INEQUALITIES IN REARRANGEMENT INVARIANT FUNCTION SPACES

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NOTATION

n	dimension
$\mathbb{R}^n$	Euclidean n-dimensional space
$\|x\|$	$\sqrt{(x_1^2 + \cdots + x_n^2)}$, if x is a vector in $\mathbb{R}^n$ and $x_1, \ldots, x_n$ are the coordinates of x
κ_n	$\pi^{n/2}[\Gamma(n/2+1)]^{-1}$, measure of the unit n-dimensional ball
m	n-dimensional Lebesgue measure
H_k	k-dimensional Hausdorff measure
$\int \ldots dx$	integral with respect to n-dimensional Lebesgue measure
∇	gradient
Δ	Laplace operator
χ_E	characteristic function of a set E
sprt	support
ess sup	essential supremum
L^p	Lebesgue space — the set of measurable functions such that $\int \|u\|^p dx < \infty$
$W^{k,p}$	Sobolev space — the set of functions from L^p that are endowed with k-th order weak derivatives in L^p

1. REARRANGEMENTS

1.1. Introduction. Set up by Hardy & Littlewood, a theory of rearrangements was popularized by the well-known book [HLP]. Rearrangements of functions are frequently used in real and harmonic analysis, in investigations about singular integrals and function spaces — see, e.g., [He], [ON], [ONW], [SW]. Pólya & Szegö and their followers demonstrated a good many isoperimetric theorems and inequalities by means of rearrangements — see [PS], a source book on this matter. More recent investigations have shown that rearrangements of functions fit well also into the theory of elliptic second-order partial differential equations — see, e.g., [Bae], [Ta3] and the bibliography therein.

Several types of rearrangements are known — presentations are in [Ka] and [Bae]. Here we limit ourselves to rearrangements à la Hardy & Littlewood.

1.2. Definitions and basic properties. Let G be a measurable subset of $\mathbb{R}^n$, and let u be a real-valued measurable function defined in G. Assume either $m(G)$ is finite or u *decays at infinity*, i.e., $m\{x \in G\colon |u(x)| > t\}$ is finite for every positive t.

Definition 1.A. The *distribution function* of u, μ, is a map which informs about the content of level sets of u; specifically,

$$\mu(t) = m\{x \in G\colon |u(x)| > t\} \tag{1.1}$$

for every nonnegative t.

The following propositions are straightforward:

(1.i) μ is a *decreasing* function defined in $[0, \infty[$.

(1.ii) μ is *right-continuous.*

(1.iii) $\mu(0) = m(\operatorname{sprt} u)$ and $\mu(+\infty) = 0$.

(1.iv) $\{t \geq 0\colon \mu(t) = 0\} = [\operatorname{ess\,sup} |u|, +\infty[$. In other words, $\operatorname{sprt} \mu$ is an interval whose end points are 0 and $\operatorname{ess\,sup} |u|$; the latter is either $+\infty$ or the smallest zero of μ.

(1.v) $\mu(t-) = m\{x \in G\colon |u(x)| \geq t\}$ for every positive t. Hence $\mu(t-) - \mu(t)$, the *jump* of μ at t, equals $m\{x \in G\colon |u(x)| = t\}$ for every positive t.

A typical situation is sketched in Fig. 1.

Definition 1.B. The *decreasing rearrangement* of u, u^*, is the distribution function of μ.

The following propositions hold:

(1.vi) u^* is a *decreasing* function defined in $[0, \infty[$.

(1.vii) u^* is *right-continuous.*

(1.viii) $u^*(0) = \operatorname{ess\,sup} |u|$ and $u^*(+\infty) = 0$.

(1.ix) $\{s \geq 0\colon u^*(s) = 0\} = [m(\operatorname{sprt} u), +\infty[$. Consequently, $\operatorname{sprt} u^*$ is an interval whose end points are 0 and $m(\operatorname{sprt} u)$.

(1.x) $\{t \geq 0\colon \mu(t) \leq s\} = [u^*(s), \infty[$ for every nonnegative s. Thus,

$$u^*(s) = \min\{t \geq 0\colon \mu(t) \leq s\} \tag{1.2}$$

for every nonnegative s — a *representation formula.*

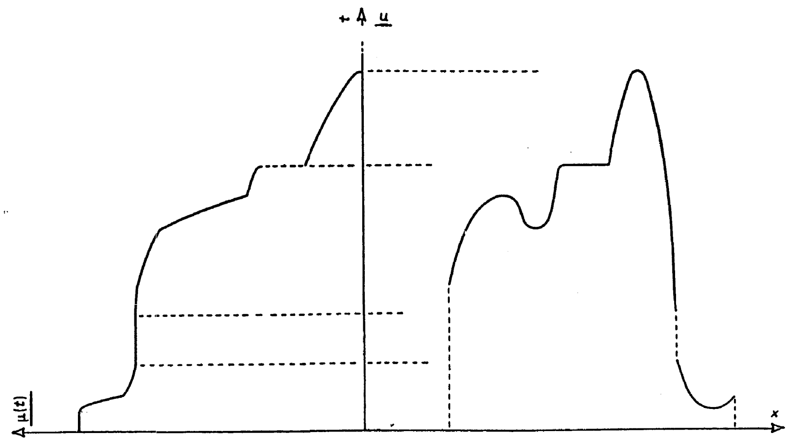

Fig. 1

Proof of Proposition (1.x). By its very definition, the value of u^* at any nonnegative s is the one-dimensional measure of the level set $\{t \geq 0\colon \mu(t) > s\}$. Distribution function μ decreases monotonically and is right-continuous. Hence $\{t \geq 0\colon \mu(t) > s\}$ is either empty or the interval $[0, u^*(s)[$. The conclusion follows. $\square$

(1.xi) $\{s \geq 0\colon u^*(s) > t\} = [0, \mu(t)[$ for every nonnegative t. In other words, the level set $\{s \geq 0\colon u^*(s) > t\}$ is, for every nonnegative t, an interval whose left end-point is 0 and whose length is exactly the measure of $\{x \in G\colon |u(x)| > t\}$. The present statement informs that *the distribution function of u^* is μ, i.e., u and u^* are equidistributed.*

Proof of Proposition (1.xi). First, $\{s \geq 0\colon u^*(s) > t\} \subseteq [0, \mu(t)[$ for every nonnegative t. Indeed, Proposition (1.x) tells us that if s is in the left-hand side, then t must be such that $\mu(t) > s$, thus s is in the right-hand side. Secondly, $\{s \geq 0\colon u^*(s) > t\} \supseteq [0, \mu(t)[$ for every nonnegative t. Indeed, Proposition (1.x) tells us that if s is in the right-hand side, then t must be less than $u^*(s)$, thus s is in the left-hand side. $\square$

(1.xii) $u^*\big(\mu(t)\big) \leq t$ for every nonnegative t, $u^*\big(\mu(t)-\big) \geq t$ for every t such that $0 \leq t < \operatorname{ess\,sup}|u|$. Thus, u^* is an inverse function of μ.

Proof of Proposition (1.xii). Proposition (1.x) yields immediately that $u^*\big(\mu(t)\big) \leq t$ if t is nonnegative. On the other hand, the very definition of u^* and a property of distribution functions tell us that $u^*(s-) =$ 1-dimensional measure of $\{t \geq 0\colon \mu(t) \geq s\}$ for every positive s. Let $0 \leq t < \operatorname{ess\,sup}|u|$. Then the limit of u^* at $\mu(t)$ from the left is the one-dimensional measure of $\{s \geq 0\colon \mu(s) \geq \mu(t)\}$ — recall that the value of μ at a point t is strictly positive if and only if t is nonnegative and strictly smaller than $\operatorname{ess\,sup}|u|$. Now, $\{s \geq 0\colon \mu(s) \geq \mu(t)\} \supseteq [0, t]$ since μ decreases monotonically. Therefore $u^*\big(\mu(t)-\big) \geq t$. $\square$

(1.xiii) $\mu\big(u^*(s)\big) \leq s$ for every nonnegative s, $\mu\big(u^*(s)-\big) \geq s$ for every s such that $0 \leq s \leq m(\operatorname{sprt} u)$. Thus, μ is an inverse function of u^*.

Proof of Proposition (1.xiii). By Proposition (1.xi), μ is the distribution function of *both* u *and* u^*. Hence $\mu(t) =$ 1-dimensional measure of $\{s \geq 0\colon u^*(s) > t\}$ for every nonnegative t, and $\mu(t-) =$ one-dimensional measure of $\{s \geq 0\colon u^*(s) \geq t\}$ for every positive t. In particular, $\mu\big(u^*(s)\big) =$ 1-dimensional measure of $\{t \geq 0\colon u^*(t) > u^*(s)\}$ for every nonnegative s, and $\mu\big(u^*(s)-\big) =$ 1-dimensional measure of $\{t \geq 0\colon u^*(t) \geq u^*(s)\}$ if $0 \leq s < m(\operatorname{sprt} u)$ — recall that u^* takes a positive value at a point s if and only if s is nonnegative and strictly smaller than $m(\operatorname{sprt} u)$. As u^* decreases monotonically, we have the inclusions $\{t \geq 0\colon u^*(t) > u^*(s)\} \subseteq [0, s[$ and $\{t \geq 0\colon u^*(t) \geq u^*(s)\} \supseteq [0, s]$ for every nonnegative s. The conclusion follows. $\square$

(1.xiv) $[\mu(t), \mu(t-)[\subseteq \{s \geq 0\colon u^*(s) = t\} \subseteq [\mu(t), \mu(t-)]$ for every positive t.

Proof of Proposition (1.xiv). $\{s \geq 0\colon u^*(s) = t\} = \bigcap_{k=1}^{\infty} \{s \geq 0\colon t(1-1/k) < u^*(s) \leq t\}$ if $t > 0$. By Proposition (1.xii), $\{s \geq 0\colon t(1-1/k) < u^*(s) \leq t\} = \big[\mu(t), \mu\big(t(1-1/k)\big)\big[$ if $t > 0$ and $k = 1, 2, \ldots$. $\square$

Proposition (1.xi) has basic consequences. For instance: If A is any continuous increasing map from $[0,\infty]$ into $[0,\infty]$ such that $A(0)=0$, then

$$\int_G A\big(|u(x)|\big)\,dx = \int_0^\infty A\big(u^*(s)\big)\,ds \tag{1.3}$$

— this equation follows from

$$\int_G A\big(|u(x)|\big)\,dx = \int_0^\infty A(t)[-d\mu(t)],$$

a form of Cavalieri's principle.

Relevant geometric aspects are illustrated in Fig. 2. An algorithm for computing and plotting decreasing rearrangements is offered in [Ta3].

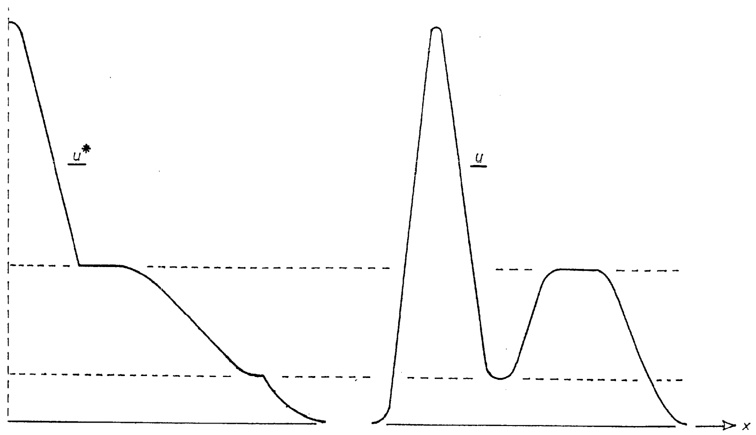

Fig. 2

Definition 1.C. The *symmetric rearrangement* of u, $u^\bigstar$, is defined by

$$u^\bigstar(x) = u^*(\kappa_n|x|^n) \tag{1.4}$$

for every x in $\mathbb{R}^n$.

Properties of u^* imply:

(1.xv) $u^\star$ is *nonnegative, radial* — i.e., invariant under rotations about the origin of $\mathbb{R}^n$ — and *radially decreasing* — i.e., $u^\star$ decreases as the distance from the origin increases.

(1.xvi) u and $u^\star$ are *equidistributed.*

Propositions (1.xv) and (1.xvi) can be summarized this way: for every nonnegative t, $\{x \in \mathbb{R}^n\colon u^\star(x) > t\}$, a level set of $u^\star$, is the ball whose center is the origin and whose measure equals the measure of $\{x \in G\colon |u(x)| > t\}$, the allied level set of $|u|$. In other words, $u^\star$ is a function whose graph results from a *Schwarz symmetrization* of the graph of $|u|$.

