RECENT DEVELOPMENTS CONCERNING ENTROPY AND APPROXIMATION NUMBERS

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INTRODUCTION

The ideas underlying the concept of entropy numbers go back a long way, certainly to the work of Kolmogorov in the 1930s on the metric entropy of compact subsets of a metric space. With the introduction of linear structures, in the form of linear mappings between Banach spaces, the study of this topic acquired fresh impetus both at the abstract and the concrete level: the work of Pietsch did much to promote the development of the theory of entropy numbers, while embeddings of Sobolev spaces were analysed from the entropy point of view by Birman and Solomyak, Triebel and numerous others. A second landmark in the growth of the subject was provided in 1980 by Carl's observation that the entropy numbers of a compact linear map T from a Banach space to itself are related to the eigenvalues of T by a simple inequality. This immediately gave the possibility of estimating eigenvalues by means of estimates for entropy numbers; and since interesting maps (such as those arising from integral operators) can often be factorised into the composition of compact embedding maps and continuous maps, it was clear that knowledge of the behaviour of the entropy numbers of compact embeddings could be translated into eigenvalue estimates. In view of this, it is perhaps a little surprising that not until the 1990s was any systematic attack made along the lines indicated on problems involving the distribution of eigenvalues of elliptic operators. A possible explanation for this delay is that to obtain really satisfactory results it is crucial to have accurate estimates for the entropy numbers of embedding maps acting between very general function spaces, even in delicate limiting situations; and these were not available until comparatively recently, when Fourier-analytical techniques based on advances in the theory of function spaces were used to establish them.

Our main object here is to describe these embedding results and to show how they can be used to study the eigenvalue distributions of degenerate elliptic differential and pseudodifferential operators: in particular, we provide a survey of the papers [31] - [34]. Although we shall be concentrating on entropy numbers we shall, where appropriate, give results concerning approximation numbers, for these numbers have independent interest and are also intimately connected with eigenvalue distributions. While we shall focus on embeddings, it should be noted that a good deal of work has been done on the analysis of various operators, such as Volterra integral operators, from the standpoint of entropy and approximation numbers, and measures of non-compactness. For details of this, and further references, we refer to [24], [25], [30] and [58].

2. Entropy and approximation numbers

2.1. Quasi-normed spaces

We begin with some basic facts about quasi-normed spaces: these can be found, for example, in [49]. A *quasi-norm* on a (real or complex) linear space B is a map $\|\cdot|B\|\colon B \to \mathbb{R}_+ = [0,\infty)$ such that (i) $\|x|B\| = 0$ if, and only if, $x = 0$; (ii) $\|\lambda x|B\| = |\lambda|\|x|B\|$ for all scalars λ and all $x \in B$; (iii) there is a constant C such that for all x, $y \in B$,

$$\|x+y|B\| \le C\big(\|x|B\| + \|y|B\|\big).$$

Clearly $C \ge 1$; if $C = 1$ is allowed then $\|\cdot|B\|$ is a norm on B. Each quasi-norm on B defines a topology on B, with a basis of neighbourhoods of any point $x \in B$ given by the sets $\{y \in B\colon \|x-y|B\| < 1/n\}$ $(n \in \mathbb{N})$. The pair $(B, \|\cdot|B\|)$ is called a *quasi-normed space*; if every Cauchy sequence (defined in the obvious way) in B converges to an element of B, we call B a quasi-Banach space.

Given any $p \in (0,1]$, a p-norm on a linear space B is a map $\|\cdot|B\|\colon B \to \mathbb{R}_+$ which satisfies conditions (i) and (ii) above and instead of (iii) satisfies

$$\text{(iii')} \qquad \|x+y|B\|^p \le \|x|B\|^p + \|y|B\|^p$$

for all $x, y \in B$. It can be shown that if $\|\cdot|B\|$ is a quasi-norm on B,then there exists $p \in (0,1]$ and a p-norm on B which is equivalent to $\|\cdot|B\|$ in the sense that one is bounded above and below by constant multiples of the other; the constant C in (iii) can be taken to be $2^{1/p-1}$. Conversely, any p-norm is a quasi-norm with $C = 2^{1/p-1}$. If $\|\cdot|B\|$ is a p-norm, the pair $(B, \|\cdot|B\|)$ is called a p-normed space; if it is complete it is called a p-Banach space. The sequence spaces ℓ_q are quasi-Banach spaces if $0 < q < 1$, and of course they are Banach spaces if $q \ge 1$.

