WEIGHTED NORM INEQUALITIES FOR INTEGRAL OPERATORS AND RELATED TOPICS

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INTRODUCTION

In the first part of the paper we study integral operators of the form

$$Kf(x) = v(x) \int_0^x k(x,y)u(y)f(y)\,dy, \quad x > 0, \tag{1}$$

where the real weight functions $v(t)$ and $u(t)$ are locally integrable and the kernel $k(x,y) \geq 0$ satisfies the following condition: there exists a constant $D \geq 1$ such that

$$D^{-1}\big(k(x,y) + k(y,z)\big) \leq k(x,z) \leq D\big(k(x,y) + k(y,z)\big), \quad x > y > z \geq 0, \tag{2}$$

where D does not depend on x, y, z. The condition (2) was introduced by R. Oinarov [O_1].

Standard examples of a kernel $k(x,y) \geq 0$ satisfying (2) are

(i) $k(x,y) = (x-y)^\alpha$, $\alpha \geq 0$

(ii) $k(x,y) = \log^\beta(1+x-y)$, $k(x,y) = \log^\beta\left(\frac{x}{y}\right)$; $\beta \geq 0$

and their various combinations.

Let $0 < p \leq \infty$. We study (1) on $\mathbb{R}^+ = (0,\infty)$, but any $(a,b) \subset \mathbb{R}$ can be taken instead of $(0,\infty)$ without any loss of generality. Also, the dual

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operator of T can be considered (see Remark 1.1 below). Let

$$L^p = L^p(0,\infty) = \Big\{ f\colon \|f\|_p = \Big(\int_0^\infty |f(x)|^p dx \Big)^{1/p} < \infty \Big\}.$$

We consider K as a map from L^p into L^q and shall characterize the following problems:

(B) $L^p - L^q$ boundedness,

(C) $L^p - L^q$ compactness and measure of non-compactness,

(S) $L^2 - L^2$ Schatten-von Neumann ideal norms.

Several factors affect the problem (B). First of these are restrictions imposed on the kernel $k(x,y) \geq 0$. Another such factor is the range of parameters p and q, because of substantial difference between the cases $p \leq q$ and $q < p$. Certain part is played also by the fact whether $1 \leq p,\, q \leq \infty$ or not. The cases when $p = 1$ or $p = \infty$; $q \geq 1$ and $q = 1$ or $q = \infty$; $p \geq 1$ follow trivially from known results ([KA], Chapter XI, Theorem 4). It also follows from the general theory of integral operators [AS], [Sc] that if $0 < p < 1$ and $K\colon L^p \to L^q$ is bounded, then $k(x,y) = 0$ almost everywhere. Among other factors, perhaps, the verifiability of a criterion is the most relevant. For instance, Muckenhoupt's criterion [M] for the $L^p - L^p$ boundedness of (1) when $k(x,y) = 1$ penetrated many areas because of its explicit form, and being so easy to verify. On the other hand, the implicit "Schur's test" [Kor], [Sz], given for arbitrary kernel $k(x,y) \geq 0$, $1 < q \leq p < \infty$, had also had effective applications [Nik], [Hern].

The problem (B) was intensively studied since 1988, when the characterization has been found for the Riemann–Liouville operator with kernel $k(x,y) = (x-y)^\alpha$, $\alpha \geq 0$ and its convolution generalizations [MS] ($1 < p \leq q < \infty$), [St$_1$] ($1 < p,\, q < \infty$), [St$_2$–St$_6$], [St$_8$], [St$_{10}$]. Unlike the Muckenhoupt ($1 < p \leq q < \infty$) or Mazja ($1 < q < p < \infty$) criteria for the case $k(x,y) = 1$ (see [B], [Ko], [M], [Ma], [Tal], [Tom], [Saw$_1$], [S$_1$] for the case $0 < q < 1 < p < \infty$, and [OK] for the full account), the $L^p - L^q$ boundedness of the Riemann–Liouville operator was characterized by two conditions, which are independent in general except the case when $u(y) = 1$ or $v(x) = 1$. Later, in the papers [BK$_1$] and [O$_2$] ($1 < p \leq q < \infty$), [St$_{12}$] ($1 < p,\, q < \infty$), the criteria have been proved for the kernels satisfying monotonicity or continuity conditions. The classical results with the power weights can be found in [HLP].

The problem (C) has the background in the spectral theory of weighted string [G], [AO], the case $k(x, y) = 1$ was proved in [Riem], and for the Riemann–Liouville operator see [St_2–St_3].

In Section 1.1 we characterize the problems (B) and (C) for the Volterra integral operators (1) under the condition (2), which probably is a balance point between generality of conditions imposed on a kernel and implicitness of a criterion. A few extensions to Lorentz and Orlicz spaces are given in Sections 1.2 and 1.3 which generalize the results of [AM], [EGP], [Saw_1] and [BK_2], [HM], [L], respectively.

The problem (S) is quite a natural step from the problems (B) and (C). The standard references in this area are the books [GK], [K], [P_1], [Sim] and the survey article [BS]. Applying the real interpolation method [BL] we give in the Section 1.4 an analog of the Hilbert–Schmidt formulae for the Schatten-von Neumann ideal σ_p-norm, $2 \le p < \infty$, for operators of the form (1) with extension to the range $1 \le p < \infty$ for the polynomial kernel.

In the second part we deal with weighted norm inequalities restricted to monotone functions. This topic was recently initiated by the papers [ArM] and [Saw_2] with further developments in [A], [Br], [CS_1], [CS_2], [G_1–G_3], [H], [HSt], [N], [St_7], [St_9], [St_{11}] and others.

Our first observation is that such inequalities are helpful in the above problem (B) for the Hardy operator, when $0 < q < p < \infty$, $p \ge 1$.

Secondly, we characterize the dual space to the classical Lorentz space given by

$$\Gamma_p(v) = \Big\{ f \colon \|f\|_{p,v} = \Big(\int_0^\infty \Big(\frac{1}{t} \int_0^t f^*(s)\, ds \Big)^p v(t)\, dt \Big)^{1/p} < \infty \Big\},$$

where $f^*(s) = \inf[z \ge 0\colon\ \mathrm{meas}\,\{x\colon |f(x)| > z\} \le s]$, and find out the criteria for a number of operators of harmonic analysis to be bounded in Γ-spaces.

Throughout the paper the expressions of the form $0 \cdot \infty$, $0/0$, ∞/∞ are taken equal to zero, the inequality $A \ll B$ means $A \le cB$, where c depends only on D and parameters of summation (p, $q \ldots$), and the relationship $A \approx B$ is interpreted as $A \ll B \ll A$ or $A = cB$. Further, χ_E denotes the characteristic function of a set E, $\mathbb{Z}$ the set of all integers, $\Box$ means the end of a proof.

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1. Problems (B) and (C)

1.1 Lebesgue spaces. Denote $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{q}+\frac{1}{q'}=1$, $\frac{1}{q}-\frac{1}{p}=\frac{1}{r}$, and

$$(3)\quad A_0=\sup_{t>0}A_0(t)=\sup_{t>0}\Big(\int_t^\infty k^q(x,t)|v(x)|^q dx\Big)^{1/q}\Big(\int_0^t|u(y)|^{p'}dy\Big)^{1/p'},$$

$$(4)\quad A_1=\sup_{t>0}A_1(t)=\sup_{t>0}\Big(\int_t^\infty |v(x)|^q dx\Big)^{1/q}\Big(\int_0^t k^{p'}(t,y)|u(y)|^{p'}dy\Big)^{1/p'},$$

$$(5)\quad B_0=\Big\{\int_0^\infty\Big(\int_t^\infty k^q(x,t)|v(x)|^q dx\Big)^{r/q}\Big(\int_0^t|u(y)|^{p'}dy\Big)^{r/q'}|u(t)|^{p'}dt\Big\}^{1/r},$$

$$(6)\quad B_1=\Big\{\int_0^\infty\Big(\int_t^\infty |v(x)|^q dx\Big)^{r/q}\Big(\int_0^t k^{p'}(t,y)|u(y)|^{p'}dy\Big)^{r/p'}|v(t)|^{q}dt\Big\}^{1/r},$$

The mapping properties of $K\colon L^p\to L^q$ for $1\le p,\ q\le\infty$ (with the usual modification if $p=1,\infty$ or $q=1,\infty$) are described by the following statement.

Theorem 1.1. *Let the integral operator K be given by* (1) *with the kernel $k(x,y)\ge 0$ satisfying* (2).

(a$_1$) *If $1<p\le q<\infty$ then the inequality*

$$(7)\qquad \|Kf\|_q\le C\|f\|_p \quad \text{for all } f\in L^p$$

is valid if, and only if, $A=\max(A_0,A_1)<\infty$ and, moreover, $\|K\|\approx A$.

(a$_2$) *If $1<p\le q<\infty$ then the operator $K\colon L^p\to L^q$ is compact if, and only if, $A<\infty$ and*

$$(8)\qquad \lim_{t\to 0}A_i(t)=\lim_{t\to\infty}A_i(t)=0,\quad i=0,1.$$

(b$_1$) *If $1<q<p<\infty$ then the inequality* (7) *holds if, and only if, $B=\max(B_0,B_1)<\infty$ and, moreover, $\|K\|\approx B$.*

(b_2) *If $1 < q < p < \infty$ then the operator $K\colon L^p \to L^q$ is compact if, and only if, $B < \infty$.*

Proof. **(a_1)** We begin with the necessity. Observe that the left hand side of (2) implies

$$k(t,y) \le Dk(x,y) \text{ and } k(x,t) \le Dk(x,y) \quad \text{for all } y < t < x. \tag{9}$$

Now, suppose that (7) and the inequality

$$\int_0^t k^{p'}(t,y)|u(y)|^{p'}\,dy < \infty \tag{10}$$

hold for some $t > 0$ (otherwise we would have $\int_t^\infty |v|^q = 0$ and the convention $0 \cdot \infty = 0$ would imply $0 = A_1(t) \le C$). Setting

$$f_t(x) = \chi_{[0,t]}(x)\big(k(t,x)|u(x)|\big)^{p'-1}\operatorname{sgn} u(x),$$

then substituting $f_t(x)$ in (7) and applying (9), we find that

$$\begin{aligned} &C\Big(\int_0^t k^{p'}(t,y)|u(y)|^{p'}\,dy\Big)^{1/p} \\ &\ge \Big(\int_0^\infty |v(x)|^q dx\Big(\int_0^x k(x,y)\chi_{[0,t]}(y)\big(k(t,y)\big)^{p'-1}|u(y)|^{p'}\,dy\Big)^q\,dx\Big)^{1/q} \\ &\ge D^{-1}\Big(\int_t^\infty |v(x)|^q dx\Big)^{1/q}\Big(\int_0^t k^{p'}(t,y)|u(y)|^{p'}\,dy\Big) \end{aligned}$$

and the estimate $D^{-1}A_1 \le C$ follows.