Fig. 3 shows an example in closed form.

$$u(x) = 8 + 2x^2 - x^4$$

$$\mu(t) = \begin{cases} 2\sqrt{1+\sqrt{9-t}} & \text{if } \ t \le 8 \\ 2\sqrt{2-2\sqrt{t-8}} & \text{if } \ 8 < t \le 9 \end{cases}$$

$$u^\star(x) = \begin{cases} 9 - x^2 + x^4/4 & \text{if } \ |x| \le \sqrt{2} \\ u(x) & \text{if } \ |x| > \sqrt{2} \end{cases}$$

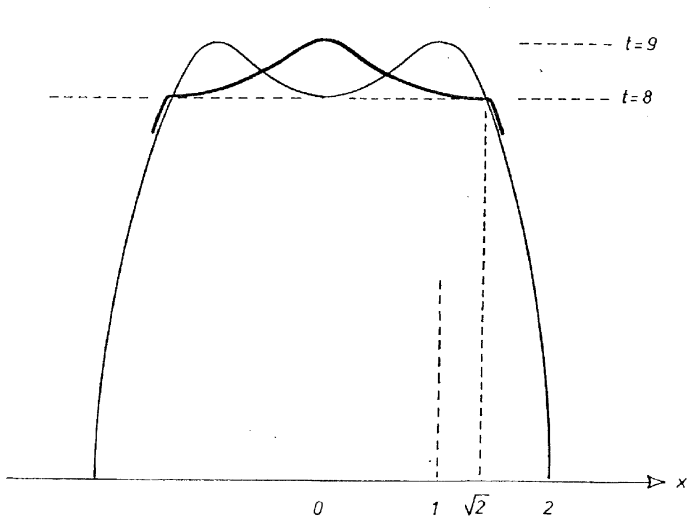

Fig. 3

The definitions of u^* and $u^\star$ can be recast in a more compact form. Recall the *layer-cake formula*: if G is any measurable subset of $\mathbb{R}^n$ and f is any nonnegative function in $L^1(G)$, then f can be recovered as the superimposition of the characteristic functions of its level sets: more precisely,

$$f = \int_0^\infty \chi_{\{x\in G:\ f(x)>t\}}\, dt \tag{1.5}$$

— the integral is à la Bochner. i.e.,

$$\Big\| f - \sum\nolimits_{k=1}^N \chi_{\{x\in G:\ f(x)>t_k\}}(t_k - t_{k-1}) \Big\|_{L^1(G)} \to 0$$

as $0 = t_0 < t_1 < \cdots < t_N$, $\max(t_k - t_{k-1}) \to 0$ and $t_N \to +\infty$. Consequently, we have the following proposition:

(1.xvii) If u is in $L^1(G)$, then

$$u^* = \int_0^\infty \chi_{[0,\mu(t)[}\, dt, \tag{1.6}$$

$$u^\star = \int_0^\infty \chi_{\{x\in\mathbb{R}^n:\ \kappa_n|x|^n<\mu(t)\}}\, dt. \tag{1.7}$$

1.3. Key theorems. Basic properties of rearrangements à la Hardy & Littlewood are summarized in Theorems 1.A, 1.B, 1.C, 1.D bellow. Roughly speaking, these theorems inform that a rearrangement, though sharing a distribution function, may play a better game than the original does.

Theorem 1.A. *Suppose u and v are measurable and nonnegative. Then*

$$\int_{\mathbb{R}^n} u(x)v(x)\, dx \le \int_0^\infty u^*(s)v^*(s)\, ds, \tag{1.8.a}$$

or

$$\int_{\mathbb{R}^n} u(x)v(x)\, dx \le \int_{\mathbb{R}^n} u^\star(x)v^\star(x)\, dx \tag{1.8.b}$$

Theorem 1.B. *Suppose f, g, h are measurable and nonnegative. Then*

$$\int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} f(x)g(y)h(x-y)\,dy \le \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} f^\bigstar(x)g^\bigstar(y)h^\bigstar(x-y)\,dy. \tag{1.9}$$

Theorem 1.C. *Suppose Φ is a Young function* — i.e., Φ maps $[0,\infty[$ into $[0,\infty[$, $\Phi(0)=0$, Φ is increasing and convex. *Suppose u is sufficiently smooth* — e.g., Lipschitz continuous — *and decays at infinity* — i.e., the measure of $\{x\in\mathbb{R}^n\colon |u(x)|>t\}$ is finite for every positive t. *Then*

$$\int_{\mathbb{R}^n} \Phi\big(|\nabla u(x)|\big)\,dx \ge \int_0^\infty \Phi\Big(-n\kappa_n^{1/n}s^{1-1/n}\frac{du^*}{ds}(s)\Big)\,ds, \tag{1.10a}$$

or

$$\int_{\mathbb{R}^n} \Phi\big(|\nabla u(x)|\big)\,dx \ge \int_{\mathbb{R}^n} \Phi\big(|\nabla u^\bigstar(x)|\big)\,dx. \tag{1.10.b}$$

Theorem 1.D. *Suppose u and v are real-valued and measurable, suppose Φ is a Young function. Then*

$$\int_{\mathbb{R}^n} \Phi\big(|u(x)-v(x)|\big)\,dx \ge \int_0^\infty \Phi\big(|u^*(s)-v^*(s)|\big)\,ds \tag{1.11.a}$$

or

$$\int_{\mathbb{R}^n} \Phi\big(|u(x)-v(x)|\big)\,dx \ge \int_{\mathbb{R}^n} \Phi\big(|u^\bigstar(x)-v^\bigstar(x)|\big)\,dx. \tag{1.11.b}$$

Theorem 1.A is by Hardy & Littlewood. A proof is in [HLP], Section 10.13; an alternative proof is offered in Subsection 1.4 below. Theorem 1.A is simple, but decisive: most theorems, that are demonstrated via rearrangements of functions, involve it.

Theorem 1.B is due to F. Riesz. Proofs appear in [Rie] and [HLP], Sections 10.14 and 10.15. Generalizations are in [BLL].

Let us sketch a simple application of Theorem 1.B. Suppose E is a three-dimensional material body, whose density is 1 and whose volume is fixed. Consider the energy of E, i.e., the energy of the gravitational field generated by E. Question: *which E renders such an energy a maximum*? We have

$$\text{energy of } E = \int_{\mathbb{R}^3} |\nabla u|^2 \, dx,$$

where u — a potential — is given by

$$u(x) = \int_{\mathbb{R}^3} \chi_E(y) \frac{1}{4\pi|x-y|} \, dy$$

and satisfies

$$-\Delta u = \chi_E.$$

Integrations by parts show

$$\text{energy of } E = \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} \chi_E(x)\chi_E(y) \frac{1}{4\pi|x-y|} \, dy.$$

Note that $(\chi_E)^\star$, the symmetric rearrangement of χ_E, is exactly the characteristic function of $E^\star$, a ball having the same volume as E. Thus Theorem 1.B tells us that

$$\text{energy of } E \leq \text{ energy of } E^\star.$$

i.e., the answer to our question: *among all homogeneous 3-dimensional bodies, whose volume and density are fixed, the ball generates the gravitational field having the largest energy.*

A proof of Theorem 1.C is detailed in Subsection 1.5 below. Theorem 1.C implies that the *total variation* and *Dirichlet type integrals* of sufficiently smooth functions decaying at infinity *decrease under the symmetric rearrangements.* Theorem 1.C is a key to proofs of isoperimetric inequalities of mathematical physics, e.g., Faber & Krahn theorem on the principal frequency of a membrane, Poincaré inequality for capacity, Saint-Venant principle for torsional rigidity — see [PS]. Theorem 1.C is also a key to a sharp proof of certain Sobolev inequalities — see, e.g., [Mos], [Ta1], [Lb] and Section 2. Exhaustive proofs of Theorem 1.C are presented in [BZ], [GR], [Hil], [Lb], [S1], [S2], [Spi], [Ta1]. An interesting derivation of Theorem 1.C is outlined in [Bae]. A variant of Theorem 1.C — where Lebesgue measure is replaced by Gauss measure — is offered in [Ehr].

Theorem 1.C implies that the symmetric rearrangement of a sufficiently smooth function, which decays fast enough at infinity, is Lipschitz continuous. Note that lots of functions exist which really differ from their symmetric rearrangements and render (1.10.b) an equality; as shown in [Ka] and [BZ], equality in (1.10.b) implies $u = u^\star$ if and only if an extra hypothesis is in force, i.e., the set of critical points of u is thin enough.

Theorem 1.D is instrumental when approximation arguments are involved. It implies that the symmetric rearrangement is a *contraction*, or a *non-expansive* map, in any Orlicz space. In the special case where $\Phi(t) = t^2$ Theorem 1.D is an immediate corollary of Theorem 1.A and the equimeasurability of rearrangements. Proofs of Theorem 1.D are in [Chi], [CZ], [Ka], [Bae].

Readers interested in further results are referred to [Bae] and [Ka].

1.4. Proof of Theorem 1.A. *First step.* Let E be a measurable subset of $\mathbb{R}^n$, and let v be a nonnegative measurable function defined in $\mathbb{R}^n$. We claim that

$$\int_E v\,dx \le \int_0^{m(E)} v^*(s)\,ds. \tag{1.12}$$

There is no loss of generality in assuming that the measure of E is finite and v is integrable. Proposition (1.xvii) gives

$$v = \int_0^\infty \chi_{\{y\in\mathbb{R}^n:\ v(y)>t\}}\,dt,$$

$$v^* = \int_0^\infty \chi_{[0,m\{y\in\mathbb{R}^n:\ v(y)>t\}[}\,dt.$$

Hence

$$\int_E v\,dx = \int_0^\infty m\big(E\cap\{y\in\mathbb{R}^n\colon v(y)>t\}\big)\,dt,$$

$$\int_0^{m(E)} v^*(s)\,ds = \int_0^\infty \min\big\{m(E), m\{y\in\mathbb{R}^n\colon v(y)>t\}\big\}\,dt.$$

Inequality (1.12) follows. Note that (1.12) coincides with inequality (1.8.a) in the case where u is the characteristic function of E — indeed, $(\chi_E)^* = \chi_{[0,m(E)[}$ by the very definition of decreasing rearrangement.

Second step. Let u and v be nonnegative and measurable. There is no loss of generality in assuming that u and v are integrable. Proposition (1.xvii) gives

$$u = \int_0^\infty \chi_{\{y\in\mathbb{R}^n:\ u(y)>t\}}\,dt,$$

$$u^* = \int_0^\infty \chi_{[0,m\{x\in\mathbb{R}^n:\ u(x)>t\}[}\,dt.$$

Hence

$$\int_{\mathbb{R}^n} u(x)v(x)\,dx = \int_0^\infty dt \int_{\{x\in\mathbb{R}^n:\ u(x)>t\}} v(x)\,dx,$$

$$\int_0^\infty u^*(s)v^*(s)\,ds = \int_0^\infty dt \int_0^{m\{x\in\mathbb{R}^n:\ u(x)>t\}} v^*(s)\,ds.$$

On the other hand, the previous step — inequality (1.12) — tells us that

$$\int_{\{x\in\mathbb{R}^n:\ u(x)>t\}} v(x)\,dx \le \int_0^{m\{x\in\mathbb{R}^n:\ u(x)>t\}} v^*(s)\,ds$$

for every nonnegative t. Inequality (1.8.a) follows. Inequality (1.8.b) follows too, since

$$\int_0^\infty u^*(s)v^*(s)\,ds = \int_{\mathbb{R}^n} u^\bigstar(x)v^\bigstar(x)\,dx$$

by the very definition of symmetric rearrangement. $\square$

1.5. Proof of Theorem 1.C. We have

$$\int_{\mathbb{R}^n} \Phi\big(|\nabla u(x)|\big)\,dx \ge \int_0^\infty ds \frac{d}{ds} \int_{\{x\in\mathbb{R}^n:\ |u(x)|>u^*(s)\}} \Phi\big(|\nabla u(x)|\big)\,dx,$$

since

$$\int_{\{x\in\mathbb{R}^n:\ |u(x)|>u^*(s)\}} \Phi\big(|\nabla u(x)|\big)\, dx$$

increases monotonically from 0 to the integral of $\Phi(|\nabla u|)$ over $\mathbb{R}^n$ as s increases from 0 to ∞. On the other hand,

$$\int_0^\infty \Phi\Big(-n\kappa_n^{1/n} s^{1-1/n} \frac{du^*}{ds}(s)\Big)\, ds = \int_{\mathbb{R}^n} \Phi\big(|\nabla u^\star(x)|\big)\, dx$$

by the very definition of $u^\star$. Thus Theorem 1.C follows immediately from the next lemma. $\square$

Lemma 1.E. *Suppose u is sufficiently smooth* — e.g., Lipschitz continuous — *and decays at infinity* — i.e., the measure of $\{x \in \mathbb{R}^n : |u(x)| > t\}$ is finite for every positive t. *Then* (i) u^* *is locally absolutely continuous in* $]0,\infty[$; (ii) *the following inequality*

$$\frac{d}{ds}\int_{\{x\in\mathbb{R}^n:\ |u(x)|>u^*(s)\}} \Phi\big(|\nabla u(x)|\big)\, dx \ge \Phi\Big(-n\kappa_n^{1/n} s^{1-1/n} \frac{du^*}{ds}(s)\Big) \tag{1.13}$$

holds for almost every positive s. Here Φ is any Young function — i.e., Φ maps $[0,\infty[$ into $[0,\infty[$, $\Phi(0)=0$, Φ is increasing and convex.

Proof of Lemma 1.E. The basic ingredients involved are: the coarea formula; the isoperimetric theorem in $\mathbb{R}^n$; Jensen's inequality.

A *coarea formula* claims that if u is Lipschitz continuous and f is integrable, then

$$\int_{\mathbb{R}^n} f(x)|\nabla u(x)|\, dx = \int_0^\infty dt \int_{\{x\in\mathbb{R}^n:\ |u(x)|=t\}} f(x) H_{n-1}(\,dx).$$

In a sense, this formula amounts to saying that the distance between level surfaces of u is inversely proportional to $|\nabla u|$. A proof of this formula appears in [Fe]. The *isoperimetric theorem* in $\mathbb{R}^n$ claims that if E is a measurable subset of $\mathbb{R}^n$ and the measure of E is finite, then

$$H_{n-1}(\partial E) \ge n\kappa_n^{1/n}[m(E)]^{1-1/n}.$$

Treatments of this theorem appear in [BZa], [Oss], [Ta5]. *Jensen's inequality* says that if Φ is convex, f is integrable and the measure of E is finite, then

$$\frac{1}{m(E)}\int_E \Phi\big(f(x)\big)\,dx \geq \Phi\Big(\frac{1}{m(E)}\int_E f(x)\,dx\Big).$$

Jensen's inequality is presented in [MPF], for instance.

First step. The following inequalities

$$\int_{\{x\in\mathbb{R}^n:\ u^*(a)>|u(x)|>u^*(b)\}} |\nabla u(x)|\,dx \geq n\kappa_n^{1/n}[u^*(a)-u^*(b)], \tag{1.14}$$

$$m\{x\in\mathbb{R}^n\colon u^*(a)>|u(x)|>u^*(b)\} \leq b-a \tag{1.15}$$

hold if $m(\operatorname{sprt} u) > b > a \geq 0$.

Proof of (1.14).