Let A, B, be quasi-Banach spaces and let $T\colon A \to B$ be linear. Just as in the Banach space case the notions of boundedness and compactness of T are defined, as is the spectrum $\sigma(T)$ of T. It can be shown (cf. [35]) that if B is a complex, infinite-dimensional quasi-Banach space and $T\colon B \to B$ is bounded and linear, then $\sigma(T) \setminus \{0\}$ consists of an at most countably infinite number of eigenvalues, each of finite algebraic multiplicity, which may accumulate only at 0.

2.2. Entropy numbers

For general information about these, in a Banach space setting, we refer to [23] and [48].

Definition. Let A, B be quasi-Banach spaces, let $U_A = \big\{a \in A\colon \|a|A\| \leq 1\big\}$ and let $T \in L(A,B)$, the space of all bounded linear maps from A to B. Then for all $k \in \mathbb{N}$, the k^{th} entropy number $e_k(T)$ of T is defined by

(2.1)

$$e_k(T) = \inf\Big\{\varepsilon > 0\colon T(U_A) \subset \bigcup_{j=1}^{2^{k-1}} (b_j + \varepsilon U_B) \text{ for some } b_1, \ldots, b_{2^{k-1}} \in B\Big\}.$$

It is easy to verify that if A, B, C are quasi-Banach spaces and $S, T \in L(A,B)$, $R \in L(B,C)$, then

(i) $\|T\| \geq e_1(T) \geq e_2(T) \geq \cdots \geq 0$; $e_1(T) = \|T\|$ if B is a Banach space;

(ii) if B is a p-Banach space $(0 < p \leq 1)$, then for all k, $\ell \in \mathbb{N}$,

$$e^p_{k+\ell-1}(S+T) \leq e^p_k(S) + e^p_\ell(T); \tag{2.2}$$

(iii) for all k, $\ell \in \mathbb{N}$,

$$e_{k+\ell-1}(R \circ S) \leq e_k(R)e_\ell(S). \tag{2.3}$$

Since the $e_k(T)$ are non-negative and decrease as $k \to \infty$, it is plain that $\lim_{k\to\infty} e_k(T)$ exists and equals

$$\inf\{\varepsilon > 0\colon T(U_A) \text{ can be covered by finitely many } B\text{-balls of radius } \varepsilon\}.$$

This limit is called the *ball measure of non-compactness* of T and is denoted by $\widetilde{\beta}(T)$; $\widetilde{\beta}(T) = 0$ if, and only if, T is compact. Note also that unless $T \in L(A,B)$ is the zero operator, none of the $e_k(T)$ is 0. The dependence of $e_k(T)$ upon k for particular maps T, and especially for embedding maps, will be examined later, but for the moment we observe that if A is a complex Banach space of finite dimension m and $I\colon A \to A$ is the identity map, then

$$1 \leq 2^{(k-1)/(2m)} e_k(I) \leq 4 \qquad (k \in \mathbb{N});$$

if A, B are Banach spaces and $T \in L(A,B)$, $T \neq 0$ has the property that there are positive numbers c, ρ such that for all $k \in \mathbb{N}$, $e_k(T) \leq c2^{-\rho k}$, then T is of finite rank (cf. [16]).

If $T \in L(A,B)$, where A, B are Banach spaces, a question which has attracted much interest is what relationship exists between the entropy numbers of T and those of its adjoint T^*. When A and B are Hilbert spaces the answer is easy, using the polar decomposition theorem: $e_k(T) = e_k(T^*)$ for all $k \in \mathbb{N}$. The question in general Banach spaces is still open, but Bourgain, Pajor, Szarek and Tomczak–Jaegermann [13] have shown that if A is uniformly convex and T is compact, then there is a positive constant c (depending only on A) such that for all $m \in \mathbb{N}$ and all $p \in [1, \infty)$,

$$c^{-1}\Big(\sum_{k=1}^{m} e_k^p(T^*)\Big)^{1/p} \leq \Big(\sum_{k=1}^{m} e_k^p(T)\Big)^{1/p} \leq c\Big(\sum_{k=1}^{m} e_k^p(T^*)\Big)^{1/p};$$

similar inequalities hold if the ℓ_p norm is replaced by any symmetric norm, such as that on the Lorentz spaces $\ell_{p,q}$, so that if $e_k(T) = O(k^{-1/p})$ as $k \to \infty$, then $e_k(T^*) = O(k^{-1/p})$.