By duality, we have

$$K : L^p \to L^q \Leftrightarrow K^* : L^{q'} \to L^{p'},$$

where

$$K^*g(y) = u(y)\int_y^\infty k(x,y)v(x)g(x)\,dx.$$

Thus (7) and $\|K^*g\|_{p'} \le C\|g\|_{q'}$, $g \in L^{q'}$, hold with the same constant C. Applying the above argument to the operator K^* and

$$g_t(y) = \chi_{[t,\infty]}(y)\big(k(y,t)|v(y)|\big)^{q-1}\operatorname{sgn} v(y),$$

we see that $D^{-1}A_0 \le C$, $D^{-1}A \le C$, and $D^{-1}A \le \inf C = \|K\|$.

To prove sufficiency we need the following

Lemma 1.1. *Let $f(y)$ be such that $u(y)f(y) \geq 0$ and $k(x,y) \geq 0$ satisfy* (2). *Put*

$$G(x) = \int_0^x k(x,y)u(y)f(y)\,dy,$$

and

$$X_k = \{x > 0\colon G(x) \geq (\delta+1)^k\}, \quad k \in \mathbb{Z},$$
$$x_k = \inf X_k \quad \text{if } X_k \neq \emptyset, \quad x_k = \infty \text{ otherwise}; \quad N = \sup\{k\colon X_k \neq \emptyset\},$$

where $\delta > 0$ is a fixed number. If $\delta > D^3$, then the inequality

$$\begin{aligned}(\delta+1)^{k-1} \leq \int_{x_{k-1}}^{x_k} k(x_k,y)u(y)f(y)\,dy + D\int_{x_{k-2}}^{x_{k-1}} k(x_{k-1},y)u(y)f(y)\,dy \\ + Dk(x_k,x_{k-1})\int_0^{x_{k-1}} u(y)f(y)\,dy \\ + D^2 l(x_{k-1},x_{k-2})\int_0^{x_{k-2}} u(y)f(y)\,dy\end{aligned}$$

holds.

Proof. By the definition we have $x_{k-1} \leq x_k$ and also, $G(x) \leq (\delta+1)^k \leq G(x_k)$, if $x \in [x_{k-1}, x_k)$. Using this, we write

$$\begin{aligned}(\delta+1)^{k+1} &= (\delta+1)^2\big((\delta+1)^k - \delta(\delta+1)^{k-1}\big) \\ &\leq (\delta+1)^2\big(G(x_k) - \delta(\delta+1)^{k-1}\big) \\ &= (\delta+1)^2\Big(\int_0^{x_k} k(x_k,y)u(y)f(y)\,dy - \delta(\delta+1)^{k-1}\Big).\end{aligned}$$

Now applying (2) twice we get

$$\begin{aligned}(\delta+1)^{k-1} \leq & Dk(x_k,x_{k-1}) \int\limits_0^{x_{k-1}} u(y)f(y)\,dy + \int\limits_{x_{k-1}}^{x_k} k(x_k,y)u(y)f(y)\,dy \\ & + D \int\limits_0^{x_{k-1}} k(x_{k-1},y)u(y)f(y)\,dy - \delta(\delta+1)^{k-1} \\ \leq & Dk(x_k,x_{k-1}) \int\limits_0^{x_{k-1}} u(y)f(y)\,dy + \int\limits_{x_{k-1}}^{x_k} k(x_k,y)u(y)f(y)\,dy \\ & + D^2 k(x_{k-1},x_{k-2}) \int\limits_0^{x_{k-2}} u(y)f(y)\,dy \\ & + D \int\limits_{x_{k-2}}^{x_{k-1}} k(x_{k-1},y)u(y)f(y)\,dy \\ & + D^2 G(x_{k-2}) - \delta(\delta+1)^{k-1}.\end{aligned}$$

Using (9) we find that $G(x_{k-2}) \leq DG(x) \leq D(\delta+1)^{k-1}$ if $x_{k-2} \leq x \leq x_{k-1}$. Consequently, $D^2G(x_{k-2}) < \delta(\delta+1)^{k-1}$, if $D^3 < \delta$, and the result follows. □

Now we continue with the proof of the sufficiency part of Theorem 1.1.

Without loss of generality we may and shall assume that f has a compact support in $\mathbb{R}^+$, $f(y)u(y) \geq 0$ and $0 < \|f\|_p < \infty$. By Lemma 1.1 we obtain

(11)
$$
\begin{aligned}
J = \int_0^\infty G^q|v|^q &= \sum_{k\in N} \int_{x_k}^{x_{k+1}} G^q|v|^q \leq \sum_{k\leq N} (\delta+1)^{q(k+1)} \int_{x_k}^{x_{k+1}} |v|^q \\
&\leq 4^{q-1}(\delta+1)^{2q} \sum_k \int_{x_k}^{x_{k-1}} |v|^q \Big\{ D^q k^q(x_k, x_{k-1}) \Big(\int_0^{x_{k-1}} uf \Big)^q \\
&\quad + \Big(\int_{x_{k-1}}^{x_k} k(x_k, y)u(y)f(y)\,dy \Big)^q + D^{2q} k^q(x_{k-1}, x_{k-2}) \Big(\int_0^{x_{k-2}} uf \Big)^q \\
&\quad + D^q \Big(\int_{x_{k-2}}^{x_{k-1}} k(x_{k-1}, y)u(y)f(y)\,dy \Big)^q \Big\} \\
&= 4^{q-1}(\delta+1)^{2q}(J_{11} + J_{12} + J_{21} + J_{22}).
\end{aligned}
$$

Using the Hölder and the Jensen inequalities, we find

$$
\begin{aligned}
J_{12} &= \sum_k \int_{x_k}^{x_{k+1}} |v|^q \Big(\int_{x_{k-1}}^{x_k} k(x_k, y)u(y)f(y)\,dy \Big)^q \\
&\leq \sum_k \int_{x_k}^{x_{k+1}} |v|^q \Big(\int_{x_{k-1}}^{x_k} k^{p'}(x_k, y)|u(y)|^{p'}\,dy \Big)^{q/p'} \Big(\int_{x_{k-1}}^{x_k} |f(y)|^p dy \Big)^{q/p} \\
&\leq A_1^q \sum_k \Big(\int_{x_{k-1}}^{x_k} |f(y)|^p dy \Big)^{q/p} \leq A_1^q \|f\|_p^q.
\end{aligned}
$$

Analogously, we get

$$
J_{22} = \sum_k \int_{x_k}^{x_{k+1}} |v|^q \Big(\int_{x_{k-2}}^{x_{k-1}} k^{p'}(x_{k-1}, y)|u(y)|^{p'}\,dy \Big)^q \leq A_1^q D^q \|f\|_p^q.
$$

In case when $k(x,y) \equiv 1$, the single condition $A_1 < \infty$ is necessary and sufficient for (7), and $A_0 \approx A_1$. To obtain upper bounds for J_{11} and J_{21} we need the following more general assertion.

Lemma 1.2 ([BK$_2$]). *Let $1 < p \le q < \infty$ and let $y = \phi(x)$ be a differentiable increasing function on $\mathbb{R}^+$ such that $\phi(0) = 0$, $\phi(\infty) = \infty$ and, thus, the inverse function $x = \phi^{-1}(y)$ exists. Then*

$$\Big\| v(x) \int_0^{\phi(x)} fu \Big\|_q \le C\|f\|_p \quad \text{for all } f \in L^p,$$

if, and only if,

$$A_\phi = \sup_{t>0} \|\chi_{[\phi^{-1}(t),\infty]} v\|_q \|\chi_{[0,t]} u\|_{p'} < \infty,$$

and, moreover, $C \approx A_\phi$.

Proof follows from the case $k(x,y) \equiv 1$ by a change of variables. □

We can now estimate J_{11} and J_{21} as follows. We have

$$J_{11} = D^q \sum_k \Big(\int_{x_k}^{x_{k+1}} |v|^q \Big) k^q(x_k, x_{k-1}) \Big(\int_0^{x_{k-1}} u(y) f(y)\, dy \Big)^q.$$

Put $\Phi(x) = \sum_k x_{k-2} \chi_{[x_k, x_{k+1}]}(x)$ and let $y = \phi(x)$ be such a function that $\phi(x_k) = x_{k-2}$, $\Phi(x) \le \phi(x) \le x$, and $\phi(x)$ satisfies the hypothesis of Lemma 1.2, that is, ϕ is an increasing C^1 function on $(0,\infty)$. Then we obtain

$$\begin{aligned} J_{11} &\ll D^q \sum_k \Big(\int_{x_k}^{x_{k+1}} |v|^q \Big) k^q(x_k, x_{k-1}) \Big[\Big(\int_{x_{k-2}}^{x_{k-1}} uf \Big)^q + \Big(\int_0^{x_{k-2}} uf \Big)^q \Big] \\ &= D^q (J_{11}^{(1)} + J_{11}^{(2)}). \end{aligned}$$

The estimate of $J_{11}^{(1)}$ is similar to J_{12}, but now we apply Hölder's inequality and (9) to get

$$\begin{aligned} J_{11}^{(1)} &\le D^q \sum_k \Big(\int_{x_k}^{x_{k+1}} k^q(x, x_{k-1}) |v(x)|^q dx \Big) \Big(\int_{x_{k-2}}^{x_{k-1}} uf \Big)^q \\ &\le D^q \sum_k \Big(\int_{x_k}^{\infty} k^q(x, x_{k-1}) |v(x)|^q dx \Big) \Big(\int_{x_{k-2}}^{x_{k-1}} |u|^{p'} \Big)^{q/p'} \Big(\int_{x_{k-2}}^{x_{k-1}} |f|^q \Big)^{q/p} \\ &\le A_1^q D^q \sum_k \Big(\int_{x_{k-2}}^{x_{k-1}} |f|^p \Big)^{q/p} \le A_1^q D^q \|f\|_p^q. \end{aligned}$$