The left-hand side of (1.14)

$=$ by $\boxed{\text{Federer's coarea formula}}$

$$\int_{u^*(b)}^{u^*(a)} H_{n-1}\{x\in\mathbb{R}^n\colon |u(x)|=t\}\,dt$$

$\geq$ by $\boxed{\text{the isoperimetric theorem in } \mathbb{R}^n}$

$$\int_{u^*(b)}^{u^*(a)} n\kappa_n^{1/n}[m\{x\in\mathbb{R}^n\colon |u(x)|\geq t\}]^{1-1/n}\,dt$$

$\geq$ by the monotonicity of the integrand

$$n\kappa_n^{1/n}[m\{x\in\mathbb{R}^n\colon |u(x)|\geq u^*(a)\}]^{1-1/n}[u^*(a)-u^*(b)]$$

$\geq$ by Proposition (1.xiii)

the right-hand side of (1.14). □

Proof of (1.15). Let μ denote the distribution function of u. Proposition (1.v) ensures that the left-hand side of (1.15) equals $\mu\big(u^*(b)\big) - \mu\big(u^*(a)-\big)$, Proposition (1.xiii) ensures that $\mu\big(u^*(b)\big) \le b$ and $\mu\big(u^*(a) - \big) \ge a$. $\square$

Recall that $\operatorname{sprt} u^* = [0, m(\operatorname{sprt} u)]$ as per Proposition (1.xi). Thus, inequalities (1.14) and (1.15) show that u^* is locally *absolutely continuous* in $]0,\infty[$. The first assertion of Lemma 1.E is demonstrated.

Incidentally, inequalities (1.14) and (1.15) give also

$$n\kappa_n^{1/n} a^{1-1/n}[u^*(a) - u^*(b)] \le (b-a) \operatorname{ess\,sup} |\nabla u|$$

whenever $m(\operatorname{sprt} u) > b > a \ge 0$. Hence

$$-n\kappa_n^{1/n} s^{1-1/n} \frac{du^*}{ds}(s) \le \operatorname{ess\,sup} |\nabla u|$$

for almost every nonnegative s, consequently

$$|\nabla u^\bigstar(x)| \le \operatorname{ess\,sup} |\nabla u|$$

for almost every x in $\mathbb{R}^n$.

Second step. The following inequality

$$\frac{d}{ds} \int\limits_{\{x\in\mathbb{R}^n:\ |u(x)|>u^*(s)\}} |\nabla u(x)|\,dx \ge -n\kappa_n^{1/n} s^{1-1/n}\frac{du^*}{ds}(s) \tag{1.16}$$

holds for almost every nonnegative s. In fact, the right-hand side of (1.16) is zero if $s \ge m(\operatorname{sprt} u)$. If $0 \le s < m(\operatorname{sprt} u)$ the left-hand side of (1.16) equals the limit of

$$\frac{1}{h} \int\limits_{\{x\in\mathbb{R}^n:\ u^*(s)\ge|u(x)|>u^*(s+h)\}} |\nabla u(x)|\,dx$$

as h is positive and tends to 0; inequality (1.14) tells us that the last quantity is greater than, or equals

$$n\kappa_n^{1/n} s^{1-1/n} \frac{1}{h}[u^*(s) - u^*(s+h)].$$

Third step. The previous step shows that inequality (1.13) holds for almost every nonnegative s in the case where $\Phi(t) = t$. We are going to show that inequality (1.13) holds for almost every nonnegative s if Φ is *any* Young function.

Arguments of real analysis tell us that there are exactly three alternatives: 1. s belongs to some exceptional set having one-dimensional measure zero; 2. du^*/ds vanishes at s; 3. a neighbourhood of s exists where u^* decreases *strictly*. If either 1 or 2 holds, there is nothing to prove. Thus, suppose alternative 3 is in force.

We claim that

$$h = m\{x \in \mathbb{R}^n : u^*(s) \geq |u(x)| > u^*(s+h)\}. \tag{1.17}$$

if h is positive and small enough. Indeed, let μ denote the distribution function of u. The right-hand side of (1.17) equals $\mu\big(u^*(s+h)\big) - \mu\big(u^*(s)\big)$. Proposition (1.v) gives $\mu(t-) - \mu(t) =$ 1-dimensional measure of $\{r \geq 0 : u^*(r) = t\}$ for every positive t, Proposition (1.xiii) gives $\mu\big(u^*(r)\big) \leq r \leq \mu\big(u^*(r)-\big)$ if $0 \leq r < m(\operatorname{sprt} u)$. Thus assumptions ensure that $\mu\big(u^*(r)\big) = r$ whenever r is close enough to s. The claim follows.

We deduce

$$\frac{1}{h} \int\limits_{\{x\in\mathbb{R}^n :\ u^*(s)\geq|u(x)|>u^*(s+h)\}} \Phi\big(|\nabla u(x)|\big)\, dx$$

$$\geq \qquad \text{by } \boxed{\text{Jensen's inequality}}$$

$$\Phi\Big(\frac{1}{h} \int\limits_{\{x\in\mathbb{R}^n :\ u^*(s)\geq|u(x)|>u^*(s+h)\}} |\nabla u(x)|\, dx\Big).$$

Consequently

$$\begin{aligned} &\frac{d}{ds} \int\limits_{\{x\in\mathbb{R}^n :\ |u(x)|>u^*(s)\}} \Phi\big(|\nabla u(x)|\big)\, dx \\ &\qquad \geq \Phi\Big(\frac{d}{ds} \int\limits_{\{x\in\mathbb{R}^n :\ |u(x)|>u^*(s)\}} |\nabla u(x)|\, dx\Big). \end{aligned} \tag{1.18}$$

Inequalities (1.16) and (1.18) yield (1.13). The proof is complete. □

2. Standard Sobolev inequalities

2.1. Background. Let us take three exponents p, q and r (greater than, or equal to 1) and ask whether a positive constant C exists such that

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\Big[\|u\|_{L^r(\mathbb{R}^n)} + \Big(\int_{\mathbb{R}^n} |\nabla u(x)|^p\,dx\Big)^{1/p}\Big] \tag{2.1.a}$$

for every *test* function u — sufficiently smooth and decaying fast enough at infinity.

An argument of dimensional analysis shows that inequality (2.1.a) is equivalent to the following
(2.1.b)

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\Big[k^{n/r-n/q}\|u\|_{L^r(\mathbb{R}^n)} + k^{n/p-n/q-1}\Big(\int_{\mathbb{R}^n} |\nabla u(x)|^p\,dx\Big)^{1/p}\Big],$$

where k is a parameter having an arbitrary positive value — simply replace $u(x)$ in (2.1.a) by $u(x/k)$, then rescale.

If the exponents of k in (2.1.b) are both positive or both negative, letting k tend to zero or infinity results in a contradiction. Thus, the question has a *negative answer* if $q < r$ and $1/q > 1/p - 1/n$ or $q > r$ and $1/q < 1/p - 1/n$. For instance, inequality (2.1.a) fails to hold if $r = p$ and $q < p$ or $r = p$, $p < n$ and $q > np/(n-p)$.

Suppose $q > r$ and $1/q \geq 1/p - 1/n$. Minimizing the right-hand side of (2.1.b) with respect to k leads to the following conclusions.

(2.i) If $1 \leq p < n$ and $q = np/(n-p)$, (2.1.a) is equivalent to the following inequality

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\Big(\int_{\mathbb{R}^n} |\nabla u(x)|^p\,dx\Big)^{1/p}. \tag{2.2}$$

(2.ii) If $n < p < \infty$ and $q = \infty$, (2.1.a) is equivalent to the following inequality

$$\sup|u| \leq \mathrm{Const}\,\|u\|_{L^r(\mathbb{R}^n)}^{\frac{(p-n)r}{(p-n)r+np}}\Big(\int_{\mathbb{R}^n} |\nabla u(x)|^p\,dx\Big)^{\frac{np}{(p-n)r+np}}, \tag{2.3}$$

where Const stands for

$$C\Big(1 - \frac{n}{p} + \frac{n}{r}\Big)\Big(\frac{r}{n}\Big)^{\frac{np}{(p-n)r+np}}\Big(\frac{p}{p-n}\Big)^{\frac{(p-n)r}{(p-n)r+np}}.$$

(2.iii) If $1/q > 1/p - 1/n$ and $r < q < \infty$, (2.1.a) is equivalent to the following inequality

$$\|u\|_{L^q(\mathbb{R}^n)} \leq \text{Const}\, \|u\|_{L^r(\mathbb{R}^n)}^{\frac{1-n/p+n/q}{1-n/p+n/r}} \Big(\int_{\mathbb{R}^n} |\nabla u(x)|^p \, dx \Big)^{\frac{n/r-n/q}{1-n/p+n/r}}, \tag{2.4}$$

where Const stands for

$$C\Big(\frac{1}{n} - \frac{1}{p} + \frac{1}{r}\Big)\Big(\frac{1}{r} - \frac{1}{q}\Big)^{-\frac{n/r-n/q}{1-n/p+n/r}} \Big(\frac{1}{n} - \frac{1}{p} + \frac{1}{q}\Big)^{-\frac{1-n/p+n/q}{1-n/p+n/r}}.$$

(2.2), (2.3) and (2.4) are samples of the so-called *Sobolev inequalities*. Sobolev inequalities are a customary tool in functional analysis, calculus of variations, partial differential equations. Basically, they inform that the membership to a Sobolev space implies ipso facto extra properties such as higher integrability or boundedness. Thus, Sobolev inequalities are prototypal regularization theorems. A presentation of Sobolev inequalities appears in [So1] and [So2]. Exhaustive proofs appear, e.g., in [Ada], [Maz], [Zie]. It can be demonstrated that (2.2) holds if $1 \leq p < n$ and $1/q = 1/p - 1/n$, (2.3) holds if $p > n$ and $r \geq 1$, (2.4) holds if $1/q > 1/p - 1/n$ and $q > r$.

2.2. A special case. The case where $p = 1$ and $q = n/(n-1)$, i.e., the following Sobolev inequality

$$\Big(\int_{\mathbb{R}^n} |u(x)|^{n/(n-1)} \, dx \Big)^{1-1/n} \leq C \int_{\mathbb{R}^n} |\nabla u(x)| \, dx, \tag{2.5.a}$$

deserves special attention. Theorem 2.A below — which appeared in [FF] — shows that (2.5.a) is nothing but an alternative version of the isoperimetric theorem in $\mathbb{R}^n$. A variant of Theorem 2.A — where Lebesgue measure is replaced by Gauss measure — is offered in [PT].

Theorem 2.A. *The smallest constant C, such that inequality* (2.5.a) *holds for every test function u, is given by*

$$1/C = n\kappa_n^{1/n}. \tag{2.5.b}$$

Proof. First step. Assume a positive constant C exists such that (2.5.a) holds for every test function u, and let E by any nice bounded subset of $\mathbb{R}^n$. We claim that

$$[m(E)]^{1-1/n} \leq C \cdot H_{n-1}(\partial E).$$

In fact, consider a mollified version, u^ϵ, of the characteristic function of E. In other words, let $\epsilon > 0$ and

$$u_\epsilon(x) = \int\limits_{\mathbb{R}^n} J_\epsilon(x-y)\chi_E(y)\,dy$$

— here $J_\epsilon(x) = \epsilon^{-n}J(x/\epsilon)$, and J is any nonnegative compactly supported smooth function whose integral over $\mathbb{R}^n$ is 1. Well-known properties of mollifiers ensure that $u_\epsilon \to \chi_E$ in $L^{n/(n-1)}(\mathbb{R}^n)$ as $\epsilon{\downarrow}0$, therefore

$$\Big(\int\limits_{\mathbb{R}^n} |u_\epsilon(x)|^{n/(n-1)}\,dx\Big)^{1-1/n} \to [m(E)]^{1-1/n}$$

as $\epsilon{\downarrow}0$. On the other hand, the Gauss–Green formulas give

$$\nabla u_\epsilon(x) = \int\limits_{\partial E} J_\epsilon(x-y) \times (\text{inner unit normal to } \partial E \text{ at } y) H_{n-1}(\,dy);$$

hence $|\nabla u_\epsilon(x)| \leq \int_{\partial E} J_\epsilon(x-y)H_{n-1}(\,dy)$, consequently

$$\int\limits_{\mathbb{R}^n} |\nabla u_\epsilon(x)|\,dx \leq H_{n-1}(\partial E)$$

for every positive ϵ. Thus, replacing u in (2.5.a) by u_ϵ then letting $\epsilon{\downarrow}0$ results in the claimed inequality.

The same inequality implies

$$1/C \leq n\kappa_n^{1/n},$$

since $H_{n-1}(\partial E)[m(E)]^{-1+1/n} = n\kappa_n^{1/n}$ if E is a ball.

In conclusion, we have shown that if inequality (2.5.a) holds for every test function u then $1/C$ cannot be larger than $n\kappa_n^{1/n}$.

Second step. Assume C is defined by (2.5.b), and let u be any test function. We claim that inequality (2.5.a) actually holds. The main ingredients involved in the present proof are: the standard isoperimetric theorem in $\mathbb{R}^n$; a coarea formula. See Subsection 1.5 for information on these topics.

Layer-cake formula — formula (1.5) — gives

$$|u| = \int_0^\infty \chi\{x \in \mathbb{R}^n \colon |u(x)| > t\}\,dt;$$

therefore we deduce via Minkowski's inequality that

$$\Big(\int_{\mathbb{R}^n} |u(x)|^{n/(n-1)}\,dx\Big)^{1/1-n} \le \int_0^\infty [m\{x\in\mathbb{R}^n\colon |u(x)|>t\}]^{1-1/n}\,dt.$$

The standard isoperimetric theorem in $\mathbb{R}^n$ yields

$$n\kappa_n^{1/n}[m\{x\in\mathbb{R}^n\colon |u(x)|>t\}]^{1-1/n} \le H_{n-1}\{x\in\mathbb{R}^n\colon |u(x)|=t\}$$

for every positive t. The coarea formula of Federer says that

$$\int_0^\infty H_{n-1}\{x\in\mathbb{R}^n\colon |u(x)|=t\}\,dt = \int_{\mathbb{R}^n} |\nabla u(x)|\,dx.$$

In conclusion

$$n\kappa_n^{1/n}\Big(\int_{\mathbb{R}^n} |u(x)|^{n/(n-1)}\,dx\Big)^{1-1/n} \le \int_{\mathbb{R}^n} |\nabla u(x)|\,dx,$$

as claimed.