The entropy numbers behave reasonably well under interpolation, at least when one end-point is fixed. Thus let A be a quasi-Banach space, let $\{B_0, B_1\}$ be an interpolation couple of p-Banach spaces, let $0 < \theta < 1$ and let B_θ be a quasi-Banach space such that $B_0 \cap B_1 \subset B_\theta \subset B_0 + B_1$ and

$$\big\|b|B_\theta\big\| \leq \big\|b|B_0\big\|^{1-\theta}\big\|b|B_1\big\|^{\theta} \text{ for all } b \in B_0 \cap B_1.$$

Let $T \in L(A, B_0 \cap B_1)$ where $B_0 \cap B_1$ is given the quasi-norm $\max\big(\|b|B_0\|, \|b|B_1\|\big)$. Then

$$e_{k+\ell-1}(T\colon A \to B_\theta) \leq 2^{1/p} e_k^{1-\theta}(T\colon A \to B_0)\, e_\ell^{\theta}(T\colon A \to B_1).$$

Corresponding results hold when the domain space A is allowed to vary and the target space is held fixed. In a Banach space setting these results, expressed in terms of entropy functions and entropy ideals, were developed by Peetre and Triebel (see [61, 1.16.2]); their reformulation in terms of entropy numbers is given in Pietsch [48, 12.1]. For a treatment of the quasi-Banach case see Haroske and Triebel [43]. It is not clear what happens when both end-points A and B can vary simultaneously: the techniques used to show that the property of compactness interpolates well in such circumstances (see [19], [20] and [22], for example) do not seem to be delicate enough to obtain the necessary estimates.

Another interesting question relates to the estimation of the entropy numbers of tensor products of operators. We refer to [37] for some recent work on the size of the sequence $\left(e_n(S\widehat{\otimes}_\alpha T)\right)$ in the scale of Lorentz sequence spaces for tensor norms α, and also for some information about the subtler 'local' question of estimating the individual entropy numbers of $S\widehat{\otimes}_\alpha T$ in terms of those of S and T. In particular, asymptotic bounds are given for the entropy numbers of tensored operators on the Schatten trace classes $c_p(\ell_2)$. References to earlier work in this area will also be found in [37].

2.3. Approximation numbers

Definition. Let $T \in L(A, B)$ where A, B are quasi-Banach spaces. Then given any $k \in \mathbb{N}$, the k^{th} approximation number $a_k(T)$ of T is given by

$$a_k(T) = \inf\{\|T - L\| \colon L \in L(A, B), \operatorname{rank} L < k\},$$

where $\operatorname{rank} L = \dim L(A)$.

The $a_k(T)$ have properties similar to (i), (ii), (iii) listed above in 2.2 for entropy numbers. Despite these resemblances, there are radical differences between the two sets of numbers. Thus if A, B are Banach spaces and $T \in L(A, B)$, then $a_k(T) = 0$ if, and only if, $\operatorname{rank} T < k$; and if $\dim A \geq n$ and $I \colon A \to A$ is the identity map, then $a_k(I) = 1$ for $k = 1, \ldots, n$. Moreover, if $T \in L(A, B)$ is compact and A, B are Banach spaces, then $a_k(T^*) = a_k(T)$ for all $k \in \mathbb{N}$; if the compactness condition is dropped, then $a_k(T^*) \leq a_k(T) \leq 5a_k(T^*)$ for all $k \in \mathbb{N}$ (cf. [37] and [23]). This contrasts sharply with the complicated current situation insofar as adjoints and entropy numbers are concerned; but by way of balance we shall see later, in 3.2, that the approximation numbers do not behave well under interpolation.

It is natural to ask whether there are connections between $e_k(T)$ and $a_k(T)$ for a compact map T: for example, is there a constant $C > 0$ such that for all $k \in \mathbb{N}$,

$$e_k(T) \leq C a_k(T)?$$

Plainly this cannot hold if T is of finite rank; but even for maps with infinite-dimensional range there are difficulties, for Carl and Stephani [16] give an example of a diagonal map $D\colon \ell_p \to \ell_p$ for which $a_k(D) = 2^{-k}$ yet

$$2^{-\sqrt{2k}-1} \leq e_{k+1}(D) \leq 3\sqrt{2}\, 2^{-\sqrt{2k}}$$

for all $k \in \mathbb{N}$. This suggests that the rate of decay of the approximation numbers may be a relevant factor in obtaining the desired inequality or some global version of it. The following result, due to Triebel [65], implements these ideas.