For the second term we obtain from Lemma 1.2

$$
\begin{aligned}
J_{11}^{(2)} &= \sum_k \Big(\int\limits_{x_k}^{x_{k+1}} |v(x)|^q dx\Big) k^q(x_k, x_{k-1})\Big) \int\limits_0^{x_{k-2}} uf\Big)^q \\
&\leq \int\limits_0^\infty \Big(\int\limits_0^{\Phi(x)} uf\Big)^q \Big(\sum_k k^q(x_k, x_{k-1})\chi_{[x_k, x_{k-1}]}(x)\Big)|v(x)|^q dx \\
&\leq \int\limits_0^\infty \Big(\int\limits_0^{\phi(x)} uf\Big)^q V(x)\, dx \ll (A_\phi^{(1)})^q \|f\|_p^q,
\end{aligned}
$$

where

$$V(x) = \Big(\sum_k k^q(x_k, x_{k-1})\chi_{[x_k, x_{k-1}]}(x)\Big)|v(x)|^q$$

and

$$A_\phi^{(1)} = \sup_{t>0} A_\phi^{(1)}(t), \quad A_\phi^{(1)}(t) = \Big(\int\limits_{\phi^{-1}(t)}^\infty V(x)\, dx\Big)^{1/q} \Big(\int\limits_0^t |u|^{p'}\Big)^{1/p'}.$$

Now by the definition of $\phi(x)$ we have that if $\phi^{-1}(t) \in [x_{k_0}, x_{k_0+1})$, then $t \in [x_{k_0-2}, x_{k_0-1})$ and, in particular, $t \leq \phi^{-1}(t)$. Applying (9) twice we find that

(12)

$$
\begin{aligned}
\int\limits_{\phi^{-1}(t)}^\infty V(x)\, dx &= k^q(x_{k_0}, x_{k_0-1}) \int\limits_{\phi^{-1}(t)}^{x_{k_0+1}} |v(x)|^q dx \\
&\quad + \sum_{k>k_0} k^q(x_{k_0}, x_{k_0-1}) \int\limits_{x_k}^{x_{k+1}} |v(x)|^q dx \\
&\leq D^{2q} \Big(\int\limits_t^{x_{k_0+1}} k^q(x,t)|v(x)|^q dx + \sum_{k>k_0} \int\limits_{x_k}^{x_{k+1}} k^q(x,t)|v(x)|^q dx\Big) \\
&= D^{2q} \int\limits_t^\infty k^q(x,t)|v(x)|^q dx.
\end{aligned}
$$

Hence, $A_\phi^{(1)} \leq D^2 A_0$, and we obtain $J_{11} \ll A^q\|f\|_p^q$. A similar argument shows that $J_{21} \ll A^q\|f\|_p^q$, and the estimate $J \ll A^q\|f\|_p^q$ is proved.

Remark 1.1. Observe that the part $(\mathbf{a}_1)$ has two natural versions:
(i) restricted to any interval of real line and (ii) with respect to the dual operator K^*. In the case (i), which we call a "restricted" version, we deal with the integral operator K given by

$$Kf(x) = v(x) \int_a^x k(x,y)u(y)f(y)\,dy, \quad -\infty \leq a < x < b \leq \infty,$$

with the kernel $k(x,y) \geq 0$ satisfying

$$D^{-1}\big(k(x,y) + k(y,z)\big) \leq k(x,z) \leq D\big(k(x,y) + k(y,z)\big), \quad b \geq x < y < z \geq a. \tag{2'}$$

Then $\|K\|_{L^p(a,b)\to L^q(a,b)} \approx A_{a,b}$ for $1 < p \leq q < \infty$, where $A_{a,b} = \max(A_{0;a,b}, A_{1;a,b})$, and

$$A_{0;a,b} = \sup_{a<t<b} A_{0;a,b}(t) = \sup_{t>0} \Big(\int_t^b k^q(x,t)|v(x)|^q dx\Big)^{1/q} \Big(\int_a^t |u(y)|^{p'} dy\Big)^{1/p'},$$

$$A_{1;a,b} = \sup_{a<t<b} A_{1;a,b}(t) = \sup_{t>0} \Big(\int_t^b |v(x)|^q dx\Big)^{1/q} \Big(\int_a^t k^{p'}(t,y)|u(y)|^{p'} dy\Big)^{1/p'}.$$

Analogously, in the "dual" version (ii) we have $\|K^*\|_{L^p(a,b)\to L^q(a,b)} \approx A^*_{(a,b)}$, where $A^*_{a,b} = \max(A_{0;a,b}, A_{1;a,b})$, and

$$A^*_{0;a,b} = \sup_{a<t<b} A^*_{0;a,b}(t) = \sup_{t>0} \Big(\int_a^t k^q(t,x)|v(x)|^q dx\Big)^{1/q} \Big(\int_t^b |u(y)|^{p'} dy\Big)^{1/p'},$$

$$A^*_{1;a,b} = \sup_{a<t<b} A^*_{1;a,b}(t) = \sup_{t>0} \Big(\int_a^t |v(x)|^q dx\Big)^{1/q} \Big(\int_t^b k^{p'}(t,y)|u(y)|^{p'} dy\Big)^{1/p'}.$$

$(\mathbf{a}_2)$ To prove necessity of (8) we use the well-known fact that a compact operator maps a weakly convergent sequence into a strongly convergent one. As before we may and shall assume (10) for some $t > 0$ and, consequently, for all $s < t$. Put

$$f_s(x) = \frac{\chi_{[0,s]}(x)\big(k(s,x)|u(x)\big)^{p'-1} \operatorname{sgn} u(x)}{\Big(\int_0^s k^{p'}(s,y)|u(y)|^{p'} dy\Big)^{1/p}}, \quad 0 < s < t.$$

Then for any fixed $g \in L^{p'}$ we have by Hölder's inequality that

$$\Big|\int_0^\infty f_s(x)g(x)\,dx\Big| \le \Big(\int_0^s |g(x)|^{p'}dx\Big)^{1/p'} \to 0, \quad s \to 0.$$

Hence, $f_s \to 0$ is a weakly convergent sequence, and, by the hypotheses, we have

$$\lim_{s\to 0} \|Kf_s\|_q = 0.$$

However, (9) yields

$$\|Kf_s\|_q = \Big(\int_0^\infty |v(x)|^q\Big(\int_0^x k(x,y)f_s(y)u(y)\,dy\Big)^q dx\Big)^{1/q}$$
$$\ge D^{-1}\Big(\int_t^\infty |v(x)|^q dx\Big)^{1/q}\Big(\int_0^s k^{p'}(s,y)|u(y)|^{p'}dy\Big)^{1/p'}.$$

Consequently, $\lim_{s\to 0} A_1(t) = 0$. Using the same argument with the sequence

$$f_s(x) = \frac{\chi_{[0,s]}(x)|u(x)|^{p'-1}\operatorname{sgn} u(x)}{\Big(\int_0^s |u(y)|^{p'}dy\Big)^{1/p}}, \quad 0 < s < t.$$

we obtain $\lim_{s\to 0} A_0(t) = 0$. The second part of (8) follows on applying similar observations with respect to the dual operator which is compact, too.

For the proof of sufficiency observe that if $A < \infty$ and there exists $t > 0$ such that

$$\int_0^t k^{p'}(t,y)|u(y)|^{p'}dy = \infty,$$

then

$$\int_t^\infty |v(x)|^q dx = 0$$

and we have to restrict our considerations to the interval $[0,t]$. Together with the similar observation at the other end of the real semiaxis this shows that without loss of generality we may and shall assume that all the factors in $A_i(t)$, $i = 0$, 1 are finite when $0 < t < \infty$.

Let $0 < a < b < \infty$ and

$$P_a f = \chi_{[0,a]} f, \quad Q_b f = \chi_{[x,\infty]} f, \quad P_{ab} f = \chi_{[a,b]} f.$$

Then we have

$$\begin{aligned} Kf &= (P_a + P_{ab} + Q_b) K (P_a + P_{ab} + Q_b) f \\ &= P_a K P_a f + Q_b K Q_b f + P_{ab} K P_{ab} f + Q_b K P_{ab} f + Q_a K P_a f. \end{aligned}$$

By $(\mathbf{a}_1)$ restricted to the intervals $[0,a]$ or $[b,\infty]$, and (8), we have

$$\begin{aligned} \|P_a K P_a\| &\le \max\{\sup_{0<t<a} A_0(t), \sup_{0<t<a} A_1(t)\} \to 0, \quad a \to 0, \\ \|Q_b K Q_b\| &\le \max\{\sup_{t>b} A_0(t), \sup_{t>b} A_1(t)\} \to 0, \quad b \to \infty. \end{aligned}$$

To finish the proof of $(\mathbf{a}_2)$ we need the following

Lemma 1.3 ([ES$_1$], Lemma 1). *Let $1 < p \le q < \infty$ and $0 < a < b < \infty$, let K be an operator of the form (1) with a kernel $k(x,y) \ge 0$ satisfying (2). Then if $A < \infty$, the maps $P_{ab} K P_{ab}$, $Q_b K P_{ab}$ and $Q_a K P_a$ are compact.*

Thus, K is compact as a limit of compact operators.

$(\mathbf{b}_1)$ We begin with the necessity part. Let $1 < q < p < \infty$, $1/r = 1/q - 1/p$, and let (7) hold. Put

$$\begin{aligned} V_0(t) &= \int_t^\infty |v(x)|^q dx, \qquad V_k(t) = \int_t^\infty k^q(x,t) |v(x)|^q dx, \\ U_0(t) &= \int_0^t |u(y)|^{p'} dy, \qquad U_k(t) = \int_0^t k^{p'}(t,y) |u(y)|^{p'} dy, \end{aligned}$$

and assume that $B_0 < \infty$. To get this inequality we can proceed by changing the weight functions u and v without changing C.