The proof of Theorem 2.A is complete. □

2.3. Sharp constants. The next theorem — which appeared in [Au], [Lb], [Ta1] — gives a *sharp form* of Sobolev inequality (2.2).

Theorem 2.B. *Assume* $1<p<n$ *and* $1/q = 1/p - 1/n$. *Then every test function* u *obeys the following inequality*

$$\Big(\int_{\mathbb{R}^n} |u(x)|^q\,dx\Big)^{1/q} \le C\Big(\int_{\mathbb{R}^n} |\nabla u(x)|^p\,dx\Big)^{1/p}, \tag{2.6.a}$$

provided C *is given by*

$$1/C = \sqrt{\pi}\,n^{1/p}\Big(\frac{n-p}{p-1}\Big)^{1-1/p}\Big[\frac{\Gamma(n/p)\Gamma(1+n-n/p)}{\Gamma(n)\Gamma(1+n/2)}\Big]^{1/n}. \tag{2.6.b}$$

Equality holds in (2.6.a) *if* C *is given by* (2.6.b) *and* u *is given by*

$$u(x) = [1+|x|^{p/(p-1)}]^{1-n/p}. \tag{2.6.c}$$

Proof, outlined. The equimeasurability of rearrangements gives

$$\int_{\mathbb{R}^n} |u|^q \, dx = \int_{\mathbb{R}^n} |u^\bigstar|^q \, dx;$$

Theorem 1.C gives

$$\int_{\mathbb{R}^n} |\nabla u|^p \, dx \geq \int_{\mathbb{R}^n} |\nabla u^\bigstar|^p \, dx.$$

We deduce a decisive information, i.e., minimizing the ratio

$$\Big(\int_{\mathbb{R}^n} |\nabla u|^p \, dx \Big)^{1/p} \Big/ \Big(\int_{\mathbb{R}^n} |u|^q \, dx \Big)^{1/q}$$

among *all* test functions u amounts to minimizing the same ratio in the special class of functions that are *nonnegative, radial, radially decreasing and Lipschitz continuous.* Hence the goal is identifying minimizers and the minimum value of the following ratio

$$(n\kappa_n)^{1/n} \Big(\int_0^\infty [-u'(r)]^p r^{n-1} \, dr \Big)^{1/p} \Big/ \Big(\int_0^\infty [u(r)]^q r^{n-1} \, dr \Big)^{1/q},$$

where r is a *real* variable and u is a *nonnegative, decreasing, Lipschitz continuous* function of r *only.* This goal can be attained via techniques of the *one-dimensional* calculus of variations. It turns out that the relevant minimum is exactly the right-hand side of equation (2.6.b) and a relevant minimizer is given by $u(r) = [1 + r^{p/(p-1)}]^{1-n/p}$. Details are in [Ta1]. $\square$

2.4. Sharp constants, continued. In this subsection we present theorems related to inequality (2.3).

Theorem 2.C. *Assume $n < p < \infty$. Then every test function u obeys the following inequality*

$$\text{(2.7.a)} \qquad \sup |u| \leq C \Big(\int_{\mathbb{R}^n} |u(x)| \, dx \Big)^{\frac{n-(n-1)q}{n+q}} \Big(\int_{\mathbb{R}^n} |\nabla u(x)|^p \, dx \Big)^{\frac{n(q-1)}{n+q}},$$

where $q = p/(p-1)$ and

$$\text{(2.7.b)} \qquad \begin{aligned} C =& \Big(\frac{1}{n} + \frac{1}{q} \Big) \Big(\frac{1}{n} - \frac{1}{p} \Big)^{\frac{(n-1)q-n}{n+q}} \\ & \times (n^n \kappa_n)^{-\frac{q}{n+q}} \Big\{ \frac{\Gamma(1+q)\Gamma\big(1-q(1-1/n)\big)}{\Gamma(2+q/n)} \Big\}^{\frac{n}{n+q}}. \end{aligned}$$

Equality holds in (2.7.a) *if* C *is given by* (2.7.b) *and* u *is given by*

$$
(2.7.c) \qquad u(x) = \begin{cases} \int\limits_{|x|}^{1} r^{-(n-1)(q-1)}(1-r^n)^{q-1}\,dr & \text{if } |x|<1, \\ 0 & \text{if } |x|\geq 1. \end{cases}
$$

Proof. We have

$$
u^*(0) = \frac{1}{k}\int\limits_0^k u^*(s)\,ds + \int\limits_0^k \Big(1-\frac{s}{k}\Big)\Big[-\frac{du^*}{ds}(s)\Big]\,ds
$$

since Lemma 1.E guarantees that u^* is absolutely continuous. Consequently,

$$
\begin{aligned}
u^*(0) \leq{}& k^{-1}\int\limits_0^\infty u^*\,ds \\
&+ k^{1/n-1/p}\Big\{\frac{\Gamma(1+q)\Gamma\big(1-q(1-1/n)\big)}{\Gamma(2+q/n)}\Big\}^{1/q} \\
&\times\Big\{\int\limits_0^\infty\Big[-s^{1-1/n}\frac{du^*}{ds}(s)\Big]^p\,ds\Big\}^{1/p}
\end{aligned}
$$

by Hölder's inequality. Here k is any positive parameter. Minimizing with respect to k yields

$$
\begin{aligned}
u^*(0) \leq{}& \Big(\frac{1}{n}+\frac{1}{q}\Big)\Big(\frac{1}{n}-\frac{1}{p}\Big)^{\frac{(n-1)q-n}{n+q}}\Big\{\frac{\Gamma(1+q)\Gamma\big(1-q(1-1/n)\big)}{\Gamma(2+q/n)}\Big\}^{\frac{n}{n+q}} \\
&\times\Big\{\int\limits_0^\infty u^*\,ds\Big\}^{\frac{n-(n-1)q}{n+q}}\Big\{\int\limits_0^\infty\Big[-s^{1-1/n}\frac{du^*}{ds}(s)\Big]^p\,ds\Big\}^{\frac{n(q-1)}{n+q}}.
\end{aligned}
$$

A property or rearrangements gives

$$
u^*(0) = \sup|u|;
$$

the equimeasurability of rearrangements gives

$$
\int\limits_0^\infty u^*\,ds = \int\limits_{\mathbb{R}^n}|u|\,dx;
$$

Theorem 1.C gives

$$\int_0^\infty \Big[-n\kappa_n^{1/n} s^{1-1/n} \frac{du^*}{ds}(s)\Big]^p \, ds \leq \int_{\mathbb{R}^n} |\nabla u(x)|^p \, dx.$$

The argument above shows that inequality (2.7.a) holds if the relevant constant is defined as in (2.7.b). It shows also that equality holds in (2.7.a) if (2.7.b) is in force and u satisfies

$$-\frac{du^*}{ds}(s) = \begin{cases} s^{-(1-1/n)q}(1-s)^{q-1} & \text{if } \ 0 < s < 1, \\ 0 & \text{if } \ s \geq 1 \end{cases}$$

The proof is complete. $\square$

Theorem 2.D. *Assume $\infty > p > n > q > 1$. Then every test function u obeys the following inequality*

(2.8.a) $$\sup |u| \leq C \Big(\int_{\mathbb{R}^n} |\nabla u(x)|^p \, dx \Big)^{\frac{n-q}{n(p-q)}} \Big(\int_{\mathbb{R}^n} |\nabla u(x)|^q \, dx \Big)^{\frac{p-n}{n(p-q)}}.$$

where

(2.8.b)
$$\begin{aligned} C =& n^{1-1/n} \kappa_n^{-1/n} (p-q)[p(n-q)]^{-\frac{p(n-q)}{n(p-q)}} [q(p-n)]^{-\frac{q(p-n)}{n(p-q)}} \\ & \times \Big(\frac{p-1}{p-n} \Big)^{\frac{(p-1)(n-q)}{n(p-q)}} \Big(\frac{q-1}{n-q} \Big)^{\frac{(q-1)(p-n)}{n(p-q)}}. \end{aligned}$$

Equality holds in (2.8.a) *if C is given by* (2.8.b) *and u is given by*
(2.8.c)
$$u^*(s) = \begin{cases} (n-1)(p-q) - (n-q)(p-1) s^{\frac{p-n}{n(p-1)}} & \textit{if } \ 0 < s < 1, \\ (q-1)(p-n) s^{-\frac{n-q}{n(q-1)}} & \textit{if } \ s \geq 1. \end{cases}$$

Proof. We have

$$u^*(0) = \int_0^\infty \Big[-\frac{du^*}{ds}(s) \Big] \, ds$$

since we know from Lemma 1.E that u^* is absolutely continuous. Therefore

$$u^*(0) = \Big(\int_0^k + \int_k^\infty \Big) s^{-1+1/n} \Big[-s^{1-1/n} \frac{du^*}{ds}(s) \Big] \, ds,$$

hence

$$\begin{aligned} u^*(0) \leq & k^{1/n-1/p}\Big[\frac{n(p-1)}{p-n}\Big]^{1-1/p}\Big\{\int_0^\infty\Big[-s^{1-1/n}\frac{du^*}{ds}(s)\Big]^p\,ds\Big\}^{1/p} \\ & + k^{1/n-1/q}\Big[\frac{n(q-1)}{n-q}\Big]^{1-1/q}\Big\{\int_0^\infty\Big[-s^{1-1/n}\frac{du^*}{ds}(s)\Big]^q\,ds\Big\}^{1/q} \end{aligned}$$

by Hölder's inequality. Here k is any positive parameter. A property of rearrangements gives

$$u^*(0) = \sup|u|;$$

Theorem 1.C gives

$$\int_0^\infty\Big[-n\kappa_n^{1/n}s^{1-1/n}\frac{du^*}{ds}(s)\Big]^p\,ds \leq \int_{\mathbb{R}^n}|\nabla u(x)|^p\,dx,$$
$$\int_0^\infty\Big[-n\kappa_n^{1/n}s^{1-1/n}\frac{du^*}{ds}(s)\Big]^q\,ds \leq \int_{\mathbb{R}^n}|\nabla u(x)|^q\,dx.$$

Thus we have shown

$$\begin{aligned} n\kappa_n^{1/n}\cdot\sup|u| \leq & k^{1/n-1/p}\Big[\frac{n(p-1)}{p-n}\Big]^{1-1/p}\Big(\int_{\mathbb{R}^n}|\nabla u(x)|^p\,dx\Big)^{1/p} \\ & + k^{1/n-1/p}\Big[\frac{n(q-1)}{n-q}\Big]^{1-1/q}\Big(\int_{\mathbb{R}^n}|\nabla u(x)|^q\,dx\Big)^{1/q}. \end{aligned}$$

Minimizing with respect to k yields (2.8.a) and (2.8.b).

The argument above shows also that equality holds in (2.8.a) if (2.8.b) is in force and u obeys

$$-\frac{du^*}{ds}(s) = \begin{cases} s^{-\frac{p(n-1)}{n(p-1)}} & \text{if } 0 < s < 1, \\ s^{-\frac{q(n-1)}{n(q-1)}} & \text{if } s \geq 1. \end{cases}$$

The proof is complete. $\square$

Theorem 2.E. *Assume $n < p < \infty$. Let u be a test function and the support of u have finite measure. Then*

$$\operatorname{ess\,sup}|u| \leq C[m(\operatorname{sprt} u)]^{1/n-1/p}\Big(\int_{\mathbb{R}^n} |\nabla u(x)|^p\,dx\Big)^{1/p}, \tag{2.9.a}$$

where

$$C = n^{-1/p}\kappa_n^{-1/n}\Big(\frac{p-1}{p-n}\Big)^{1-1/p}. \tag{2.9.b}$$

Equality holds in (2.8.a) *if C is given by* (2.8.b) *and u is given by*

$$u(x) = \begin{cases} 1-|x|^{(p-n)/(p-1)} & \text{if } |x|<1,\\ 0 & \text{if } |x|\geq 1. \end{cases} \tag{2.9.c}$$

Proof. We have

$$u^*(0) = \int_0^{m(\operatorname{sprt} u)} \Big[-\frac{du^*}{ds}(s)\Big]\,ds$$

since u^* is absolutely continuous. Hence

$$u^*(0) \leq n^{1-1/p}\Big(\frac{p-1}{p-n}\Big)^{1-1/p}[m(\operatorname{sprt} u)]^{1/n-1/p} \times \Big\{\int_0^\infty \Big[-s^{1-1/n}\frac{du^*}{ds}(s)\Big]^p\,ds\Big\}^{1/p}$$

by Hölder's inequality. Furthermore,

$$u^*(0) = \operatorname{ess\,sup}|u|$$

by a property of rearrangements, and

$$\int_0^\infty \Big[-n\kappa_n^{1/n}s^{1-1/n}\frac{du^*}{ds}(s)\Big]^p\,ds \leq \int_{\mathbb{R}^n}|\nabla u(x)|^p\,dx$$

by Theorem 1.C.

The argument above shows that inequality (2.9.a) holds if the relevant constant is defined as in (2.9.b). It shows also that equality holds in (2.9.a) if (2.9.b) is in force and u satisfies

$$-\frac{du^*}{ds}(s) = \begin{cases} s^{-(1-1/n)/(1-1/p)} & \text{if } 0<s<1,\\ 0 & \text{if } s\geq 1. \end{cases}$$

The proof is complete. $\square$

2.5. Remarks. (2.5.i) A sharp form of inequality (2.3) can be stated as follows. If $n < p < \infty$ and $1 \le r < \infty$, the smallest constant C such that the inequality

$$(\sup|u|)^{np+(p-n)r} \le C\Big(\int_{\mathbb{R}^n} |u|^r dx\Big)^{p-n}\Big(\int_{\mathbb{R}^n} |\nabla u|^p\, dx\Big)^n \tag{2.10.a}$$

holds for every test function u is given by
(2.10.b)

$$\frac{1}{C} = n^{np}\kappa_n^p\Big[\frac{p-n}{n(p-1)}\Big]^{(n-1)p}\Big(\int_0^\infty U^r t^{\frac{(n-1)p}{p-n}}\, dt\Big)^{p-n}\Big(\int_0^\infty (-U')^p\, dt\Big)^n,$$

where U is a nonnegative decreasing solution of the differential equation

$$\frac{d}{dt}[-U'(t)]^{p-1} + t^{\frac{(n-1)p}{p-n}}[U(t)]^{r-1} = 0 \tag{2.10.c}$$

such that $U(0) = 1$ and $U(t) \to 0$ as $t \to +\infty$.