Theorem. *Let A and B be quasi-Banach spaces and let $T \in L(A, B)$ be compact.*

(i) *Suppose that for some $c > 0$,*

$$a_{2^{j-1}}(T) \leq c a_{2^j}(T) \text{ for all } j \in \mathbb{N}.$$

Then there is a positive constant C such that for all $j \in \mathbb{N}$,

$$e_j(T) \leq C a_j(T).$$

(ii) *Let $f\colon \mathbb{N} \to \mathbb{R}$ be positive and increasing, and suppose that for some $c > 0$,*

$$f(2^j) \leq c f(2^{j-1}) \quad \text{for all } j \in \mathbb{N}.$$

Then there is a positive constant C such that for all $n \in \mathbb{N}$,

$$\sup_{1 \leq j \leq n} f(j) e_j(T) \leq C \sup_{1 \leq j \leq n} f(j) a_j(T).$$

As particular consequences of this theorem we see immediately that if $a_j(T) = O(j^{-\rho})$ (resp. $a_j(T) = O((\log j)^{-\rho})$) as $j \to \infty$, where ρ is a positive constant, then $e_j(T) = O(j^{-\rho})$ (resp. $e_j(T) = O((\log j)^{-\rho})$). This will be of interest because, as we shall see, the approximation numbers of embeddings between function spaces typically have power-type or logarithmic-type decay.

2.4. Eigenvalues

To conclude this section we mention the connections between eigenvalues of a compact map and entropy and approximation numbers. Let A be a complex quasi-Banach space and let $T \in L(A)$ $(:= L(A, A))$ be compact.

We know that the spectrum of T, apart from the point 0, consists solely of eigenvalues of finite algebraic multiplicity: let $\{\lambda_k(T)\}$ be the sequence of all non-zero eigenvalues of T, each repeated according to algebraic multiplicity and ordered so that

$$|\lambda_1(T)| \geq |\lambda_2(T)| \geq \cdots \geq 0.$$

If T has only $m(< \infty)$ distinct eigenvalues and M is the sum of their algebraic multiplicities, we put $\lambda_n(T) = 0$ for all $n > M$.

So far as the entropy numbers of T are concerned, the most useful connection between them and the eigenvalues of T, from our point of view, is given by the following theorem, proved by Carl and Triebel [17] when A is a Banach space.

Theorem. *Let T and $\{\lambda_k(T)\}_{k\in\mathbb{N}}$ be as above. Then for all $k \in \mathbb{N}$,*

$$\Big(\prod_{m=1}^{k} |\lambda_m(T)|\Big)^{1/k} \leq \inf_{n\in\mathbb{N}} 2^{n/(2k)} e_n(T).$$

The proof is a modification of the volume-covering argument used by Carl and Triebel to handle the case when A is a Banach space: cf. [35]. Note that by taking $n = k$ we obtain the following

$$|\lambda_k(T)| \leq \sqrt{2} e_k(T). \tag{2.4}$$

This was originally proved by Carl [14], using a different method of proof, in the Banach space setting. It is this corollary which will be extensively used later on to estimate eigenvalues of certain (pseudo-) differential operators.

The approximation numbers also have strong connections with eigenvalues. For example, if H is a complex Hilbert space and $T \in L(H)$ is compact, then T^*T has a positive, self-adjoint, compact square root $|T|$ and for all $k \in \mathbb{N}$,

$$a_k(T) = \lambda_k(|T|)$$

(see [23], for example). (Hence the approximation numbers of T coincide with its eigenvalues if T is compact and positive.) Also, a famous inequality of Weyl states that for all $n \in \mathbb{N}$ and all $p \in (0, \infty)$,

$$\sum_{j=1}^{n} |\lambda_j(T)|^p \leq \sum_{j=1}^{n} a_j^p(T).$$

These are Hilbert space results, but even in general Banach spaces much can be established. Thus König (cf. [45]) has shown that if A is a complex Banach space and $T \in L(A)$ is compact, then

$$|\lambda_n(T)| = \lim_{k\to\infty} a_n^{1/k}(T^n)$$

and

$$\Big(\sum_{k=1}^{n} |\lambda_k(T)|^p\Big)^{1/p} \le K_p\Big(\sum_{k=1}^{n} a_k^p(T)\Big)^{1/p} \qquad (n \in \mathbb{N},\ 0 < p < \infty)$$

where

$$K_p = \begin{cases} 2e/\sqrt{p} & \text{if } 0 < p < 1, \\ 2^{1/p}\sqrt{2e} & \text{if } 1 \le p < \infty. \end{cases}$$

There are Lorentz space analogues of this last inequality which, in particular, show that if for some $\rho > 0$, $a_n(T) = O(n^{-\rho})$ (resp. $o(n^{-\rho})$), then $|\lambda_n(T)| = O(n^{-\rho})$ (resp. $o(n^{-\rho})$) as $n \to \infty$.

3. Embeddings; nonlimiting cases

The importance of embedding maps from one function space to another, typified by the Sobolev embedding theorems, arises from the possibility of factorising the maps derived from differential or integral operators into the composition of maps, one of the components being an embedding map. We shall see later on in detail how this procedure works. Accepting this for the moment, we introduce the scale of function spaces that will be used.