We set

$$f_k = (V_k^{1/q} U_0^{1/q'})^{r/p} |u|^{p'/p} \operatorname{sgn} u.$$

Then $B_0 = \|f_k\|_p^{p/r}$, and replacing in (7) f by f_k, we get

$$C\|f_k\|_p = CB_0^{r/p} \ge \|Kf_k\|_q$$
$$= \Big(\int_0^\infty |v(t)|^q dt \int_0^t k(t,y)u(y)f_k(y)\,dy\Big(\int_0^t k(t,x)u(x)f_k(x)\,dx\Big)^{q-1}\Big)^{1/q}$$
$$= \Big(\int_0^\infty f_k(y)u(y)\,dy \int_y^\infty k(t,y)|v(t)|^q dt\Big(\int_0^t k(t,x)u(x)f_k(x)\,dx\Big)^{q-1}\Big)^{1/q}$$

(by (9))

$$\ge D^{-1/q'}\Big(\int_0^\infty f_k(y)u(y)\,dy \int_y^\infty k^q(t,y)|v(t)|^q dt\Big(\int_0^y u(x)f_k(x)\,dx\Big)^{q-1}\Big)^{1/q}$$

(applying (9) again we find that $V_k(x) \ge D^{-q}V_k(y)$ if $0 < x < y$)

$$\ge D^{-1}\Big(\int_0^\infty f_k(y)u(y)\big(V_k(y)\big)^{(p-1)/(p-q)}dy\Big(\int_0^y |u(x)|^{p'}U_0(x)\,dx\Big)^{q-1}\Big)^{1/q}$$
$$\approx B_0^{r/q}.$$

The above estimate gives

$$C \ge \Big(\frac{p-q}{p-1}\Big)^{1/q'} D^{-1}B_0,$$

and the temporary assumption $B_0 < \infty$ can be removed thanks to the Fatou lemma. A similar argument applied to the dual operator gives

$$C \ge \Big(\frac{p-q}{p-1}\Big)^{1/p} D^{-1}B_1,$$

and the required lower bound follows.

The proof of the sufficiency part begins exactly in the same way as the proof of $(\mathbf{a}_1)$. Preserving the notation (11) we write

$$J \equiv \|Kf\|_q^q \ll (J_{11} + J_{12} + J_{21} + J_{22}).$$

Applying twice Hölder's inequality and then Jensen's inequality we find that

$$J_{12} = \sum_k \int\limits_{x_k}^{x_{k+1}} |v|^q \Big(\int\limits_{x_{k-1}}^{x_k} k(x_k, y) u(y) f(y)\, dy \Big)^q$$

$$\le \sum_k \int\limits_{x_k}^{x_{k+1}} |v|^q \Big(\int\limits_{x_{k-1}}^{x_k} k^{p'}(x_k, y) |u(y)|^{p'} dy \Big)^{q/p'} \Big(\int\limits_{x_{k-1}}^{x_k} |f(y)|^p dy \Big)^{q/p}$$

$$\le \Big(\sum_k \Big(\int\limits_{x_k}^{x_{k+1}} |v|^q \Big)^{r/q} \Big(\int\limits_{x_{k-1}}^{x_k} k^{p'}(x_k, y) |u(y)|^{p'} dy \Big)^{r/p'} \Big)^{q/r} \Big(\sum_k \int\limits_{x_{k-1}}^{x_k} |f(y)|^p dy \Big)^{q/p}$$

$$\le \frac{r}{q} \Big(\sum_k \int\limits_{x_k}^{x_{k+1}} \Big(\int\limits_x^\infty |v|^q \Big)^{r/p} |v(x)|^q dx \Big(\int\limits_{x_{k-1}}^{x_k} k^{p'}(x_k, y) |u(y)|^{p'} dy \Big)^{r/p'} \Big)^{q/r} \|f\|_p^q$$

$$\le \frac{r}{q} D^q \Big(\sum_k \int\limits_{x_k}^{x_{k+1}} \Big(\int\limits_0^x k^{p'}(x, y) |u(y)|^{p'} dy \Big)^{r/p'} \Big(\int\limits_x^\infty |v|^q \Big)^{r/q} |v(x)|^q dx \Big)^{q/r} \|f\|_p^q$$

$$\le \frac{r}{q} B_1^q D^q \|f\|_p^q.$$

Similarly we get

$$J_{22} \le \frac{r}{q} B_1^q D^q \|f\|_p^q.$$

As in the proof of $(\mathbf{a}_1)$ it is clear at the moment that in the case $k(x,y) = 1$, the single condition $B_1 < \infty$ is necessary and sufficient for (7), and $B_0 \ll \|K\| \ll B_1$. Let us show that $B_1 \ll B_0$ in this case. This is obvious if $B_0 = \infty$, therefore suppose $B_0 < \infty$. Then

$$\begin{aligned} \Big(\int\limits_t^\infty |v|^q \Big)^{r/q} \Big(\int\limits_0^t |u|^{p'} \Big)^{r/p'} &= \Big(\int\limits_t^\infty |v|^q \Big)^{r/q} \int\limits_0^t d \Big(\int\limits_0^s |u|^{p'} \Big)^{r/p'} \\ &\le \int\limits_0^t \Big(\int\limits_t^\infty |v|^q \Big)^{r/q} d \Big(\int\limits_0^s |u|^{p'} \Big)^{r/p'} \\ &= \frac{r}{p'} \int\limits_0^t \Big(\int\limits_s^\infty |v|^q \Big)^{r/q} \Big(\int\limits_0^s |u|^{p'} \Big)^{r/q'} |u(s)|^{p'} ds \\ &\le \frac{r}{p'} B_0 < \infty, \end{aligned}$$

and, consequently,

$$\Big(\int_t^\infty |v|^q\Big)^{r/q}\Big(\int_0^t |u|^{p'}\Big)^{r/p'} \to 0 \quad \text{for } t \to 0$$

by the Lebesgue Dominated Convergence Theorem. Hence, integrating by parts, we see that

$$\begin{aligned}
B_0^r &= \int_0^\infty \Big(\int_t^\infty |v|^q\Big)^{r/q}\Big(\int_0^t |u|^{p'}\Big)^{r/p'} |u(t)|^{p'}dt \\
&= \frac{p'}{r}\int_0^\infty \Big(\int_t^\infty |v|^q\Big)^{r/q} d\Big(\int_0^t |u|^{p'}\Big)^{r/p'} \\
&\le \frac{p'}{r}\int_0^\infty \Big(\int_t^\infty |v|^q\Big)^{r/p}\Big(\int_0^t |u|^{p'}\Big)^{r/p'} |v(t)|^q dt = \frac{p'}{r}B_1^r.
\end{aligned}$$

Also, as in the proof of $(\mathbf{a}_1)$, we need the slightly extended version of this case which can be proved easily by the change of variables.

Lemma 1.4. *Let $1 < q < p < \infty$ and let $y = \phi(x)$ be a differentiable increasing function on $\mathbb{R}^+$, such that $\phi(0) = 0$, $\phi(\infty) = \infty$, and thus the inverse function $x = \phi^{-1}(y)$ exists. Then*

$$\Big\| v(x) \int_0^{\phi(x)} fu \Big\|_q \le C\|f\|_p \quad \textit{for all } f \in L^p,$$

if, and only if,

$$B_\phi = \Big(\int_0^\infty \Big(\int_{\phi^{-1}(t)}^\infty |v|^q\Big)^{r/p}\Big(\int_0^t |u|^{p'}\Big)^{r/p'} |u(t)|^{p'}dt\Big)^{1/r} < \infty,$$

and, moreover, $C \approx B_\phi$.

Now we continue with the proof of sufficiency in $(\mathbf{b}_1)$ by applying the construction from $(\mathbf{a}_1)$, involving the functions $\Phi(x)$ and $\phi(x)$ which have been used for the upper bound of J_{11} and J_{21}. As before we have

$$J_{11} \ll D^q(J_{11}^{(1)} + J_{11}^{(2)}).$$

The estimate of $J_{11}^{(1)}$ is similar to that of J_{12}, so we have $J_{11}^{(1)} \ll B_1^q D^q \|f\|_p^q$. Applying Lemma 1.4 for the upper bound of $J_{11}^{(2)}$ we find $J_{11}^{(2)} \ll (B_\phi^{(1)})^q \|f\|_p^q$, where

$$B_\phi^{(1)}(t) = \Big(\int_0^\infty \Big(\int_{\phi^{-1}(t)}^\infty V(x)\,dx \Big)^{r/q} \Big(\int_0^t |u|^{p'} \Big)^{r/q'} |u(t)|^{p'}\,dt \Big)^{1/r},$$

and using (12) we obtain $J_{11}^{(2)} \ll B_0^q \|f\|_p^q$. The upper bound of J_{21} follows by a similar argument. Thus, combining the above estimates, we get $J \ll B^q \|f\|_p^q$ and the part $(\mathbf{b}_1)$ is established.

$(\mathbf{b}_2)$ Necessity follows immediately from $(\mathbf{b}_1)$, and the Ando theorem [An] implies sufficiency.

Theorem 1.1 is proved. $\square$

Remark 1.2. The measure of non-compactness of $K\colon L^p \to L^q$ is given by

$$\alpha(K) = \inf \|K - P\|,$$

where the infimum is taken over all bounded linear maps $P\colon L^p \to L^q$ of finite rank. Using the restricted version of Theorem 1.1 we can show that in the case $1 < p \le q < \infty$ we have

$$\alpha(K) \approx \max(A_L, A_R),$$

where $A_L = \lim_{a\to 0} \|P_a K P_a\|$; $A_R = \lim_{b\to\infty} \|P_b K P_b\|$. (See [$\mathrm{ES}_1$] for details.)