(2.5.ii) Inequality (2.4) is easily derived from (2.2), (2.3) and Hölder's inequality. Thus, Theorem 2.A tells us that inequality (2.4) does hold if $p = 1$ and $1 < q < n/(n-1)$, and the smallest relevant constant is given by

$$1/C = n\kappa_n^{1/n}.$$

However, a sharp form of (2.4) is not known in general — partial results are in [Le].

3. Inequalities relating u^*, $|\nabla u|^*$ and $(\Delta u)^*$

3.1. Introduction. A *rearrangement invariant function space* is a Banach space $\mathcal{X}$ of real-valued measurable functions having the following properties:
(i) If u belongs to $\mathcal{X}$ and v is a measurable function such that $u \ge |v|$, then v is in $\mathcal{X}$ and $\|u\| \ge \|v\|$;
(ii) If u belongs to $\mathcal{X}$ and v is equidistributed with u, then v is in $\mathcal{X}$ and $\|u\| = \|v\|$.

Rearrangement invariant function spaces were introduced in [LZ], and include Lebesgue, Orlicz and Lorentz spaces.

The theorems from the present Section provide with tools for investigating inequalities à la Sobolev, i.e.,

$$\frac{\text{a norm of } |\nabla u|}{\text{a stronger norm of } u} \ge \text{Const independent of } u$$

or

$$\frac{\text{a norm of } \Delta u}{\text{a stronger norm of } u} \geq \text{Const independent of } u,$$

in the case where rearrangement invariant function spaces are involved.

Applications are given in Section 4.

3.2. Statements.

Theorem 3.A. *Let u be a real-valued function defined in $\mathbb{R}^n$. Assume u is sufficiently smooth* — e.g., Lipschitz continuous — *and decays at infinity* — i.e., the measure of $\{x \in G\colon |u(x)| > t\}$ is finite for every positive t. *Let $V = m(\operatorname{sprt} u)$. Then*

$$u^*(s) \leq n^{-1}\kappa_n^{-1/n} \int_0^{V-s} (s+t)^{-1+1/n} |\nabla u|^*(t)\, dt \tag{3.1.a}$$

and

$$u^*(s) \geq u^*(0) - n^{-1}\kappa_n^{-1/n} \int_0^{s} t^{-1+1/n} |\nabla u|^*(t)\, dt \tag{3.1.b}$$

for almost every s such that $0 \leq s < V$. Equality holds in (3.1.a) *and* (3.1.b) *if u is given by*

$$u(x) = \begin{cases} 1 - |x| & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases} \tag{3.1.c}$$

Theorem 3.B. *Let G be an open subset of $\mathbb{R}^n$, u a real-valued function defined in G. Assume u is smooth enough* — e.g., twice continuously differentiable — *and vanishes on the boundary of G* — i.e., the measure of $\{x \in G\colon |u(x)| > t\}$ is finite and the closure of $\{x \in G\colon |u(x)| > t\}$ is contained in G for every positive t. *Let v be defined by*

$$v(s) = n^{-2}\kappa_n^{-2/n} \int_s^{m(G)} dr\; r^{-2+2/n} \int_0^{r} (\Delta u)^*(t)\, dt \tag{3.2}$$

for every s such that $0 \leq s < m(G)$. Assertions: (i) *the following inequality*

$$u^*(s) \leq v(s) \tag{3.3.a}$$

holds for every s *such that* $0 \le s < m(G)$*:* (ii) *the following inequality*

$$\int_G |\nabla u|^p \, dx \le \int_0^{m(G)} \Big[- n\kappa_n^{1/n} s^{1-1/n} \frac{dv}{ds}(s) \Big]^p \, ds \tag{3.3.b}$$

holds for every p *such that* $0 < p \le 2$*:* (iii) *equality holds in* (3.3.a) *and* (3.3.b) *if* G *is a ball and* Δu *is radial and radially decreasing.*

Theorem 3.B is a prototype: generalizations of it are available, in which Laplace operator Δ is replaced by elliptic second-order partial differential operators (linear or nonlinear) having a divergence structure. These generalizations are a tool for investigating a priori estimates of solutions to elliptic second-order boundary value problems — see, e.g., [Ta3] and the references therein.

A version of Theorem 3.A appeared in [Ta4]; a shortened proof is offered in the next subsection. For a proof of Theorem 3.B we refer to [Ta2].

3.3. A proof of Theorem 3.A. Lemma 1.E tells us that u^* is absolutely continuous and

$$-n\kappa_n^{1/n} \frac{du^*}{dt}(t) \le t^{-1+1/n} \frac{d}{dt} \int_{\{x \in \mathbb{R}^n :\ |u(x)| > u^*(t)\}} |\nabla u(x)| \, dx$$

for almost every nonnegative t. A property of rearrangements gives sprt u^* $= [0, V]$. We deduce

$$n\kappa_n^{1/n} u^*(s) \le \int_s^V dt \ t^{-1+1/n} \frac{d}{dt} \int_{\{x \in \mathbb{R}^n :\ u^*(t) < |u(x)| < u^*(s)\}} |\nabla u(x)| \, dx$$

for almost every s such that $0 \le s < V$.

4. Lorentz spaces

4.1. Introduction. Rearrangement invariant function spaces are mentioned in Subsection 3.1. *Lorentz spaces* are significant examples of these spaces. Lorentz spaces play a role in the theory of interpolation of operators and in partial differential equations; they were introduced in [Lo1] and [Lo2] and exhaustively treated in [Hu].

4.2. Definitions and basic properties. (4.i) Let G be a measurable subset of $\mathbb{R}^n$. If u is a real-valued measurable function defined in G, then $\overline{u}$ is defined by

(4.1)
$$\overline{u}(s) = \begin{cases} \sup\Big\{[m(E)]^{-1} \int_E |u(x)|\,dx \colon E \subseteq G, m(E) \geq s\Big\} & \text{if } \ 0 \leq s \leq m(G), \\ \frac{1}{s}\int_G |u(x)|\,dx & \text{if } \ s > m(G) \end{cases}$$

for every nonnegative s.

(4.ii) Let $1 < p < \infty$ and $1 \leq q < \infty$. The *Lorentz* $L(p,q)$ *space* is the collection of all real-valued measurable functions u defined in G such that

$$\Big\{ \int_0^\infty [s^{1/p}\overline{u}(s)]^q \frac{ds}{s} \Big\}^{1/q} < \infty; \tag{4.2.a}$$

$L(p,q)$ is a linear space and an appropriate norm in $L(p,q)$ is defined by

$$\|u\|_{L(p,q)} = \Big\{ \int_0^\infty [s^{1/p}\overline{u}(s)]^q \frac{ds}{s} \Big\}^{1/q}. \tag{4.2.b}$$

The usual modification has to be made if $q = \infty$. Accordingly, $L(p,\infty)$ — also called *weak* L^p *space* — is the collection of all real-valued measurable functions u such that

$$\sup\Big\{[m(E)]^{-1+1/p} \int_E |u(x)|\,dx \colon E \subseteq G\Big\} < \infty,$$

and

$$\|u\|_{L(p,\infty)} = \sup\Big\{[m(E)]^{-1+1/p} \int_E |u(x)|\,dx \colon E \subseteq G\Big\}.$$

The following propositions hold:

(4.iii) $\overline{u}$ is a *decreasing* function defined in $[0,\infty[$, $\overline{u}(0) = \operatorname{ess\,sup}|u|$ and $\overline{u}(+\infty) = 0$.

(4.iv) We have

$$\overline{u}(s) = \frac{1}{s}\int_0^s u^*(t)\,dt \tag{4.3}$$

for every positive s.

Proof of Proposition (4.iv). Propositions from Subsection 1.2 inform that $\operatorname{sprt} u^* = [0, m(\operatorname{sprt} u)]$ and $\int_0^\infty u^*(t)\,dt = \int_G |u(x)|\,dx$. Hence equation (4.3) does hold if $s > m(G)$. We shall demonstrate that if $0 < s \le m(G)$ the sup in (4.1) is actually a maximum and equals the right-hand side of (4.3).

First step. Let $0 < s \le m(G)$. We have

$$[m(E)]^{-1} \int_E |u(x)|\,dx \le s^{-1} \int_0^s u^*(t)\,dt$$

for every measurable set E such that $E \subseteq G$ and $m(E) \le s$. Indeed, Theorem 1.A tells us that

$$\int_E |u(x)|\,dx \le \int_0^{m(E)} u^*(t)\,dt$$

if E is any measurable subset of G. As is easy to see, the monotonicity of u^* implies

$$[m(E)]^{-1} \int_0^{m(E)} u^*(t)\,dt \le s^{-1} \int_0^s u^*(t)\,dt$$

if $m(E) \ge s > 0$.

Second step. Let $0 \le s \le m(G)$. A measurable set E exists such that

$$\int_E |u(x)|\,dx = \int_0^{m(E)} u^*(t)\,dt,$$

and $E \subseteq G$ and $m(E) = s$. Indeed, observe that

$$m\{x \in G \colon |u(x)| > u^*(s)\} \le s \le m\{x \in G \colon |u(x)| \ge u^*(s)\}$$

and recall that the *Lebesgue n-dimensional measure m is free from atoms.* The former property is a consequence of Propositions (1.ix) and (1.xiii): the latter is crucial in the present setting. Then a measurable set E exists such that

$$\{x \in G \colon |u(x)| > u^*(s)\} \subseteq E \subseteq \{x \in G \colon |u(x)| \ge u^*(s)\}$$

and $m(E) = s$.

Such a set E satisfies the equation

$$\int_E |u(x)|\,dx$$
$$= \int_{\{x\in G:\ |u(x)|>u^*(s)\}} |u(x)|\,dx + u^*(s)[m(E) - m\{x \in G\colon |u(x)| > u^*(s)\}].$$

We have also

$$\int_0^{m(E)} u^*(t)\,dt$$
$$= \int_0^{m\{x\in G:\ |u(x)|>u^*(s)\}} u^*(t)\,dt + u^*(s)[m(E) - m\{x \in G\colon |u(x)| > u^*(s)\}]$$

by Proposition (1.xiv). The equimeasurability of u and u^* — Proposition (1.xi) — implies

$$\int_{\{x\in G:\ |u(x)|>u^*(s)\}} |u(x)|\,dx = \int_0^{m\{x\in G:\ |u(x)|>u^*(s)\}} u^*(t)\,dt.$$

Therefore the set E in hand does the job.

The proof is complete. □

(4.v) If $p > 1$ then

$$\|u\|_{L(p,1)} = \frac{p}{p-1}\int_0^\infty s^{1/p}u^*(s)\frac{ds}{s}; \tag{4.4.a}$$

if $p > 1$ and $1 < q \le \infty$, then

$$\Big(1 - \frac{1}{p}\Big)\|u\|_{L(p,q)} \le \Big\{\int_0^\infty \big(s^{1/p}u^*(s)\big)^q\frac{ds}{s}\Big\}^{1/q} \le \|u\|_{L(p,q)} \tag{4.4.b}$$

— thus, if $p > 1$ the functional whose values are

$$\Big\{ \int_0^\infty [s^{1/p} u^*(s)]^q \frac{ds}{s} \Big\}^{1/q}$$

is equivalent to the standard norm in $L(p, q)$.

Proof of Proposition (4.v). Equation (4.4.a) is an immediate consequence of equation (4.2.b) and Proposition (4.iv). Proposition (4.iv) yields $\overline{u} \geq u^*$, since u^* decreases monotonically. Proposition (4.iv) and Theorem 330 from [HLP] give

$$\Big\{ \int_0^\infty [s^{1/p} \overline{u}(s)]^q \frac{ds}{s} \Big\}^{1/q} \leq \frac{p}{p-1} \Big\{ \int_0^\infty [s^{1/p} u^*(s)]^q \frac{ds}{s} \Big\}^{1/q}$$

if $p > 1$ and $q \geq 1$. Inequalities (4.4.b) follow. $\square$

(4.vi) If $p > 1$, then $L(p, p) = L^p(G)$.

(4.vii) If $p > 1$ and $1 \leq q < r \leq \infty$, then $L(p, q) \subset L(p, r)$ and the relevant embedding is continuous.

Proof of Proposition (4.vii). If r is not ∞, then clearly

$$\frac{d}{ds} \Big\{ \int_0^s \big(t^{1/p} \overline{u}(t)\big)^q \frac{dt}{t} \Big\}^{r/q} = \frac{r}{q} \Big\{ \int_0^s \Big(t^{1/p} \overline{u}(t)\Big)^q \frac{dt}{t} \Big\}^{r/q-1} [\overline{u}(s)]^q s^{q/p-1}$$

for every positive s. The right-hand side of the last equation

$$\geq \frac{r}{q} \Big\{ \int_0^s \Big(t^{1/p} \overline{u}(s)\Big)^q \frac{dt}{t} \Big\}^{r/q-1} [\overline{u}(s)]^q s^{q/p-1},$$

since $r/q > 1$ and Proposition (4.iii) guarantees that $\overline{u}$ decreases monotonically. Therefore

$$\frac{d}{ds} \Big\{ \int_0^s \big(t^{1/p} \overline{u}(t)\big)^q \frac{dt}{t} \Big\}^{r/q} \geq p^{r/q-1} q^{-r/q} r \big(s^{1/p} \overline{u}(s)\big)^r \frac{1}{s}$$

for every positive s. We deduce

$$\left\{ \int_0^\infty \left(t^{1/p}\overline{u}(t)\right)^q \frac{dt}{t} \right\}^{r/q} \geq p^{r/q-1} q^{-r/q} r \int_0^\infty \left(s^{1/p}\overline{u}(s)\right)^r \frac{ds}{s},$$

in other words

$$(q/p)^{1/q} \|u\|_{L(p,q)} \geq (r/p)^{1/r} \|u\|_{L(p,r)}.$$

This inequality implies also that

$$(q/p)^{1/q} \|u\|_{L(p,q)} \geq \|u\|_{L(p,\infty)},$$

therefore concludes the proof. □

4.3. An embedding theorem. The following theorem improves a classical result.

Theorem 4.A. *Suppose* $1 \leq p < n$. *Then* $W^{1,p}(\mathbb{R}^n)$ *is continuously embedded into Lorentz space* $L\left(\frac{np}{n-p}, p\right)$.