3.1. Function spaces

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and its dual space $\mathcal{S}'(\mathbb{R}^n)$ of all complex-valued tempered distributions are supposed to have the usual meaning here. Also, if $0 < p \le \infty$, $L_p(\mathbb{R}^n)$ is the usual complex quasi-Banach space with respect to Lebesgue measure, quasi-normed by $\big\| \cdot |L_p(\mathbb{R}^n)\big\|$. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\operatorname{supp}\phi \subset \{y \in \mathbb{R}^n\colon |y| < 2\} \text{ and } \phi(x) = 1 \text{ if } |x| \le 1; \tag{3.1}$$

put $\phi_0 = \phi$ and $\phi_j(x) = \phi(2^{-j}x) - \phi(2^{-j+1}x)$ for $j \in \mathbb{N}$. Then $\{\phi_j\}$ give a dyadic partition of unity: $\sum_{j=0}^{\infty} \phi_j(x) = 1$ for all $x \in \mathbb{R}^n$. Given any $f \in \mathcal{S}'(\mathbb{R}^n)$, we denote by $\widehat{f}$ and $f^\vee$ its Fourier transform and its inverse Fourier transform respectively, and recall that $(\phi_j\widehat{f})^\vee$ is an entire analytic function on $\mathbb{R}^n$.

Definition 1. (i) Let $s \in \mathbb{R}$, $0 < p \leq \infty$ and $0 < q \leq \infty$. Then $B^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f|B^s_{pq}(\mathbb{R}^n)\|_\phi := \Big(\sum_{j=0}^{\infty} 2^{jsq}\|(\phi_j \widehat{f})^\vee|L_p(\mathbb{R}^n)\|^q\Big)^{1/q} \tag{3.2}$$

(with the usual modification if $q = \infty$) is finite.

(ii) Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. Then $F^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f|F^s_{pq}(\mathbb{R}^n)\|_\phi := \Big\|\Big(\sum_{j=0}^{\infty} 2^{jsq}|(\phi_j \widehat{f})^\vee|^q\Big)^{1/q}|L_p(\mathbb{R}^n)\Big\| \tag{3.3}$$

(with the usual modification if $q = \infty$) is finite.

The theory of these spaces has been systematically developed in [62], [63], and in particular it has been shown that $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ are quasi-Banach spaces (Banach spaces if $p, q \geq 1$) which are independent of the particular $\phi \in \mathcal{S}(\mathbb{R}^n)$ chosen according to (3.1), in the sense of equivalent quasi-norms. We shall therefore omit the subscript ϕ in (3.2) and (3.3) in what follows.

Definition 2. Let Ω be a bounded domain in $\mathbb{R}^n$ with C^∞ boundary.

(i) Let $s \in \mathbb{R}$, $0 < p \leq \infty$ and $0 < q \leq \infty$. Then $B^s_{pq}(\Omega)$ is the restriction of $B^s_{pq}(\mathbb{R}^n)$ to Ω.

(ii) Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. Then $F^s_{pq}(\Omega)$ is the restriction of $F^s_{pq}(\mathbb{R}^n)$ to Ω.

In other words, $f \in \mathcal{D}'(\Omega)$ $\big(\mathcal{D}(\Omega) = C_0^\infty(\Omega)\big)$ is an element of $B^s_{pq}(\Omega)$ if, and only if, there exists $g \in B^s_{pq}(\mathbb{R}^n)$ with $f = g\big|_\Omega$ in $\mathcal{D}'(\Omega)$; and

$$\|f|B^s_{pq}(\Omega)\| = \inf \|g|B^s_{pq}(\mathbb{R}^n)\| \tag{3.4}$$

where the infimum is taken over all such g. Analogous considerations hold for the spaces $F^s_{pq}(\Omega)$. For each of the spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ there is an extension operator from Ω to $\mathbb{R}^n$, that is, a bounded linear map ext: $B^s_{pq}(\Omega) \to B^s_{pq}(\mathbb{R}^n)$ with ext $f|_\Omega = f$ for all $f \in B^s_{pq}(\Omega)$; and similarly for $F^s_{pq}(\Omega)$. If $\chi \in \mathcal{S}(\mathbb{R}^n)$ is such that $\chi(x) = 1$ for all $x \in \Omega$, then $\chi\cdot$ext is also an extension operator.