1.2. Lorentz spaces. For $0 < r < \infty$, $0 < s \le \infty$, and a locally integrable function $\phi(x)$ on $\mathbb{R}^+$, the Lorentz space $L_\phi^{rs} \equiv L_\phi^{rs}(\mathbb{R}^+)$ consists of all measurable functions f such that $\|f\|_{rs,\phi} < \infty$, where

$$\|f\|_{rs,\phi} = \Big(\int_0^\infty \big(t^{1/r} f^{**}(t)\big)^s \frac{dt}{t} \Big)^{1/s} \qquad \text{for } 0 < s < \infty,$$

$$\|f\|_{r\infty,\phi} = \sup_{t>0} t^{1/r} f^{**}(t) \qquad \text{for } s = \infty,$$

and

$$f^{**}(t) = \int_0^t f^*(s)\,ds,$$

$$f^*(t) = \inf\Big\{ x > 0\colon \lambda_f(x) = \int_{\{y\in\mathbb{R}^+\colon\ |f(y)|>x\}} \phi(z)\,dz \le t \Big\}.$$

If $r = s$, then

$$\|f\|_{rr,\phi} = \Big(\int_0^\infty |f(x)|^r \phi(x)\,dx\Big)^{1/r}.$$

There is a natural extension of Theorem 1.1, (**a**), to Lorentz spaces. We set

$$A_0 = \sup_{t>0} A_0(t) = \sup_{t>0} \big\|\chi_{[0,t]}(.)k(t,.)\big(u(.)/\phi(.)\big)\big\|_{r's',\phi}\|\chi_{[t,\infty]}v\|_{pq,\psi},$$
$$A_1 = \sup_{t>0} A_1(t) = \sup_{t>0} \|\chi_{[0,t]}(u/\phi)\|_{r's',\phi}\|\chi_{[t,\infty]}(.)k(.,t)v(.)\|_{pq,\psi},$$

Theorem 1.2 ([LS]). *Let K be the integral operator* (1) *with a kernel $k(x,y) \geq 0$ satisfying* (2), *and let $1 < r,\ p < \infty$, $1 \leq s,\ q \leq \infty$ be such that*

(13) $$\max(r,s) \leq \min(p,q).$$

(**a**$_1$) *Then*

(14) $$\|Kf\|_{pq,\psi} \leq C\|f\|_{rs,\phi}, \quad f \in L^{rs}_\phi$$

if, and only if,

$$A = \max(A_0, A_1) < \infty$$

and, moreover, $\|K\| \approx A$, where $\|K\|$ is the norm of K, i.e., the least possible constant C in (14).

(**a**$_2$) *If $1 < r,\ p < \infty$, $1 \leq s,\ q < \infty$ and* (13) *holds, then $K\colon L^{rs}_\phi \to L^{pq}_\psi$ is compact if, and only if, $A < \infty$ and*

$$\lim_{t\to 0} A_i(t) = \lim_{t\to\infty} A_i(t) = 0, \quad i = 0, 1\ldots.$$

Proof of Theorem 1.2 can be obtained by applying the scheme of the proof of Theorem 1.1 and Lemma 3 from [EGP], which is a substitute for Jensen's inequality in the Lorentz spaces. □

Remark 1.3. (a) The case $k(x,y) = 1$ of Theorem 1.2 has recently been proved in [EGP], Theorem 3.

(b) The results on the measure of non-compactness, analogous to those mentioned in Remark 1.2, are valid for Lorentz spaces. Moreover, using the "restricted" version of Theorem 1.2 and schemes from [EEH] and [ES$_1$] it is possible to obtain upper and lower bounds of the approximation numbers of K (see [LS] for details).

(c) In Section 2.1 below a criterion is given for the boundedness of the Hardy operator in Lorentz spaces in the case when (13) is not satisfied.

1.3. Orlicz spaces. Let $\Phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a nonnegative, convex function such that
$$\lim_{x\to 0}\frac{\Phi(x)}{x}=0,\quad \lim_{x\to\infty}\frac{\Phi(x)}{x}=\infty.$$
Following [KR] we call Φ an N-function. Several authors ([BK$_2$], [HM], [L]) have recently established criteria for the validity of the inequality

$$\Phi_2^{-1}\Big(\int_0^\infty \Phi_2\big(|Kf(x)|\big)|\omega(x)|\,dx\Big) \le \Phi_1^{-1}\Big(\int_0^\infty \Phi_1\big(C|f(x)|\big)|\rho(x)|\,dx\Big), \tag{15}$$

where $\omega(x)$ and $\rho(x)$ are locally integrable, Φ_1, Φ_2 are two N-functions satisfying appropriate conditions, and K is an integral operator of type (1) with a kernel monotone with respect to x and y. Using the arguments from [BK$_2$], [L] and Lemma 1.1 we obtain the following

Theorem 1.3. *Let K be the integral operator* (1) *with a kernel $k(x,y)\ge 0$ satisfying* (2)*, let Φ_1, Φ_2 be two N-functions with complementary N-functions Ψ_1, Ψ_2, respectively, and such that $\Phi_2\circ\Phi_1^{-1}$ is convex. Then* (15) *holds if, and only if, $\mathbb{A}<\infty$, where $\mathbb{A}=\mathbb{A}_0+\mathbb{A}_1$ and*

$$\mathbb{A}_0=\inf\Big\{\tau>0\colon \sup_{t>0}\sup_{\lambda>0}\int_0^t \Psi_1\Big(\frac{\alpha(\lambda,t)k(t,x)|u(x)|}{\tau\lambda|\rho(x)|}\Big)\frac{|\rho(x)|}{\alpha(\lambda,t)}\,dx\le 1\Big\},$$
$$\mathbb{A}_1=\inf\Big\{\tau>0\colon \sup_{t>0}\sup_{\lambda>0}\int_0^t \Psi_1\Big(\frac{\beta(\lambda,t)|u(x)|}{\tau\lambda|\rho(x)|}\Big)\frac{|\rho(x)|}{\beta(\lambda,t)}\,dx\le 1\Big\}$$

with
$$\alpha(\lambda,t)=\Phi_1\circ\Phi_2^{-1}\Big(\int_t^\infty \Phi_2\big(\lambda|v(x)|\big)|\omega(x)|\,dx\Big),$$
and
$$\beta(\lambda,t)=\Phi_1\circ\Phi_2^{-1}\Big(\int_t^\infty \Phi_2\big(\lambda k(x,t)|v(x)\big)|\omega(x)|\,dx\Big).$$
Moreover, the best possible constant from (15) satisfies $C\approx\mathbb{A}$.

Remark 1.4. The statement of Theorem 1.3 is essentially taken from [BK$_2$], Theorem 1.7. An alternative version is given in [L], Theorem 1, which also has an extension for the kernels satisfying (2) without monotonicity conditions (we omit the details). Both papers as well as [O_2] and the present one make use of the Martín–Reyes and Sawyer method ([MS]), which is applicable to the non-monotone kernels in view of the technical Lemma 1.1, which, in particular, allows to resist the temptation to reduce the problem to a continuous kernel.

1.4 Schatten-von Neumann ideal norms. Let H be a separable Hilbert space. Then the set of all linear bounded operators $T\colon H \to H$ forms the normed algebra $\mathbf{B}$, where σ_∞ is the ideal of all compact operators. The theory of symmetrically normed (s.n.) ideals $\sigma_\Phi \subset \sigma_\infty$ was developed by using the s.n. functions Φ defined on the space of sequences with a finite number of non-zero terms ([GK], Chapter 3). If $T \in \sigma_\infty$, then $T^* \in \sigma_\infty$ and $(T^*T)^{1/2} \in \sigma_\infty$. To construct σ_∞ the sequences of singular numbers $s_j(T) = \lambda_j[(T^*T)^{1/2}]$ were used with eigenvalues $\lambda_j \geq 0$ taken with respect to their multiplicity and in a decreasing ordering. The formula $\|T\|_{\sigma_\Phi} = \Phi\big(s_j(T)\big)$ defines the norm (quasinorm) in the s.n. ideal σ_Φ. The most well-known s.n. ideals are those, related to the space of sequences l_p, $0 < p \leq \infty$, and called σ_p. The norm (quasinorm) $\|T\| = \big(\sum_j s_j^p(T)\big)^{1/p}$ is usually called the Schatten-von Neumann norm (quasinorm). Thus, $\|T\|_{\sigma_\infty} = \|T\|$ and $\|T\|_{\sigma_2}$ is the Hilbert-Schmidt norm defined for an integral operator $Tf(x) = \int T(x,y)f(y)\,dy$ by the formula $\|T\|_{\sigma_2} = \big(\iint |T(x,y)|^2 dx\,dy\big)^{1/2}$. It is known [BS], that in general the norm $\|T\|_{\sigma_p}$ of an integral operator essentially depends on the smoothness of its kernel when $p < 2$. The aim of this section is to present a brief account of some results from [ES_2] about the Schatten-von Neumann ideal norms for the integral operator (1) under the condition (2) for its kernel.

Let $H = L^2(0,\infty)$ and

$$A_0^2 = \sup_{t>0} \int_t^\infty k^2(x,t)|v(x)|^2 dx \int_0^t |u(y)|^2 dy,$$

$$A_1^2 = \sup_{t>0} \int_t^\infty |v(x)^2 dx \int_0^t k^2(t,y)|u(y)|^2 dy.$$

Theorem 1.1 and the Hilbert-Schmidt formula yield

$$
\begin{aligned}
\|K\|_{\sigma_\infty} &\approx A_0 + A_1,\\
\|K\|_{\sigma_2} &= \Big(\int_0^\infty |v(x)|^2 dx \int_0^x k^2(t,y)|u(y)|^2 dy\Big)^{1/2}\\
&= \Big(\int_0^\infty |u(y)|^2 dy \int_y^\infty k^2(x,y)|v(x)|^2 dx\Big)^{1/2}.
\end{aligned}
$$

Using these formulas and applying the real interpolation method we obtain the following

Theorem 1.4. *Let K be an operator of the form* (1) *with a kernel satisfying* (2) *and $K \in \sigma_\infty$. Then*

(16)

$$
\begin{aligned}
\|K\|_{\sigma_p} \approx \Big(\int_0^\infty \Big[\Big(\int_0^x k^2(x,y)|u(y)|^2 dy\Big)^{p/2}\Big(\int_x^\infty |v(y)|^2 dy\Big)^{p/2-1}|v(x)|^2\\
+ \Big(\int_x^\infty k^2(y,x)|v(y)|^2 dy\Big)^{p/2}\Big(\int_0^x |u(y)|^2 dy\Big)^{p/2-1}|u(x)|^2\Big]\, dx\Big)^{1/p},\\
2 \le p < \infty.
\end{aligned}
$$

Remark 1.5. In the case $k(x,y) \equiv 1$ the formula (16) can be simplified as follows. If

$$
Hf(x) = v(x)\int_0^x f(y)u(y)\, dy,
$$

then

(17)

$$
\begin{aligned}
\frac{1}{2}\|H\|_{\sigma_p} &\le \Big(\int_0^\infty \Big(\int_0^x |u(y)|^2 dy\Big)^{p/2}\Big(\int_x^\infty |v(y)|^2 dy\Big)^{\frac{p}{2}-1}|v(x)|^2 dx\Big)^{1/p}\\
&\le \|H\|_{\sigma_p}, \quad 2 \le p < \infty.
\end{aligned}
$$

For the values $1 \le p < 2$ we obtain the following necessary conditions, using the approach of the recent paper by K. Nowak [Now].