Proof. First step. We consider here the $p = 1$ case and show that the following inequality

$$(1 - 1/n)\kappa_n^{1/n} \|u\|_{L(\frac{n}{n-1},1)} \leq \int_{\mathbb{R}^n} |\nabla u(x)|\, dx \tag{4.5}$$

holds for every function u.

The coarea formula of Federer says that

$$\int_{\mathbb{R}^n} |\nabla u(x)|\, dx = \int_0^\infty H_{n-1}\{x \in \mathbb{R}^n \colon |u(x)| = t\}\, dt.$$

The isoperimetric theorem in $\mathbb{R}^n$ gives

$$H_{n-1}\{x \in \mathbb{R}^n \colon |u(x)| = t\} \geq n\kappa_n^{1/n} [m\{x \in \mathbb{R}^n : |u(x)| > t\}]^{1-1/n}$$

for every positive t. Proposition (1.xvii) yields

$$\int_0^\infty [m\{x \in \mathbb{R}^n \colon |u(x)| > t\}]^{1-1/n}\, dt = (1 - 1/n) \int_0^\infty s^{-1/n} u^*(s)\, ds.$$

Hence

$$\int_{\mathbb{R}^n} |\nabla u(x)|\,dx \geq (n-1)\kappa_n^{1/n} \int_0^\infty s^{1-1/n} u^*(s) \frac{ds}{s}.$$

Inequality (4.5) follows since

$$n \int_0^\infty s^{1-1/n} u^*(s) \frac{ds}{s} = \|u\|_{L(\frac{n}{n-1},1)},$$

according to equation (4.4.a).

Incidentally, one may observe that inequality (4.5) is sharp and can be derived also from Theorem 4.B below.

Second step. We assume here $1 < p < n$ and $1 \leq q \leq \infty$, and show that the following inequality

$$\text{(4.6)} \qquad \Big\{ \int_0^\infty [s^{1/p-1/n} u^*(s)]^q \frac{ds}{s} \Big\}^{1/q} \leq \frac{\Gamma(1-1/p)\Gamma(1/p-1/n)}{\Gamma(1-1/n)} \Big\{ \int_0^\infty [s^{1/p} |\nabla u|^*(s)]^q \frac{ds}{s} \Big\}^{1/q}$$

holds for every test function u.

Theorem 3.A plays a role at this point. It gives

$$u^*(s) \leq n^{-1}\kappa_n^{-1/n} \int_0^\infty (s+t)^{-1+1/n} |\nabla u|^*(t)\,dt$$

for almost every positive s.

The last inequality reads

$$\phi(s) \leq \int_0^\infty K(s,t)\psi(t)\,dt$$

for almost every positive s, if the abbreviations $\phi(s) = s^{1/p-1/n}u^*(s)$ and $\psi(s) = s^{1/p}|\nabla u|^*(s)$ are used and K is defined by

$$K(s,t) = n^{-1}\kappa_n^{-1/n} \frac{1}{(s+t)} \Big(\frac{s}{t}\Big)^{1/p} \Big(\frac{t}{s}+1\Big)^{1/n}.$$

As K is homogeneous of degree -1, a change of variables gives

$$\phi(s) \leq \int_0^\infty K(1,r)\psi(rs)\,dr$$

for almost every positive s. Consequently, Minkowski's inequality and the equation $\int_0^\infty [\psi(rs)]^q\,ds/s = \int_0^\infty [\psi(s)]^q\,ds/s$ imply

$$\Big\{\int_0^\infty [\phi(s)]^q \frac{ds}{s}\Big\}^{1/q} \leq \int_0^\infty K(1,r)\,dr \Big\{\int_0^\infty [\psi(s)]^q \frac{ds}{s}\Big\}^{1/q}.$$

A computation gives

$$\int_0^\infty K(1,r)\,dr = \frac{\Gamma(1-1/p)\Gamma(1/p-1/n)}{\Gamma(1-1/n)}.$$

Inequality (4.6) follows. We stress that inequality (4.6) holds for every q such that $1 \leq q \leq \infty$, and a sharp form of (4.6) appears in Theorem 4.B below in the case where $1 \leq q \leq p$.

The proof of Theorem 4.A is complete. □

4.4. Sharp constants. The following theorem is in [Al].

Theorem 4.B. *Suppose* $1 \leq q \leq p < n$. *If* u *is any test function* — i.e., is sufficiently smooth and decays fast enough at infinity — *then*

$$\left(\frac{n}{p}-1\right)^q \kappa_n^{q/n} \int_0^\infty [s^{1/p-1/n}u^*(s)]^q \frac{ds}{s} \leq \int_0^\infty [s^{1/p}|\nabla u|^*(s)]^q \frac{ds}{s}. \tag{4.7.a}$$

Equality holds in (4.7.a) *if* $q = 1$ *and* u *is any test function obeying*

$$u(x) \quad \text{is a decreasing convex function of } |x|; \tag{4.7.b}$$

the ratio between the two sides of (4.7.a) *is arbitrarily close to* 1 *if*

$$u(x) = \begin{cases} 1 + k(1-|x|) & \text{if } |x| < 1, \\ |x|^{-k} & \text{if } |x| \geq 1, \end{cases} \tag{4.7.c}$$

and k is larger than, but close enough to $n/p-1$.

Proof. As $p \geq q$, $s^{q/p-1}$ decreases as s increases from 0 to $+\infty$. Consequently, Lemma 4.C below tells us that

$$\int_0^\infty [s^{1/p}|\nabla u|^*(s)]^q \frac{ds}{s} \geq \int_0^\infty \Big[-n\kappa_n^{1/n} s^{1/p-1/n+1} \frac{du^*}{ds}(s)\Big]^q \frac{ds}{s}.$$

Recall from Lemma 1.E that u^* is absolutely continuous — thus

$$u^*(s) = \int_0^\infty \Big[-\frac{du^*}{dt}(t)\Big]\, dt$$

for every positive s. An integration by parts shows

$$\int_0^\infty \Big[-s^{1/p-1/n+1}\frac{du^*}{ds}(s)\Big]\frac{ds}{s} = \Big(\frac{1}{p}-\frac{1}{n}\Big)\int_0^\infty s^{1/p-1/n}u^*(s)\frac{ds}{s},$$

whereas Theorem 330 from [HLP] yields

$$\int_0^\infty \Big[-s^{1/p-1/n+1}\frac{du^*}{ds}(s)\Big]^q\frac{ds}{s} \geq \Big(\frac{1}{p}-\frac{1}{n}\Big)\int_0^\infty [s^{1/p-1/n}u^*(s)]^q\frac{ds}{s}$$

if $q > 1$.

Inequality (4.7.a) follows. The remaining assertions are easily checked by inspection. $\square$

Lemma 4.C. *Suppose Φ is a Young function, u is a test function. Then*

$$\int_0^\infty \Phi\big(|\nabla u|^*(s)\big)\phi(s)\,ds \geq \int_0^\infty \Phi\Big(-n\kappa_n^{1/n}s^{1-1/n}\frac{du^*}{ds}(s)\Big)\phi(s)\,ds. \tag{4.8}$$

provided ϕ is a nonnegative decreasing function.

Proof of Lemma 4.C. Since ϕ is nonnegative, Lemma 1.E gives

$$\int_0^\infty \Phi\Big(-n\kappa_n^{1/n}s^{1-1/n}\frac{du^*}{ds}(s)\Big)\phi(s)\,ds$$
$$\leq \int_0^\infty ds\phi(s)\frac{d}{ds}\int_{\{x\in\mathbb{R}^n:\ |u(x)|>u^*(s)\}} \Phi\big(|\nabla u(x)|\big)\,dx.$$

An integration by parts shows that the last right-hand side equals

$$\phi(+\infty) \int\limits_{\mathbb{R}^n} \Phi(|\nabla u|)\, dx + \int\limits_0^\infty [-d\phi(s)] \int\limits_{\{x \in \mathbb{R}^n :\ |u(x)| > u^*(s)\}} \Phi\big(|\nabla u(x)|\big)\, dx.$$

Notice the following proposition:

(4.viii) Let f be a real-valued *continuous increasing* function defined in $[0,\infty[$ such that $f(0) = 0$ and $f(+\infty) = +\infty$. Suppose w is a nonnegative measurable function defined in $\mathbb{R}^n$ that decays at infinity. Then the decreasing rearrangement of $f(w)$ is less than, or equals $f(w^*)$.

Proof of Proposition (4.vii). As f increases monotonically,

$$\big\{x \in \mathbb{R}^n : f\big(w(x)\big) > f(t)\big\} \subseteq \{x \in \mathbb{R}^n : w(x) > t\}$$

for every nonnegative t. We deduce successively

$$m\big\{x \in \mathbb{R}^n : f\big(w(x)\big) > f(t)\big\} \le m\{x \in \mathbb{R}^n : w(x) > t\}$$

for every nonnegative t,

$$\big\{t \ge 0 : m\big\{x \in \mathbb{R}^n : f\big(w(x)\big) > f(t)\big\} \le s\big\} \\ \supseteq \big\{t \ge 0 : m\{x \in \mathbb{R}^n : w(x) > t\} \le s\big\}$$

for every nonnegative s,

$$\min\big\{t \ge 0 : m\big\{x \in \mathbb{R}^n : f\big(w(x)\big) > f(t)\big\} \le s\big\} \\ \le \min\big\{t \ge 0 : m\{x \in \mathbb{R}^n : w(x) > t\} \le s\big\}$$

for every nonnegative s. Propositions from Subsection 1.2 yield

$$w^*(s) = \min\big\{t \ge 0 : m\{x \in \mathbb{R}^n : w(x) > t\} \le s\big\}$$

and

$$\big(f(w)^*\big)(s) = \min\big\{t \ge 0 : m\big\{x \in \mathbb{R}^n : f\big(w(x)\big) > t\big\} \le s\big\}$$

for every nonnegative s. As f is increasing and continuous and the range of f is $[0,+\infty[$, we have

$$\big(f(w)^*\big)(s) = f\Big(\min\big\{t \ge 0 : m\big\{x \in \mathbb{R}^n : f\big(w(x)\big) > f(t)\big\} \le s\big\}\Big)$$

for every nonnegative s. In conclusion,

$$\big(f(w)^*\big)(s) \le f\big(w^*(s)\big)$$

for every nonnegative s. $\square$

Proposition (4.viii) tells us that the decreasing rearrangement of $\Phi(|\nabla u|)$ is less than, or equals $\Phi(|\nabla u|^*)$. Proposition (1.xiii) ensures that $m\{x \in \mathbb{R}^n : |u(x)| > u^*(s)\} \le s$ for every nonnegative s. Therefore Theorem 1.A gives

$$\int\limits_{\{x\in\mathbb{R}^n :\ |u(x)|>u^*(s)\}} \Phi\big(|\nabla u(x)|\big)\,dx \le \int\limits_0^s \Phi\big(|\nabla u|^*(t)\big)\,dt$$

for every positive s.

Thus,

$$\begin{aligned}\int\limits_0^\infty \Phi\Big(-n\kappa_n^{1/n}s^{1-1/n}\frac{du^*}{ds}(s)\Big)\phi(s)\,ds \le \phi(+\infty)\int\limits_{\mathbb{R}^n}\Phi(|\nabla u|)\,dx \\ + \int\limits_0^\infty [-d\phi(s)]\int\limits_0^\infty \Phi\big(|\nabla u|^*(t)\big)\,dt\end{aligned}$$

because $[-d\phi]$ is a nonnegative measure. An integration by parts shows that the last right-hand side equals

$$\int\limits_0^\infty \Phi\big(|\nabla u|^*(s)\big)\phi(s)\,ds.$$

Inequality (4.8) follows. $\square$

Theorem 4.D. *If u is any test function then*

$$\sup|u| \le (n-1)n^{-2}\kappa_n^{-1/n}\big\||\nabla u|\big\|_{L(n,1)}. \tag{4.9.a}$$

Equality holds in (4.9.a) *if u is given by*

$$u(x) = \begin{cases} 1-|x| & \text{if } |x|<1, \\ 0 & \text{if } |x|\ge 1. \end{cases} \tag{4.9.b}$$

Proof. Theorem 3.A yields

$$u^*(0) \le n^{-1}\kappa_n^{-1/n}\int_0^\infty t^{-1+1/n}|\nabla u|^*(t)\,dt.$$

A property or rearrangements gives

$$u^*(0) = \sup|u|;$$

equation (4.4.a) gives

$$\int_0^\infty t^{-1+1/n}|\nabla u|^*(t)\,dt = \Big(1-\frac{1}{n}\Big)\big\||\nabla u|\big\|_{L(n,1)}.$$

Inequality (4.9.a) follows. The remaining assertion follows from an immediate inspection. □

Theorem 4.E. *Let p, q and r satisfy $1 < p < n/2$ and $1 < q < r$. Let G be any open subset of $\mathbb{R}^n$, u a real-valued function defined in G. Assume u is smooth enough* — e.g., twice continuously differentiable — *and vanishes on the boundary of G* — i.e., the measure of $\{x \in G\colon |u(x)| > t\}$ is finite and the closure of $\{x \in G\colon |u(x)| > t\}$ is contained in G for every positive t. *Then*

$$\Big\{\int_0^\infty [s^{1/p-2/n}u^*(s)]^r \frac{ds}{s}\Big\}^{1/r} \le C\|\Delta u\|_{L(p,q)}, \tag{4.10.a}$$

where C is given by
(4.10.b)

$$n^2\kappa_n^{2/n}C = \Big(\frac{1}{p}-\frac{2}{n}\Big)^{-1+1/q-1/r}\Big(1-\frac{1}{q}\Big)^{1-1/q} r^{-1/r}\Big\{\frac{\Gamma(\frac{rq}{r-q})}{\Gamma(\frac{r}{r-q})\Gamma(q\frac{r-1}{r-q})}\Big\}^{1/q-1/r}.$$

Moreover

$$\|u\|_{L(\frac{np}{n-2p},q)} \le \frac{p^2\kappa_n^{-2/n}}{(np+2p-n)(n-2p)}\|\Delta u\|_{L(p,q)}. \tag{4.11}$$

The constants displayed are the smallest ones which bound the ratios between the left-hand side and the norm on the right-hand side.