The B and F scales cover many of the well-known classical spaces. Thus $B^s_{\infty,\infty}(\Omega)$ for $s > 0$ is the Hölder-Zygmund space; $B^s_{pq}(\Omega)$, with $s > 0$, $1 < p < \infty$, $1 \le q \le \infty$, is the classical Besov space; $F^0_{p2}(\Omega)$, with $0 < p < \infty$, is the (non-homogeneous) Hardy space $h_p(\Omega)$, which coincides with $L_p(\Omega)$ if $1 < p < \infty$; and $F^s_{p2}(\Omega)$, when $s \in \mathbb{R}$ and $0 < p < \infty$, is the (fractional) Sobolev (-Hardy) space $H^s_p(\Omega)$, which is the (fractional) Sobolev space (or Bessel potential space) when $1 < p < \infty$ and coincides with the classical Sobolev space $W^s_p(\Omega)$ when $s \in \mathbb{N}$ and $1 < p < \infty$. We refer to [62], [63] for a systematic account of this.

Alternative characterisations of these spaces are possible. In particular, an intrinsic characterisation of $H^s_p(\Omega) = F^s_{p2}(\Omega)$ ($s \in \mathbb{R}$, $0 < p < \infty$) can be given. Given any $f \in L_p(\Omega)$ ($0 < p < \infty$) we define the difference $(\Delta^1_h f)(x)$ (x, $h \in \mathbb{R}^n$) by

$$(\Delta^1_h f)(x) = \begin{cases} f(x+h) - f(x) & \text{if } x, x+h \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Higher-order differences $(\Delta^m_h f)(x)$ are defined by iteration:

$$(\Delta^m_h f)(x) = (\Delta^{m-1}_h \Delta^1_h f)(x) \qquad (m \in \mathbb{N}, m \ge 2).$$

We also need mean values of differences and set

$$(d^m_t f)(x) = t^{-n} \int\limits_{|h| \le t} |\Delta^m_h f(x)|\, dh$$

for $m \in \mathbb{N}$, $x \in \Omega$, $t > 0$. It turns out (cf. Triebel [63]) that if $0 < p < \infty$, $s > n\big(\frac{1}{p} - 1\big)_+$, $m \in \mathbb{N}$, $m > s$, then

(3.5)
$$H^s_p(\Omega) = \Big\{ f \in L_{\max(p,1)}(\Omega) : \\ \big\| f | L_p(\Omega) \big\| + \Big\| \Big(\int\limits_0^1 t^{-2s} (d^m_t f)^2(\cdot) \frac{dt}{t} \Big)^{1/2} | L_p(\Omega) \Big\| < \infty \Big\}.$$

The choice of $m \in \mathbb{N}$, $m > s$ is unimportant, different choices merely giving equivalent (quasi-) norms. If $s \le n\big(\frac{1}{p} - 1\big)_+$, we choose $m \in \mathbb{N}$ so that $s + 2m > n\big(\frac{1}{p} - 1\big)_+$ and define $H^s_p(\mathbb{R}^n)$ to be $(-\operatorname{id} - \Delta)^m\ H^{s+2m}_p(\mathbb{R}^n)$, where Δ is the Laplace operator and $H^{s+2m}_p(\mathbb{R}^n)$ is defined as in (3.5) but with Ω replaced by $\mathbb{R}^n$. Then $H^s_p(\Omega)$ may be defined to be the restriction of $H^s_p(\mathbb{R}^n)$ to Ω, and defined in this way it again coincides with $F^s_{p2}(\Omega)$ (see Triebel [63]). We remind the reader that when $s \in \mathbb{N}$ and $1 < p < \infty$ the space $H^s_p(\Omega) = F^s_{p2}(\Omega)$ also coincides, up to equivalent norms, with the classical Sobolev space

$$W^s_p(\Omega) = \{u \colon D^\alpha u \in L_p(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq s\}.$$

Triebel [63] has also given intrinsic characterisations of $\mathrm{B}^s_{pq}(\Omega)$ and $\mathrm{F}^s_{pq}(\Omega)$ via oscillations, derivatives and differences of functions if, as we have assumed in this paper, Ω is a bounded domain with C^∞ boundary and $s > n\left(\frac{1}{p} - 1\right)_+$; and with Winkelvoss [67] he has recently given atomic characterisations of these spaces under very mild and natural restrictions on Ω (allowing a quite rough boundary $\partial\Omega$) and for the whole range of values of s, p and q.