Theorem 1.5. *Let $1 \le p < \infty$ and let the integral operator $K \in \sigma_p$ be given by* (1) *with a kernel $k(x,y) \ge 0$ satisfying* (2). *Then*

$$\begin{aligned} &\|K\|_{\sigma_p} \\ &\le (2D)^{-1} \sup_{\{a_m\}} \Big\{ \sum_m \Big[\Big(\int\limits_{a_{m-1}}^{a_m} k^2(a_{mk}, y)|u(y)|^2 dy \Big)^{p/2} \Big(\int\limits_{a_m}^{a_{m+1}} |v(x)|^2 dx \Big)^{p/2} \\ &+ \Big(\int\limits_{a_{m-1}}^{a_m} |u(y)|^2 dy \Big)^{p/2} \Big(\int\limits_{a_m}^{a_{m+1}} k^2(x, a_m)|v(x)|^2 dx \Big)^{p/2} \Big] \Big\}^{1/p}, \end{aligned}$$

where the supremum is taken over all sequences $0 < \cdots < a_{m-1} < a_m < a_{m+1} < \ldots$.

Corollary 1.1. *In addition to* (17) *we have*

$$\|H\|_{\sigma_p} \ge p2^{-p-1} \Big(\int\limits_0^\infty \Big(\int\limits_0^x |u(y)|^2 dy \Big)^{p/2} \Big(\int\limits_x^\infty |v(y)|^2 dy \Big)^{\frac{p}{2}-1} |v(x)|^2 dx \Big)^{1/p}, \quad 1 \le p < 2.$$

Now we restrict our consideration to the convolution operators with polynomial kernels of the form

$$Tf(x) = v(x) \int\limits_0^x P_n(x-y)u(y)f(y)\,dy,$$

where

$$P_n(x) = \alpha_n x^n + \cdots + \alpha_1 x + \alpha_0, \alpha_n \ne 0$$

is a polynomial with real coefficients. Using the scheme developed in [ES_1], we get an upper bound for $\|T\|_{\sigma_p}$. Denote $|P_n|(x) = |\alpha_n|x^n + \cdots + |\alpha_1|x + |\alpha_0|$.

Theorem 1.6. *Let $0 < p < \infty$. Then*

$$\begin{aligned} &\|T\|_{\sigma_p} \\ &\ll \sup_{\{a_m\}} \Big\{ \sum_m \Big[\Big(\int\limits_{a_{m-1}}^{a_m} |P_n|^2(a_m - y)|u(y)|^2 dy \Big)^{p/2} \Big(\int\limits_{a_m}^{a_{m+1}} |v(x)|^2 dx \Big)^{p/2} \\ &+ \Big(\int\limits_{a_{m-1}}^{a_m} |u(y)|^2 dy \Big)^{p/2} \Big(\int\limits_{a_m}^{a_{m+1}} |P_n|^2(x - a_m)|v(x)|^2 dx \Big)^{p/2} \Big] \Big\}^{1/p}. \end{aligned}$$

When $\alpha_n = 1$, $\alpha_{n-1} = 0, \dots, \alpha_0 = 0$, so that

$$T_n f(x) = v(x) \int_0^x (x-y)^n u(y) f(y)\, dy, \quad n \geq 0,$$

we obtain the following supplement to the results of Theorems 1.4 and 1.5.

Corollary 1.2. *Let $1 \leq p < \infty$ and let $n \geq 0$ be an integer. Then*

$$\|T_n\|_{\sigma_p} \approx \sup_{\{a_m\}} \Big\{ \sum_m \Big[\Big(\int_{a_{m-1}}^{a_m} (a_m - y)^{2n} |u(y)|^2 dy \Big)^{p/2} \Big(\int_{a_m}^{a_{m+1}} |v(x)|^2 dx \Big)^{p/2}$$
$$+ \Big(\int_{a_{m-1}}^{a_m} |u(y)|^2 dy \Big)^{p/2} \Big(\int_{a_m}^{a_{m+1}} (x - a_m)^{2n} |v(x)|^2 dx \Big)^{p/2} \Big] \Big\}^{1/p}.$$

Remark 1.6. An alternative method to obtain the estimates of the Schatten-von Neumann norms for T_n, $n > \frac{1}{2}$, in the case when $u(y) = 1$, is given in [NS].

2. Weighted inequalities on the cones of monotone functions

2.1. Let $0 < p$, $q < \infty$, and let $v(x) \geq 0$ and $u(x) \geq 0$ be locally integrable weight functions on $\mathbb{R}^+$, let $f \downarrow$ denote a non-negative non-increasing function on $\mathbb{R}^+$, and let the similar notation $f \uparrow$ stand for a non-decreasing function. Put

$$V(t) = \int_0^t v, \quad W(t) = \int_0^t w.$$

Our first result is the following

Theorem 2.1. **(a)** *Let $0 < p \leq q < \infty$. Then*

$$\sup_{f\downarrow} \frac{\left(\int_0^\infty f^q w \right)^{1/q}}{\left(\int_0^\infty f^p v \right)^{1/p}} \approx \sup_{t>0} W^{1/q}(t) V^{1/p}(t) \equiv \mathsf{A}. \tag{18}$$

(b) *Let* $0 < q < p < \infty$, $1/r = 1/q - 1/p$. *Then*

$$\sup_{f\downarrow} \frac{\left(\int_0^\infty f^q w\right)^{1/q}}{\left(\int_0^\infty f^p v\right)^{1/p}} \approx \mathsf{B} \approx \mathbf{B} + \frac{W^{1/q}(\infty)}{V^{1/p}(\infty)}, \tag{19}$$

where

$$\mathsf{B} = \Big(\int_0^\infty W^{r/p} V^{-r/p} w\Big)^{1/r}, \quad \mathbf{B} = \Big(\int_0^\infty W^{r/q} V^{-r/q} v\Big)^{1/r}.$$

(c) *The same results hold for non-decreasing functions.*

Of special interest is the particular case $q = 1$, which provides the principle of duality, that is, reverse Hölder inequality for monotone functions.

Corollary 2.1. **(a)** *Let* $1 < p < \infty$, $1/p + 1/p' = 1$, *and let* $g(x) \geq 0$ be locally integrable. *Then*

$$\begin{aligned} \sup_{f\downarrow} \frac{\int_0^\infty fg}{\left(\int_0^\infty f^p v\right)^{1/p}} &\approx \Big(\int_0^\infty \Big(\int_0^s g\Big)^{p'-1} V^{1-p'}(s) g(s)\, ds\Big) \\ &\approx \Big(\int_0^\infty \Big(\int_0^s g\Big)^{p'} V^{-p'}(s) v(s)\, ds\Big)^{1/p'} + \frac{\int_0^\infty g}{\left(\int_0^\infty v\right)^{1/p}}. \end{aligned} \tag{20}$$

(b) *If* $0 < p \leq 1$, *then*

$$\sup_{f\downarrow} \frac{\int_0^\infty fg}{\left(\int_0^\infty f^p v\right)^{1/p}} \approx \sup_{t>0} V^{-1/p}(t) \int_0^t g. \tag{21}$$

Remark 2.1. The original proof of (20) is due to E. Sawyer ([Saw$_2$], Theorem 1), as well as Theorem 2.1 for the range $1 < p, q < \infty$, which was established in [Saw$_2$] in terms of embeddings of classical Lorentz spaces. In full scope one can find the proof of Theorem 2.1 in [St$_9$], Proposition 1. Also, (21) was independently proved in [CS$_1$].

Corollary 2.1 is useful in many ways. In particular, it allows to reduce the weighted inequalities for monotone functions to those for arbitrary functions. Another, rather curious application is that, conversely, ordinary inequalities can be proved by using (20), too. The following theorem demonstrates this fact. Recall that the Lorentz space $L_\phi^{rs} \equiv L_\phi^{rs}(\mathbb{R}^+)$ (see Section 1.2) is equipped with an equivalent quasinorm [SW]

$$\|f\|^*_{rs,\phi} = \Big(\int_0^\infty \big(t^{1/r} f^*(t)\big)^s \frac{dt}{t}\Big)^{1/s}, \quad 0 < r, s < \infty.$$

Theorem 2.2. *Let $0 < q < r < \infty$, $r \geq 1$, $0 < p < \infty$, $1/\gamma = 1/q - 1/r$, and let the operator H be given by $Hf(x) = \int_0^x f(y)u(y)\,dy$. Then*

$$\sup_{f\neq 0} \frac{\|Hf\|^*_{pq,\psi}}{\|f\|^*_{rr,\phi}} \approx \Big(\int_0^\infty \big(\|\chi_{[t,\infty]}\|^*_{pq,\psi}\big)^\gamma d\big(\|\chi_{[0,t]}(u/\phi)\|^*_{r'r',\phi}\big)^\gamma\Big)^{1/\gamma} \equiv \mathsf{B}_0.$$

Proof. We begin with the sufficiency part. Suppose that $f(x) \geq 0$ is a function with a compact support and such that $0 < \|f\|^*_{rr,\phi} < \infty$. We have

$$\|Hf\|^*_{pq,\psi} = \Big(\frac{1}{q}\int_0^\infty \Big(\int_t^\infty \psi\Big)^{q/p} d\big(Hf(t)\big)^q\Big)^{1/q}.$$