The present theorem informs that $W^{2,p}(\mathbb{R}^n)$ is continuously imbedded in Lorentz space $L\left(\frac{np}{n-2p}, p\right)$ if $1 < p < n/2$.

Proof, outlined. Since $\operatorname{sprt} u^* \subseteq [0, m(G)]$, Theorem 3.B gives

$$u^*(s) \le n^{-2}\kappa_n^{-2/n} \int_s^\infty dt\; t^{-2+2/n} \int_0^t (\Delta u)^*(z)\,dz$$

for almost every positive s. Therefore Proposition (4.iv) gives successively

$$u^*(s) \le n^{-2}\kappa_n^{-2/n} \int_s^\infty t^{-1+2/n}\overline{(\Delta u)}(t)\,dt \tag{4.12}$$

and

$$\overline{u}(s) \le n^{-2}\kappa_n^{-2/n} \int_0^\infty \frac{t^{2/n}}{\max\{t,s\}}\overline{(\Delta u)}(t)\,dt \tag{4.13}$$

for almost every positive s.

Slight changes in an inequality by Bliss [Bl] ensure that

$$\left\{ \int_0^\infty \left[s^{1/p-2/n} \int_0^\infty f(t)\,dt\right]^r \frac{ds}{s} \right\}^{1/r} \le n^2\kappa_n^{2/n} C \left\{ \int_0^\infty [s^{1+1/p-2/n} f(s)]^q \frac{ds}{s} \right\}^{1/q}$$

if f is any nonnegative measurable function, $1/p - 2/n > 0$ and $1 < q < r$, and C is given by (4.10.b). Hence inequality (4.12) gives (4.10.a) and (4.10.b).

Inequality (4.13) reads

$$[s^{1/p-2/n}\overline{u}(s)] \le \int_0^\infty K(s,t)[t^{1/p}\overline{(\Delta u)}(t)]\,dt$$

for almost every positive s, if the following abbreviation

$$K(s,t) = n^{-2}\kappa_n^{-2/n} \frac{(s/t)^{1/p-2/n}}{\max\{s,t\}}$$

is used. Observe that K is positive and homogeneous of degree -1. Quite the same argument used while proving Theorem 4.A yields

$$\Big\{\int_0^\infty [s^{1/p-2/n}\overline{u}(s)]^q \frac{ds}{s}\Big\}^{1/q} \le \int_0^\infty K(1,r)\,dr \Big\{\int_0^\infty [s^{1/p}\overline{(\Delta u)}(s)]^q \frac{ds}{s}\Big\}^{1/q}.$$

As

$$\int_0^\infty K(1,r)\,dr = \frac{p^2\kappa_n^{-2/n}}{(np+2p-n)(n-2p)},$$

inequality (4.11) follows.

We skip further details and break off. □

5. Equidistributed gradients

5.1. Introduction. Consider a cylindrical rod made up of a number of plastic materials and subject to torsion. Suppose the rod has a given length; suppose the number of the materials, and the quantity and the plastic yield limit of each material are given. Then the rod withstands the largest twisting moment if and only if its cross-section is a disk and the materials are arranged in concentric annuli — the softest material innermost, the hardest material outermost, the other materials orderly in between. Recall from the theory of plasticity that the moment in question is proportional to the integral of a real-valued function u — the stress function — and the relevant data are stored in the distribution function of $|\nabla u|$ — the length of the gradient of u. See [Ar] for details. The above assertion can be derived from the following theorem.

Theorem 5.A. *Let M be a nonnegative decreasing right-continuous function defined in $[0,\infty[$ that decays fast enough at the infinity, and let V be a number larger than or equal to $M(0)$. Consider any real-valued function u defined in $\mathbb{R}^n$ that is nice enough and satisfies the following conditions:*
(i) *the distribution function of $|\nabla u|$ is M, i.e.,*

$$m\{x\in\mathbb{R}^n : |\nabla u(x)|>t\} = M(t) \tag{5.1}$$

for every nonnegative t;
(ii) *the support of u has measure V, i.e.,*

$$m\{x\in\mathbb{R}^n : |u(x)|>0\} = V. \tag{5.2}$$

Then

$$(5.3) \qquad \Big| \int_{\mathbb{R}^n} u(x)\,dx \Big| \le (n+1)^{-1} \kappa_n^{-1/n} \int_0^\infty \left[V^{1+1/n} - \big(V - M(t)\big)^{1+1/n} \right] dt.$$

Equality holds in (5.3) *if and only if either u or $-u$ is a* dome *function (i.e., a* nonnegative radial *function whose restriction to its support is* concave*).*

A full proof of Theorem 5.A, including a derivation of the "only if" clause, was given in [AT]. Related proofs appeared in [Ar] and [GN].

Theorem 2.1 from [GN] implies the following theorem.

Theorem 5.B. *Let M, V, u as in Theorem* 5.A. *Then*

$$(5.4) \qquad \sup |u| \le \kappa_n^{-1/n} \int_0^\infty \big(M(t)\big)^{1/n}\,dt.$$

Equality holds in (5.4) *if u is any* spire *function (i.e., a* nonnegative radial *function whose restriction to any ray is* decreasing *and* convex*).*

Motivated by the preceding result, one may consider the variational problem

$$(5.5) \qquad \begin{aligned} &\|u\| = \text{ maximum,} \\ &\text{under conditions (i) and (ii) above} \end{aligned}$$

— i.e., the problem of rendering $\|u\|$ a maximum within a class of functions u whose support has a prescribed measure and such that $|\nabla u|$ has a prescribed rearrangement. Here $\|\,.\,\|$ stands for a norm in some Banach function space. According to usage, we say that two functions are equidistributed, or a rearrangement of each other, if they have the same distribution function. Theorems 5.A and 5.B give a complete answer if $\|\,.\,\|$ is either the norm in $L^1(\mathbb{R}^n)$ or the norm in $L^\infty(\mathbb{R}^n)$. Theorem 3.1 from [ALT] claims that, if $\|\,.\,\|$ is the norm in $L^p(\mathbb{R}^n)$, then a solution to problem (5.5) actually exists and is radial (provided p is suitably related to dimension n and the decay of distribution function M at infinity). A characterization of maximizers is left out in [ALT], however. Further investigations about problem (5.5) are in [FP], Section 3, and [Po].

In this section we display the geometry of solutions to problem (5.5) in the case where $\|\,.\,\|$ is (equivalent to) a norm in Lorentz space $L(p,1)$. Our main result — Theorem 5.C below — includes both Theorem 5.A and Theorem 5.B. It may be viewed as an approach to problems from the calculus of variations in which side conditions constrain a rearrangement. Problems of such a sort are worked out, e.g., in [ALT], [Bu], [EST], [FP], [LS], [McL].

For the sake of brevity we shall abuse notations and set

$$\|u\|_{L(p,q)} = \Big\{ \int_0^\infty [s^{1/p} u^*(s)]^q \frac{ds}{s} \Big\}^{1/q}. \tag{5.6}$$

Thus $\|u\|_{L(p,p)} = \{ \int_{\mathbb{R}^n} |u(x)|^p \, dx \}^{1/p}$, the norm of u in $L^p(\mathbb{R}^n)$; arguments used while proving Proposition (4.vii) give

$$(q/p)^{1/q} \|u\|_{L(p,q)} \geq (r/p)^{1/r} \|u\|_{L(p,r)} \tag{5.7}$$

if $q < r$ — hence $\|u\|_{L(p,1)} \geq p \|u\|_{L^p(\mathbb{R}^n)}$ if $p \geq 1$.

5.2. Statement. The following theorem can be found in [Ta6].

Theorem 5.C. *Let u be a real-valued function defined in $\mathbb{R}^n$. Suppose u is nice enough* — e.g., Lipschitz continuous — *and the support of u has a finite measure. Let M and V denote the distribution function of $|\nabla u|$ and the measure of the support of u, respectively — in the other words, suppose equations* (5.1) *and* (5.2) *are in force.*

Let v and w be the real-valued functions defined in $\mathbb{R}^n$ that obey the following conditions: (i) *$|\nabla v|$ and $|\nabla w|$ are rearrangements of $|\nabla u|$, i.e.,*

$$M(t) = m\{x \in \mathbb{R}^n : |\nabla v(x)| > t\} = m\{x \in \mathbb{R}^n : |\nabla w(x)| > t\} \tag{5.8}$$

for every nonnegative t; (ii) *the support of v and the support of w have the same measure as the support of u, i.e.,*

$$V = m\{x \in \mathbb{R}^n : |v(x)| > 0\} = m\{x \in \mathbb{R}^n : |w(x)| > 0\}; \tag{5.9}$$

(iii) *v and w are* radial *and radially* decreasing*; moreover the restriction of $|\nabla v|$ to the support of v is radially* increasing, *while $|\nabla w|$ is radially* decreasing — *in the other words, v is a* dome *function and w is a* **spire** *function.*

Assertions:

$$\|u\|_{L(p,1)} \leq \|v\|_{L(p,1)} \qquad \textit{if } \; n = 1 \;\textit{ or }\; 0 < p \leq n/(n-1), \tag{5.10.a}$$

(5.10.b) $\qquad \|u\|_{L(p,1)} \le \|w\|_{L(p,1)} \qquad$ *if* $n > 1$ *and* $p \ge n/(n-1)$;

furthermore

$$\text{(5.11.a)}\quad \|v\|_{L(p,1)} = \frac{p}{n\kappa_n^{1/n}(\frac{1}{p}+\frac{1}{n})}\int_0^\infty \left[V^{1/p+1/n} - \big(V - M(t)\big)^{1/p+1/n}\right] dt,$$

$$\text{(5.11.b)}\quad \|w\|_{L(p,1)} = \frac{p}{n\kappa_n^{1/n}(\frac{1}{p}+\frac{1}{n})}\int_0^\infty \big(M(t)\big)^{1/p+1/n}\, dt.$$

Proof of Theorem 5.C. Let φ and ψ be nice real-valued functions defined in $]0,\infty[$ such that

$$\text{(5.12.a)}\quad \varphi(s) \ge 0, \quad s\varphi(s) \ge (1-1/n)\int_0^s \varphi(t)\, dt,$$

and

$$\text{(5.12.b)}\quad \psi(s) \ge 0, \quad s\psi(s) \le (1-1/n)\int_0^s \psi(t)\, dt$$

for every positive s. We are going to prove

$$\text{(5.13.a)}\quad \int_0^\infty \varphi(s)u^*(s)\, ds \le \int_0^\infty \varphi(s)v^*(s)\, ds,$$

$$\text{(5.13.b)}\quad \int_0^\infty \psi(s)u^*(s)\, ds \le \int_0^\infty \psi(s)w^*(s)\, ds.$$

Notice these facts:

(i) We have

$$\text{(5.14)}\quad u^*(s) = \begin{cases} \int_s^V \left[-\frac{du^*}{dt}(t)\right] dt & \text{if } 0 < s < V, \\ 0 & \text{if } s \ge V. \end{cases}$$

Indeed, we know from Subsection 1.2 that the support of u^* is $[0, V]$, and Lemma 1.E informs that the restriction of u^* to every compact subinterval of $]0, \infty[$ is absolutely continuous.

(ii) The following inequalities

$$(5.15.a)\quad -\frac{du^*}{ds}(s) \le \frac{1}{n\kappa_n^{1/n} s^{1-1/n}} \Big\{ -\frac{d}{ds} \int\limits_{\{x\in\mathbb{R}^n:\ 0<|u(x)|<u^*(s)\}} |\nabla u(x)|\, dx \Big\}$$

$$(5.15.b)\quad -\frac{du^*}{ds}(s) \le \frac{1}{n\kappa_n^{1/n} s^{1-1/n}} \Big\{ \frac{d}{ds} \int\limits_{\{x\in\mathbb{R}^n:\ |u(x)|>u^*(s)\}} |\nabla u(x)|\, dx \Big\}$$

hold for almost every positive s. This assertion follows from Lemma 1.E. Indeed,

$$\int\limits_{\{x\in\mathbb{R}^n:\ 0<|u(x)|<u^*(s)\}} |\nabla u(x)|\, dx$$

$$= \int\limits_{\{x\in\mathbb{R}^n:\ |u(x)|>0\}} |\nabla u(x)|\, dx - \int\limits_{\{x\in\mathbb{R}^n:\ |u(x)|>u^*(s)\}} |\nabla u(x)|\, dx,$$

since Theorem 3.2.2(c) from [Mrr] ensure that either $\{x \in \mathbb{R}^n : |u(x)| = \text{Constant}\}$ has measure zero or ∇u vanishes almost everywhere in such a set. Hence (5.15.a) is a consequence of (5.15.b). The latter follows from (1.13).

