S_n^+\overline{T}v_r\|_{\ell_\infty(2^{-m}F_m)} + \|S_n^+\overline{T}v_r\|_{\ell_\infty(2^{-m}F_m)}\right\} \\
&\quad\leq 2\sigma_1 + \varepsilon.
\end{aligned}$$

Hence

$$\|S_n^+\overline{T}Q_n^+\|_{\ell_1(2^{-m}G_m),\ell_\infty(2^{-m}F_m)} \leq 2\gamma_{\mathcal{K}}(T : A_1 \longrightarrow B_1) + \varepsilon$$

for n sufficiently big and (5) follows.

The $(-,-)$-term can be estimated in a similar way.

The remaining terms require more elaborated arguments. The outcome is

$$\begin{aligned}
\gamma_{\mathcal{K}}\left(\overline{T}P_n\right) &\leq 2\delta\gamma_{\mathcal{K}}(T : A_0 \longrightarrow B_0)^{1-\theta}\gamma_{\mathcal{K}}(T : A_1 \longrightarrow B_1)^{\theta}, \\
\gamma_{\mathcal{K}}\left(R_n\overline{T}(Q_n^+ + Q_n^-)\right) &\leq 2\delta\gamma_{\mathcal{K}}(T : A_0 \longrightarrow B_0)^{1-\theta}\gamma_{\mathcal{K}}(T : A_1 \longrightarrow B_1)^{\theta}.
\end{aligned}$$

Details can be found in [9]. □

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