Applying (20) in the form

$$\int_0^\infty G\,d\mu \ll \Big(\int_0^\infty G^s dv\Big)^{1/s} \Big(\int_0^\infty \Big(\int_0^t d\mu\Big)^{s'-1} \Big(\int_0^t dv\Big)^{1-s'} d\mu(t)\Big)^{1/s'}, \quad G\downarrow,$$

where

$$s = \frac{\gamma}{q} > 1, \quad G = \Big(\int_t^\infty \psi\Big)^{q/p}, \quad d\mu = d(Hf)^q,$$

$$dv(t) = d\big(\|\chi_{[0,t]}(u/\phi)\|^*_{r'r',\phi}\big)^\gamma,$$

and using Muckenhoupt's criterion [M], we get

$$\begin{aligned}\|Hf\|^*_{pq,\psi} \ll&\Big(\int_0^\infty (\|\chi_{[t,\infty]}\|^*_{pq,\psi})^\gamma d\big(\|\chi_{[0,t]}(u/\phi)\|^*_{r'r',\phi}\big)^\gamma\Big)^{1/\gamma}\\ &\times\Big(\int_0^\infty \big(\|\chi_{[0,t]}(u/\phi)\|^*_{r'r',\phi}\big)^{\gamma(1-s')} d(Hf)^{qs'}\Big)^{1/qs'}\\ =&\mathsf{B}_0\Big(\int_0^\infty (Hf)^r d\big(\|\chi_{[0,t]}(u/\phi)\|^*_{r'r',\phi}\big)^{1-r}\Big)^{1/r}\\ \ll&\mathsf{B}_0\Big(\int_0^\infty f^r\phi\Big)^{1/r} = \mathsf{B}_0\|f\|^*_{rr,\phi}.\end{aligned}$$

Let $r>1$ and $\|Hf\|^*_{pq,\psi} \le C\|f\|^*_{rr,\phi}$ be held for all $f\in L^{rr}_\phi$. Suppose $\mathsf{B}_0<\infty$ and let f_0 be defined by the formula

$$f_0(t) = (\|\chi_{[t,\infty]}\|^*_{pq,\psi})^{\gamma/r}\big(\|\chi_{[0,t]}(u/\phi)\|^*_{r'r',\phi}\big)^{\gamma r'/q'r}\big(u(t)/\phi(t)\big)^{r'/r},$$

then $\|f_0\|_{rr,\phi} = \mathsf{B}_0^{\gamma/r}<\infty$. We may and shall assume for the time being without changing the constant C that $\operatorname{supp}\psi\subset\mathbb{R}^+$. Integrating by parts we find

$$\begin{aligned}C\|f_0\|^*_{rr,\phi} = C\mathsf{B}_0^{\gamma/r} &\ge \|Hf_0\|^*_{pq,\psi}\\ &=\Big(\int_0^\infty \big(Hf_0(t)\big)^{q-1}\Big(\int_t^\infty\psi\Big)^{q/p} f_0(t)u(t)\,dt\Big)^{1/q}\\ &=\Big(\frac{1}{q}\int_0^\infty \big(Hf_0(t)\big)^q d\Big[-\Big(\int_t^\infty\psi\Big)^{q/p}\Big]\Big)^{1/q}.\end{aligned}$$

Employing

$$\begin{aligned}\big(Hf_0(t)\big)^q &= \Big(\int_0^t\Big(\int_0^\infty\psi\Big)^{\gamma/rp}\Big(\int_0^s u^{r'}\phi^{1-r'}\Big)^{\gamma/rq'}u^{r'}(s)\phi^{1-r'}(s)\,ds\Big)^q\\ &\ge\Big(\frac{q'r}{q(\gamma+q'r)}\Big)^q\Big(\int_t^\infty\psi\Big)^{\gamma q/rp}\Big(\int_0^t u^{r'}\phi^{1-r'}\Big)^{\gamma/r'},\end{aligned}$$

we continue

$$
\begin{aligned}
&C\mathsf{B}_0^{\gamma/r}\\
&\geq \Big(\frac{q'r}{q(\gamma+q'r)}\Big)\Big(\frac{1}{q}\int_0^\infty \Big(\int_t^\infty \psi\Big)^{\gamma q/rp}\Big(\int_0^t u^{r'}\phi^{1-r'}\Big)^{\gamma/r'} d\Big[-\Big(\int_t^\infty \psi\Big)^{q/p}\Big]\Big)^{1/q}\\
&= \Big(\frac{q'r}{q(\gamma+q'r)}\Big)\Big(\frac{1}{\gamma}\int_0^\infty \Big(\int_0^t u^{r'}\phi^{1-r'}\Big)^{\gamma/r'} d\Big[-\Big(\int_t^\infty \psi\Big)^{\gamma/p}\Big]\Big)^{1/q}\\
&= \Big(\frac{q'r}{q(\gamma+q'r)}\Big)\Big(\frac{1}{r'}\int_0^\infty \Big(\int_t^\infty \psi\Big)^{\gamma/p} d\Big(\int_0^t u^{r'}\phi^{1-r'}\Big)^{\gamma/r'}\Big)^{1/q}\\
&= \Big(\frac{q'r}{q(\gamma+q'r)}\Big)(r')^{-1/q}\mathsf{B}_0^{\gamma/q},
\end{aligned}
$$

and $C \geq \Big(\frac{q'r}{q(\gamma+q'r)}\Big)(r')^{-1/q}\mathsf{B}_0$ follows.

The limiting case $r=1$ is proved in [SS]. $\square$

Remark 1.7. (i) Theorem 2.2 has been proved in a different form in [Saw$_1$], Theorem 3 and an alternative proof of it in the case $0<q<r<\infty$, $r>1$, $p=q$, was given in [S$_1$].

(ii) The same theorem for the dual operator follows from the inequalities for nondecreasing functions.

2.2. If $p>0$ and $\nu(t)\geq 0$ is a locally integrable function on $\mathbb{R}^+$ and the *nonincreasing rearrangement* on $\mathbb{R}^+$ of a measurable function $f(x)$ is defined by

$$f^*(t)=\inf\big\{s\colon \operatorname{meas}\big(x\in R^n; |f(x)|>s\big)\leq t\big\},\quad t>0,$$

then the classical Lorentz spaces $\Lambda_p(\nu)$ and $\Gamma_p(\nu)$ are defined by

$$\|f\|^*_{p,\nu}=\Big(\int_0^\infty \big(f^*(t)\big)^p\nu(t)\,dt\Big)^{1/p}<\infty,$$

and

$$\|f\|_{p,\nu}=\Big(\int_0^\infty \Big(\frac{1}{t}\int_0^t f^*\Big)^p\nu(t)\,dt\Big)^{1/p}<\infty,$$

respectively. These spaces were introduced by G. G. Lorentz [L$_1$], [L$_2$]. E. Sawyer [Saw$_2$] found that (20) gives a powerful tool for the study of a number of problems in $\Lambda_p(\nu)$-spaces. In particular, it follows from (20) and (21) that $\Lambda_p(\nu)$ has the dual space of the form

$$\Lambda_p^*(\nu) = \Gamma_{p'}\big(\big(s^{-1}V(s)\big)^{-p'}\nu(s)\big), \quad 1 < p < \infty,$$

provided $V(\infty) = \infty$ [Saw$_2$], and

$$\Lambda_p^*(\nu) = \Gamma_\infty\big(\big(s^{-p}V(s)\big)^{1/p}\big), \quad 0 < p \le 1.$$

The same and some other problems for $\Gamma_p(\nu)$-spaces were recently solved in [GHS] by using suitable criteria like (20), (21) for the functions representable as $f(x) = \frac{1}{x}\int_0^x g$, $0 \le g \downarrow$. To this end the discretization method from [G$_1$], G$_2$], [G$_3$] was used for the class of functions Ω_{01} of the form

$$\Omega_{01} = \Big\{f(x) \ge 0, f(x) \uparrow, \frac{1}{x}f(x) \downarrow\Big\},$$

and a Borel measure $d\beta$ on R^+ with the following nondegeneracy properties

$$\int_0^\infty \Big(\frac{s}{s+1}\Big)^p d\beta(x) < \infty; \tag{22}$$

$$\int_0^1 d\beta(s) = \int_1^\infty s^p d\beta(s) = \infty. \tag{23}$$

Under these conditions *the fundamental function* of the measure $d\beta$ of the form

$$\rho_{\beta,p}(t) = \Big(\int_0^\infty \Big(\frac{s}{s+1}\Big)^p d\beta(s)\Big)^{1/p}$$

for any fixed number $a > 1$ has a *discretising sequence* $\{\mu_k\}$ such that

$$\mu_0 = 1,$$
$$\mu_{k+1} = t\colon\ \min\Big\{\frac{\rho_{\beta,p}(\mu_k)}{\rho_{\beta,p}(t)}, \frac{t\rho_{\beta,p}(t)}{\mu_k\rho_{\beta,p}(\mu_k)}\Big\} = a,\ \ k \ge 0,$$
$$\mu_{k-1} = t\colon\ \min\Big\{\frac{\rho_{\beta,p}(t)}{\rho_{\beta,p}(\mu_k)}, \frac{\mu_k\rho_{\beta,p}(\mu_k)}{t\rho_{\beta,p}(t)}\Big\} = a,\ \ k \le 0.$$

Such a sequence was first used by K. I. Oskolkov and then by a number of authors (see [G$_2$] and the references given there).

Applying the methods developed in [G$_1$], [G$_2$], [G$_3$], we obtain the following

Theorem 2.3. *Let $d\beta$ and $d\gamma$ be Borel measures such that for $d\beta$ the nondegeneracy conditions* (22) *and* (23) *are valid. Then for any $p > 0$ there exists $a > 1$ and a discretizing sequence $\{\mu_k\}$ of the fundamental function $\rho_{\beta,p}$ such that for $0 < q < p < \infty$ the following formula is satisfied*

$$J \equiv \sup_{f\in\Omega_{01}} \frac{\left(\int\limits_0^\infty f^q d\gamma\right)^{1/q}}{\left(\int\limits_0^\infty f^p d\beta\right)^{1/p}} \approx \Big\{ \sum_k \Big[\frac{\rho_{\gamma,q}(\mu_k)}{\rho_{\beta,p}(\mu_k)}\Big]^r \Big\}^{1/r}, \tag{24}$$

where $1/r = 1/q - 1/p$. If $0 < p \le q < \infty$, then $J \approx \sup\limits_{t>0} \rho_{\gamma,q}(t)/\rho_{\beta,p}(t)$.