(iii) The inequality

$$(5.16.a)\quad \int\limits_{\{x\in\mathbb{R}^n:\ 0<|u(x)|<u^*(s)\}} |\nabla u(x)|\, dx \le \int_0^{V-s} |\nabla u|^*(t)\, dt$$

holds for every s such that $0 \le s < V$, the inequality

$$(5.16.b)\quad \int\limits_{\{x\in\mathbb{R}^n:\ |u(x)|>u^*(s)\}} |\nabla u(x)|\, dx \le \int_0^{s} |\nabla u|^*(t)\, dt$$

holds for every nonnegative s. Indeed, Theorem 1.A yields

$$\int_E f(x)\, dx \le \int_0^{m(E)} f^*(s)\, ds$$

if f is a nonnegative and E is measurable. On the other hand, recall from Subsection 1.2 that the distribution function of u, μ, is defined by $\mu(t) = m\{x \in \mathbb{R}^n \colon |u(x)| > t\}$ for every nonnegative t, and obeys $\mu(t-) = m\{x \in \mathbb{R}^n \colon |u(x)| \geq t\}$ for every positive t. Proposition (1.xiii) gives $\mu\big(u^*(s)\big) \leq s$ if $s \geq 0$ and $\mu\big(u^*(s)-\big) \geq s$ if $0 \leq s < V$. Therefore

$$m\{x \in \mathbb{R}^n \colon 0 < |u(x)| < u^*(s)\} \leq V - s$$

if $0 < s < V$, and

$$m\{x \in \mathbb{R}^n \colon |u(x)| > u^*(s)\} \leq s$$

if $s \geq 0$.

The proof goes ahead this way.

$$\int_0^\infty \varphi(s) u^*(s)\, ds$$

$=$ by formula (5.14)

$$\int_0^V \Big\{ \int_0^s \varphi(t)\, dt \Big\} \{ -\frac{du}{ds}(s) \Big\}\, ds$$

$\leq$ by the first inequality in (5.12.a) and inequality (5.15.a)

$$\int_0^V \Big\{ \frac{1}{n\kappa_n^{1/n} s^{1-1/n}} \int_0^s \varphi(t)\, dt \Big\} \Big\{ -\frac{d}{dt} \int_{\{x \in \mathbb{R}^n \colon\, 0 < |u(x)| < u^*(s)\}} |\nabla u(x)|\, dx \Big\}\, ds$$

$\leq$

$$\int_0^V \Big\{ \frac{1}{n\kappa_n^{1/n} s^{1-1/n}} \int_0^s \varphi(t)\, dt \Big\} \Big\{ -d \int_{\{x \in \mathbb{R}^n \colon\, 0 < |u(x)| < u^*(s)\}} |\nabla u(x)|\, dx \Big\}$$

$\leq$ integrations by parts

$$\int_0^V \frac{1}{n\kappa_n^{1/n} s^{2-1/n}} \Big\{ s\varphi(s) - \Big(1 - \frac{1}{n}\Big) \int_0^s \varphi(t)\, dt \Big\}$$

$$\times \Big\{ \int\limits_{\{x\in\mathbb{R}^n :\ 0<|u(x)|<u^*(s)\}} |\nabla u(x)|\,dx \Big\}\,ds$$

$\leq$ by the second inequality in (5.12.a) and inequality (5.16.a)

$$\int\limits_0^V \frac{1}{n\kappa_n^{1/n} s^{2-1/n}} \Big\{ s\varphi(s) - \Big(1-\frac{1}{n}\Big)\int\limits_0^s \varphi(t)\,dt \Big\}\Big\{ \int\limits_0^{V-s} |\nabla u|^*(t)\,dt \Big\}\,ds$$

$=$ integrations by parts

$$\int\limits_0^V \Big\{ \frac{1}{n\kappa_n^{1/n} s^{1-1/n}} \int\limits_0^s \varphi(t)\,dt \Big\} |\nabla u|^*(V-s)\,ds$$

$$= \int\limits_0^V \varphi(s) \Big\{ \int\limits_s^V |\nabla u|^*(V-t) \frac{dt}{n\kappa_n^{1/n} t^{1-1/n}} \Big\}\,ds.$$

Thus we have shown

$$\int\limits_0^\infty \varphi(s) u^*(s)\,ds \leq \int\limits_0^V \varphi(s) \Big\{ \int\limits_s^V |\nabla u|^*(V-t) \frac{dt}{n\kappa_n^{1/n} t^{1-1/n}} \Big\}\,ds. \tag{5.17.a}$$

Parallel arguments, that will be omitted here, show

$$\int\limits_0^\infty \psi(s) u^*(s)\,ds \leq \int\limits_0^V \psi(s) \Big\{ \int\limits_s^V |\nabla u|^*(t) \frac{dt}{n\kappa_n^{1/n} t^{1-1/n}} \Big\}\,ds. \tag{5.17.b}$$

The very definitions of v and w give

$$|\nabla v(x)| = \begin{cases} |\nabla u|^*(V - \kappa_n |x|^n) & \text{if } |x| < (V/\kappa_n)^{1/n}, \\ 0 & \text{if } |x| \geq (V/\kappa_n)^{1/n}, \end{cases} \tag{5.18.a}$$

and

$$|\nabla w(x)| = |\nabla w|^*(\kappa_n |x|^n), \tag{5.18.b}$$

as well as

$$|\nabla v|^* = |\nabla w|^* = |\nabla u|^*. \tag{5.19}$$

We deduce the following representation formulas

(5.20.a)

$$v(x) = \begin{cases} \int\limits_{\kappa_n |x|^n}^{V} |\nabla u|^*(V-s)\frac{ds}{n\kappa_n^{1/n} s^{1-1/n}} & \text{if } |x| < (V/\kappa_n)^{1/n}, \\ 0 & \text{if } |x| \geq (V/\kappa_n)^{1/n}, \end{cases}$$

and

$$w(x) = \begin{cases} \int\limits_{\kappa_n |x|^n}^{V} |\nabla u|^*(s)\frac{ds}{n\kappa_n^{1/n} s^{1-1/n}} & \text{if } |x| < (V/\kappa_n)^{1/n}, \\ 0 & \text{if } |x| \geq (V/\kappa_n)^{1/n}. \end{cases} \tag{5.20.b}$$

We deduce consequently

$$v^*(s) = \begin{cases} \int\limits_{s}^{V} |\nabla u|^*(V-t)\frac{dt}{n\kappa_n^{1/n} t^{1-1/n}} & \text{if } 0 < s < V, \\ 0 & \text{if } s \geq V, \end{cases} \tag{5.21.a}$$

and

$$w^*(s) = \begin{cases} \int\limits_{s}^{V} |\nabla u|^*(t)\frac{dt}{n\kappa_n^{1/n} t^{1-1/n}} & \text{if } 0 < s < V, \\ 0 & \text{if } s \geq V, \end{cases} \tag{5.21.b}$$

Inequalities (5.13.a) and (5.13.b) follow via inequalities (5.17.a) and (5.17.b), and equations (5.21.a) and (5.21.b).

Observe that, if $n = 1$ or $n > 1$ and $0 < p \leq n/(n-1)$, and φ is given by

$$\varphi(s) = s^{1/p-1}, \tag{5.22.a}$$

then conditions (5.12.a) are satisfied; if $n > 1$ and $p \geq n/(n-1)$, and ψ given by

$$\psi(s) = s^{1/p-1}, \tag{5.22.b}$$

then conditions (5.12.b) are satisfied. Coupling (5.13.a) and (5.22.a) results in inequality (5.10.a); coupling (5.13.b) and (5.22.b) results in inequality (5.10.b) – equation (5.6) comes here into play.

Equation (5.6) and equations (5.21.a) and (5.21.b) yield

$$\|v\|_{L(p,1)} = \frac{p}{n\kappa_n^{1/n}} \int_0^V (V-s)^{1/p+1/n-1} |\nabla u|^*(s)\, ds,$$

$$\|w\|_{L(p,1)} = \frac{p}{n\kappa_n^{1/n}} \int_0^V s^{1/p+1/n-1} |\nabla u|^*(s)\, ds.$$

Layer-cake formula (1.5) implies

$$|\nabla u|^* = \int_0^\infty \chi_{[0,M(t)[}\, dt, \tag{5.23}$$

since the very definition of decreasing rearrangement and the definition of M ensure that $\{s \geq 0 \colon |\nabla u|^*(s) > t\}$ is, for every positive t, precisely the interval $[0, M(t)[$. Formula (5.23) gives

$$\int_0^V (V-s)^{1/p+1/n-1} |\nabla u|^*(s)\, ds$$
$$= \frac{1}{\frac{1}{p}+\frac{1}{n}} \int_0^\infty \left[V^{1/p+1/n} - \big(V - M(t)\big)^{1/p+1/n}\right] dt,$$

$$\int_0^V s^{1/p+1/n-1} |\nabla u|^*(s)\, ds = \frac{1}{\frac{1}{p}+\frac{1}{n}} \int_0^\infty \big(M(t)\big)^{1/p+1/n}\, dt.$$

Equations (5.11.a) and (5.11.b) follow.

The proof is complete. □

5.3. Remarks. An analog of Theorem 5.C cannot hold verbatim if Lorentz space $L(p,1)$ is replaced by Lebesgue space $L^p(\mathbb{R}^n)$. The following are apropos examples.

(5.i) Let $n = 2$ and consider the function u defined thus

$$u(x) = \begin{cases} 1 - |x|^2 & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1 \end{cases} \tag{5.24}$$

— a *dome* function.

An inspection shows that M, the distribution function of $|\nabla u|$, obeys $M(t) = \pi(1-t^2/4)$ if $0 \le t < 2$, $M(t) = 0$ if $t \ge 2$; and that V, the measure of the support of u, equals π. Then $|\nabla u|^*(s) = 2\sqrt{1-s/\pi}$ if $0 \le s < \pi$, $|\nabla u|^*(s) = 0$ if $s \ge \pi$. Formulas (5.20.a) and (5.20.b) yield

$$v = u, \tag{5.25.a}$$

$$w(x) = \begin{cases} \arccos(|x|) - |x|\sqrt{1-|x|^2} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \ge 1. \end{cases} \tag{5.25.b}$$

Formulas (5.24), (5.25.a) and (5.25.b) give

$$\int_{\mathbb{R}^2} |v(x)|^p\,dx = \pi/(p+1), \qquad \int_{\mathbb{R}^2} |w(x)|^p\,dx = 2^{-p-1}\pi \int_0^\pi (\theta - \sin\theta)^p\,d\theta.$$

Hence numerical analysis shows that

$$\|w\|_{L^p(\mathbb{R}^2)}/\|v\|_{L^p(\mathbb{R}^2)} > 1 \tag{5.26.a}$$

if and only if

$$p > 2.871649\ldots. \tag{5.26.b}$$

Now consider the function u defined thus

$$u(x) = \begin{cases} (1-|x|)^2 & \text{if } |x| < 1, \\ 0 & \text{if } |x| \ge 1 \end{cases} \tag{5.27}$$

— a *spire* function. This time $M(t) = \pi(1-t/2)^2$ if $0 \le t < 2$, $M(t) = 0$ if $t \ge 2$; $V = \pi$; $|\nabla u|^*(s) = 2(1-\sqrt{s/\pi})$ if $0 \le s < \pi$, $|\nabla u|^*(s) = 0$ if $s \ge \pi$. Hence formulas (5.20.a) and (5.20.b) give

(5.28.a)

$$v(x) = \begin{cases} \arccos(|x|) + |x|\sqrt{1-|x|^2} + 2(1-|x|) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \ge 1; \end{cases}$$

$$w = u. \tag{5.28.b}$$

Formulas (5.27), (5.28.a) and (5.28.b) imply that the inequality

$$\|w\|_{L^p(\mathbb{R}^2)}/\|v\|_{L^p(\mathbb{R}^2)} > 1 \tag{5.29.a}$$

holds *if and only if*

$$p > 2.777883\ldots. \tag{5.29.b}$$

Note that *the right-hand side of* (5.26.b) *differs from the right-hand side of* (5.29.b), *and both differ from $n/(n-1)$, the index appearing the Theorem* 5.C.

(5.ii) Let $\{p_1, p_2, \ldots, p_l\}$ be a decreasing sequence of l positive numbers. Consider the problem

$$\int_{\mathbb{R}^n} \big(u(x)\big)^2\, dx = \text{ maximum}, \tag{5.30}$$

under the following conditions: u is real-valued and Lipschitz continuous, and

$$m(\operatorname{sprt} u) = \kappa_n, \tag{5.31}$$

$$|\nabla u|^* = \sum_{i=1}^{l} p_i \chi_{[\kappa_n(i-1)l^{-1}, \kappa_n i l^{-1}[} \tag{5.32}$$

— the right-hand side of the last equation is the step function that takes the value p_i at every point from $[\kappa_n(i-1)l^{-1}, \kappa_n i l^{-1}[$ and takes the value 0 at every point from $[\kappa_n, \infty[$.

Clearly, competing functions include Lipschitz continuous functions u having the following properties:

$$u \text{ is radial and radially decreasing}, \tag{5.33.a}$$

$$u(x) = 0 \quad \text{if} \quad |x| \geq 1, \tag{5.33.b}$$

$$|\nabla u| = \sum\nolimits_{i=1}^{l} q_i \chi_{\{(i-1)l^{-1} < |x|^n < il^{-1}\}} \tag{5.33.c}$$

— here $\{q_1, q_2, \ldots, q_l\}$ is any *permutation* of $\{p_1, p_2, \ldots, p_l\}$, and the right-hand side of equation (5.33.c) is the piecewise constant function that takes the value q_i where $(i-1)l^{-1} < |x|^n < il^{-1}$ and takes the value 0 where $|x| \geq 1$.

Let $l = 7$ and assume $\{p_1, p_2, \ldots, p_l\}$ is

$$\{100, 85, 70, 55, 40, 25, 10\}. \tag{5.34}$$

Computations show the following. If $n = 2$, the Lipschitz function specified by (5.33.a), (5.33.b) and (5.33.c) satisfies (5.30) if $\{q_1, q_2, \ldots, q_l\}$ is

$$\{10, 25, 40, 70, 100, 85, 55\}. \tag{5.35}$$

If $n = 3$, the Lipschitz continuous function specified by (5.33.a), (5.33.b) and (5.33.c) satisfies (5.30) if $\{q_1, q_2, \ldots, q_l\}$ is

$$\{40, 100, 85, 70, 55, 25, 10\}. \tag{5.36}$$

Notice that neither (5.35) nor (5.36) is a monotone rearrangement of (5.34). In other words, *if $n = 2$ or $n = 3$ the solution of problem* (5.30), (5.33.a) *and* (5.33.b) *is neither a dome nor a spire function.*

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