Now, to obtain analogues of (20) and (21) for functions represented by $f(x) = \frac{1}{x}\int_0^x g$, $0 \le g \downarrow$ we consider the particular case of Theorem 2.3, when $q = 1 < p < \infty$, $d\beta(s) = s^{-p}\nu(s)\,ds$, and $\{\nu_k\}$ is the discretizing sequence of the function

$$\rho_{\beta,p}(t) = \Big(\int\limits_0^\infty \frac{\nu(s)\,ds}{(s+t)^p}\Big)^{1/p} \equiv \frac{1}{t}V^{1/p}(t). \tag{25}$$

Theorem 2.4. **(a)** *Let $1 < p < \infty$, $0 \le g(x) \downarrow$, and let the nondegeneracy conditions* (23) *and* (24) *be valid for the measure $d\beta(s) = s^{-p}\nu(s)\,ds$, where $\nu(s) \ge 0$ is a locally integrable function on R^+. Then*

$$\sup_{0\le f\downarrow} \frac{\int\limits_0^\infty fg}{\left(\int\limits_0^\infty \left(\frac{1}{x}\int\limits_0^x f\right)\nu(x)\,dx\right)^{1/p}} \approx \Big(\int\limits_0^\infty \Big(\int\limits_0^x g\Big)^{p'} \mathcal{V}(x)\,dx\Big)^{1/p'}$$

where

$$\mathcal{V}(x) = \sum \delta_{\nu_k}(x) V^{-p'+1}(x) \tag{26}$$

and $\delta_{\nu_k}(x)$ is the Dirac δ-function at the point ν_k.

(b) *Let $0 < p \le 1$, $g(x) \ge 0$. Then*

$$\sup_{0\le f\downarrow} \frac{\int\limits_0^\infty fg}{\left(\int\limits_0^\infty \left(\frac{1}{x}\int\limits_0^x f\right)^p \nu(x)\,dx\right)^{1/p}} \approx \sup_{t>0}\Big(\int\limits_0^t g\Big)V^{-1/p}(t).$$

2.3. This section is devoted to some applications of Theorem 2.4 to the theory of operators in the $\Gamma_p(\nu)$-Lorentz spaces and, in particular, we obtain analogs of some results from [Saw$_2$] proven for $\Lambda_p(\nu)$-spaces. Throughout this section we assume that the nondegeneracy conditions for a weight function $\nu(s) \geq 0$ of the form

$$\int_0^\infty \frac{\nu(s)\,ds}{(s+1)^p} < \infty; \quad \int_0^1 s^{-p}\nu(s)\,ds = \int_1^\infty \nu(s)\,ds = \infty \tag{27}$$

are satisfied and the measure $d\mathcal{V}(x)$ is given by (26).

Theorem 2.5. *Let the nondegeneracy conditions* (27) *be valid for a locally integrable function* $\nu(s) \geq 0$. *Then*

$$\Gamma_p^*(\nu) = \Gamma_{p'}\big(t^{p'}\mathcal{V}(t)\big), \quad 1 < p < \infty,$$
$$\Gamma_p^*(\nu) = \Gamma_\infty\big(tV^{1/p}(t)\big), \quad 0 < p \leq 1.$$

Now we consider the problem of the boundedness of the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\,dy, \quad x \in R^n,$$

where the supremum is taken over all the cubes $Q \subset R^n$, containing the point x and having sides parallel to the coordinate axes. Using the results from [St$_8$], [St$_{10}$], we obtain the following

Theorem 2.6. *Let* $1 < p,\, q < \infty$ *and let* $\nu(x) \geq 0$, $w(x) \geq 0$ *be locally integrable functions such that the nondegeneracy conditions* (27) *are satisfied for* $\nu(x)$.

(a) *If* $1 < p \geq q < \infty$, *then the inequality*

$$\|Mf\|_{q,w} \leq C\|f\|_{p,\nu}, \quad f \in \Gamma_{p,\nu} \tag{28}$$

is valid if, and only if, $\mathcal{M} = \max(A_0, A_1, B_0, B_1) < \infty$. *Moreover,* $C \approx \mathcal{M}$, *where*

$$A_0 = \sup_{t>0} \Big(\int_0^t w \Big)^{1/q} \Big(\int_t^\infty \mathcal{V} \Big)^{1/p'},$$

$$A_1 = \sup_{t>0} \Big(\int_t^\infty s^{-q} w(s)\, ds \Big)^{1/q} \Big(\int_t^\infty s^{p'} \mathcal{V}(s)\, ds \Big)^{1/p'},$$

$$B_0 = \sup_{t>0} \Big(\int_t^\infty s^{-q} \log^q \Big(\frac{s}{t}\Big) w(s)\, ds \Big)^{1/q} \Big(\int_t^\infty s^{p'} \mathcal{V}(s)\, ds \Big)^{1/p'},$$

$$B_1 = \sup_{t>0} \Big(\int_t^\infty s^{-q} w(s)\, ds \Big)^{1/q} \Big(\int_t^\infty s^{p'} \log^{p'} \Big(\frac{t}{s}\Big) \mathcal{V}(s)\, ds \Big)^{1/p'}.$$

(b) *If* $1 < q < p < \infty$, $1/r = 1/q - 1/p$, *then the inequality* (28) *is true if, and only if,* $\mathcal{L} = \max(\mathcal{A}_0, \mathcal{A}_1, \mathcal{B}_0, \mathcal{B}_1) < \infty$. *Moreover,* $C \approx \mathcal{L}$, *where*

$$\mathcal{A}_0 = \Big(\int_0^\infty \Big(\int_0^t w \Big)^{r/p} \Big(\int_t^\infty \mathcal{V} \Big)^{r/p'} w(t)\, dt \Big)^{1/r},$$

$$\mathcal{A}_1 = \Big(\int_0^\infty \Big(\int_t^\infty s^{-q} w(s)\, ds \Big)^{r/p} \Big(\int_0^t s^{p'} \mathcal{V}(s)\, ds \Big)^{r/p'} t^{-q} w(t)\, dt \Big)^{1/r},$$

$$\mathcal{B}_0 = \Big(\int_0^\infty \Big(\int_t^\infty s^{-q} w(s)\, ds \Big)^{r/p} \Big(\int_0^t s^{p'} \log^{p'} \Big(\frac{t}{s}\Big) \mathcal{V}(s)\, ds \Big)^{r/p'} t^{-q} w(t)\, dt \Big)^{1/r},$$

$$\mathcal{B}_1 = \Big(\int_0^\infty \Big(\int_t^\infty s^{-q} w(s)\, ds \Big)^{r/p} \Big(\int_0^t s^{p'} \log^{p'} \Big(\frac{t}{s}\Big) \mathcal{V}(s)\, ds \Big)^{r/q'} t^{p'} \mathcal{V}(t)\, dt \Big)^{1/r}.$$

(c) *If* $0 < p \leq 1 < q < \infty$, *then the inequality* (28) *is valid if, and only* *if,* $\mathcal{N} = \max(G_0, G_1.G_2) < \infty$. *Moreover,* $C \approx \mathcal{N}$, *where*

$$G_0 = \sup_{t>0} \Big(\int_0^t w \Big)^{1/q} V^{-1/p}(t),$$

$$G_1 = \sup_{t>0} \Big(\int_t^\infty s^{-q} w(s)\, ds \Big)^{1/q} tV^{-1/p}(t),$$

$$G_2 = \sup_{t>0} \Big(\int_t^\infty s^{-q} \log^q \Big(\frac{s}{t}\Big) w(s)\, ds \Big)^{1/q} tV^{-1/p}(t).$$

An analogous assertion holds for the Hilbert transform

$$\mathcal{H}f(x) = \text{ p.v. } \int_{-\infty}^{\infty} \frac{f(y)\, dy}{x-y}.$$

Theorem 2.7. *Let the hypotheses and the notation of Theorem 2.6 be preserved.*

(a) *If* $1 < p \le q < \infty$, *then the inequality*

$$\|\mathcal{H}f\|_{q,w} \le C\|f\|_{p,\nu}, \quad f \in \Gamma_{p,\nu} \tag{29}$$

is valid if, and only if,

$$\mathcal{M}_{\mathcal{H}} = \max(A_0, A_1, B_0, B_1, D_0, D_1) < \infty.$$

Moreover, $C \approx \mathcal{M}_{\mathcal{H}}$, *where*

$$D_0 = \sup_{t>0} \Big(\int_0^t w \Big)^{1/q} \Big(\int_t^\infty \log^{p'} \Big(\frac{s}{t}\Big) \mathcal{V}(s)\, ds \Big)^{1/p'},$$

$$D_1 = \sup_{t>0} \Big(\int_0^t \log^q \Big(\frac{t}{s}\Big) w(s)\, ds \Big)^{1/q} \Big(\int_t^\infty \mathcal{V} \Big)^{1/p'}.$$

(b) *If* $1 < q < p < \infty$, $1/r = 1/q - 1/p$, *then* (29) *is true if, and only if,* $\mathcal{L}_{\mathcal{H}} = \max(\mathcal{A}_0, \mathcal{A}_1, \mathcal{B}_0, \mathcal{B}_1, \mathcal{D}_0, \mathcal{D}_1) < \infty$. *Moreover,* $C \approx \mathcal{L}_{\mathcal{H}}$, *where*

$$\mathcal{D}_0 = \Big(\int_0^\infty \Big(\int_0^t w \Big)^{r/p} \Big(\int_t^\infty \log^{p'} \Big(\frac{s}{t}\Big) \mathcal{V}(s)\, ds \Big)^{r/p'} w(t)\, dt \Big)^{1/r},$$

$$\mathcal{D}_1 = \Big(\int_0^\infty \Big(\int_0^t \log_q \Big(\frac{t}{s}\Big) w(s)\, ds \Big)^{r/q} \Big(\int_t^\infty \mathcal{V} \Big)^{r/q'} \mathcal{V}(s)\, dt \Big)^{1/r}.$$

(c) *If $0 < p \le 1 < q < \infty$, then (29) is fulfilled if, and only if, $\mathcal{N}_{\mathcal{H}} = \max(G_0, G_1, G_3, G_4) < \infty$. Moreover, $C \approx \mathcal{N}_{\mathcal{H}}$, where*

$$G_3 = \sup_{t>0} \Big(\int_0^t \log^q \Big(\frac{s}{t}\Big) w(s)\, ds\Big)^{1/q} V^{-1/p}(t).$$

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