3.2. Hölder inequalities

Throughout this section Ω will stand for a bounded domain in $\mathbb{R}^n$ with C^∞ boundary. The classical Hölder inequality may be written as

$$L_{r_1}(\Omega)L_{r_2}(\Omega) \subset L_r(\Omega), \tag{3.6}$$

where r_1, $r_2 \in [1,\infty]$ and $1/r = 1/r_1 + 1/r_2 \leq 1$. This may be interpreted as a statement that any $g \in L_{r_2}(\Omega)$ is a pointwise multiplier $f \longmapsto gf$ from $L_{r_1}(\Omega)$ to $L_r(\Omega)$. Sickel and Triebel [53] generalised this to the setting of the spaces $\mathrm{B}^s_{pq}(\Omega)$ and $\mathrm{F}^s_{pq}(\Omega)$, and we shall need some of their results later on, in the context of the space $\mathrm{H}^s_p(\Omega)$. This is contained in the following

Theorem (cf. [53], [34]). *Let r_1, $r_2 \in (1,\infty)$ and suppose that $1/r = 1/r_1 + 1/r_2 \leq 1$. Suppose also that*

$$\mathrm{s} \in \mathbb{R} \text{ and } \frac{1}{r_1} + \frac{s}{n} > 0. \tag{3.7}$$

Then

$$H^s_{r_1^s}(\Omega)H^{|s|}_{r_2^{|s|}}(\Omega) \subset H^s_{r^s}(\Omega), \tag{3.8}$$

where

$$\frac{1}{r^s} = \frac{1}{r} + \frac{s}{n} \tag{3.9}$$

and $r_1^s, r_2^{|s|}$ *are defined analogously.*

Note that if $s = 0$, then (3.8) coincides with (3.6), apart from limiting cases. If $s > 0$, (3.8) gives what we shall describe as a Hölder inequality at the level s. To aid the understanding of this result we refer to Fig. 1 and remark that any line of slope n in $\big(1/p, s\big)$ diagram is characterised by the point at which it meets the axis $s = 0$: if this point is $\big(1/r, 0\big)$, then any

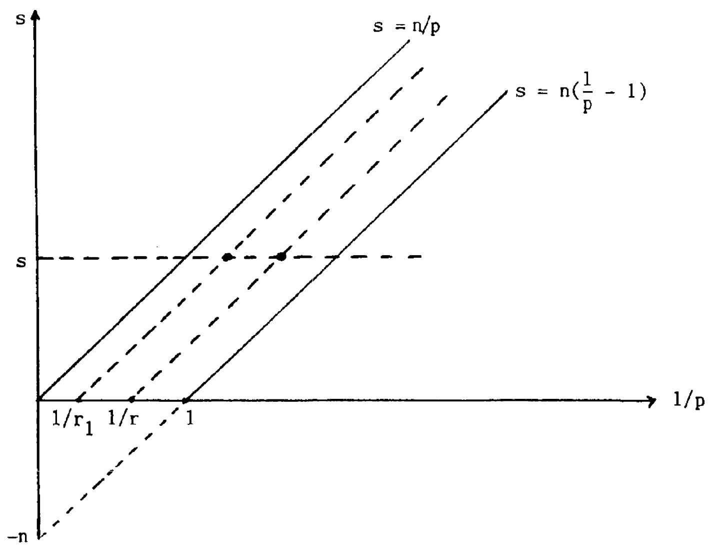

Fig. 1

point on the line has coordinates $\big(1/r^s, s\big)$, where r^s is defined as in (3.9) above. Thus any $g \in H^s_{r_2^s}$ gives a mapping $f \longmapsto gf \colon H^s_{r_1^s}(\Omega) \to H^s_{r^s}(\Omega)$, characterised by the large dots in Fig. 1. The result (3.8) with $s > 0$ is a special case of Theorem 4.2 of [53]. If $s < 0$, (3.8) follows from (3.8) with $|s|$ instead of s together with duality

$$(H^s_p(\mathbb{R}^n))' = H^{-s}_{p'}(\mathbb{R}^n), \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad 1 < p < \infty,$$

and restriction to Ω. Briefly, (3.8) simply means that Hölder's inequality is shifted along the lines of slope n in the $\big(1/p, s\big)$-diagram to the appropriate s-level.

In the limiting case $r_1 = \infty$, we have $1/r_1^s = s/n$ and $1 < r_2 = r < \infty$. As a consequence of more general results (cf. [53, Theorem 4.3]) we then have

$$H^s_{r_1^s}(\Omega) H^s_{r^s}(\Omega) \subset H^s_{r^s}(\Omega) \text{ if, and only if, } s \geq n. \tag{3.10}$$

3.3 Entropy and approximation numbers

Here we give estimates for these numbers when the mapping in question is an embedding from one function space in our scales of spaces to another. To be able to present the results in a succinct form we shall let $A^s_{pq}(\Omega)$ stand for either $B^s_{pq}(\Omega)$ or $F^s_{pq}(\Omega)$, with the understanding that if $A^s_{pq}(\Omega) = F^s_{pq}(\Omega)$ then p must be finite. Once again Ω will stand for a bounded domain in $\mathbb{R}^n$ with C^∞ boundary. The principal result in the non-limiting situations which we are here handling is the following:

Theorem ([31,32]). *Let $-\infty < s_2 < s_1 < \infty$, suppose that p_1, p_2, q_1, $q_2 \in (0, \infty]$ and assume that*

$$\delta := s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > 0. \tag{3.11}$$

Let $\mathrm{id}\colon A^{s_1}_{p_1 q_1}(\Omega) \to A^{s_2}_{p_2 q_2}(\Omega)$ be the natural embedding. Then for the entropy numbers $e_k(\mathrm{id})$ we have

$$e_k(\mathrm{id}) \asymp k^{-(s_1 - s_2)/n}; \tag{3.12}$$

that is, there are positive numbers c_1 and c_2 such that for all $k \in \mathbb{N}$,

$$c_1 k^{-(s_1-s_2)/n} \leq e_k(\mathrm{id}) \leq c_2 k^{-(s_1-s_2)/n}.$$

For the approximation numbers $a_k(\mathrm{id})$ the position is more complicated: if in addition to the general hypotheses, either

$$0 < p_1 \leq p_2 \leq 2 \tag{3.13}$$

or

$$2 \leq p_1 \leq p_2 \leq \infty \tag{3.14}$$

or

$$0 < p_2 \leq p_1 \leq \infty, \tag{3.15}$$

then

$$a_k(\mathrm{id}) \asymp k^{-\delta/n}; \tag{3.16}$$

if in addition to the general hypotheses,

$$0 < p_1 \le 2 \le p_2 < \infty \text{ and } \lambda = \frac{s_1 - s_2}{n} - \max\left(\frac{1}{2} - \frac{1}{p_2}, \frac{1}{p_1} - \frac{1}{2}\right) > \frac{1}{2}, \tag{3.17}$$

then

$$a_k(\mathrm{id}) \asymp k^{-\lambda}; \tag{3.18}$$

and if in addition to the general hypotheses,

$$0 < p_1 \le 2 \le p_2 \le \infty; \tag{3.19}$$

then there are positive constants c_1 *and* c_2 *such that for all* $k \in \mathbb{N}$,

$$c_1 k^{-\varepsilon-\delta/n} \le a_k(\mathrm{id}) \le c_2 k^{-\delta/n}, \tag{3.20}$$

where $\varepsilon = \min\Big(1/2 - 1/p_2, 1/p_1 - 1/2\Big)$.

Remarks. 1. The theorem as stated is proved in [31] and [32]. Of course, special cases of this result hve been known for many years, and go back to the work of Birman and Solomjak [7], [8], [9], [10], who showed that if $s \in \mathbb{N}$, $1 < p_1 < p_2 \le \infty$, $s - n/p_1 > -n/p_2$, then the entropy number e_k of the natural embedding of the Sobolev space $W^s_{p_1}(\Omega)$ in $\mathrm{L}_{p_2}(\Omega)$ satisfies

$$e_k \asymp k^{-s/n},$$

with the same estimate for the approximation numbers. Their method of proof used the technique of piecewise polynomial approximations which is due to them and which has been often used in modified and refined versions since its introduction in 1967. Of these versions, that of spline approximation which, when combined with the so-called Ciesielski isomorphism, enables problems of our type for Besov spaces $\mathrm{B}^s_{pq}(\Omega)$ with p, $q \in [1, \infty]$ to be reduced to corresponding problems for diagonal operators acting in ℓ_p sequence spaces. For this approach and other earlier work on classical Besov spaces we refer the reader to Carl [15], König [45], Linde [47], Pietsch [49] and Triebel [61]. The approach of [31], [32] is completely different from this earlier work, and is based on Fourier-analytical techniques for the spaces

$\mathrm{B}^s_{pq}(\Omega)$ which work for all values of the parameters p and q; that is, for all $p,\ q \in (1, \infty]$.

2. While the results in the theorem concerning entropy numbers are aesthetically pleasing, those for the approximation numbers appear less elegant. Of course, this is partly due to the curious role played by the number 2 in (3.13)–(3.20), but this is simply a fact of life so far as these and other width numbers are concerned and is well established in the literature: see [45, 3.c.2, 3.c.7], [49, 6.4.14] and [47]. There is still a gap in the results: what is the true rate of decay of the $a_k(\mathrm{id})$, if there is one, if (3.19) holds but $s_1 - s_2 \le n \max(1 - 1/p_2, 1/p_2)$?

3. The different rates of decay given in (3.16) and (3.18) show that there can be no unrestricted interpolation property for approximation numbers and that there is no common best approximating operator covering all cases.

3.4. Rough indication of the proof of the Theorem

In view of the embeddings

$$B^s_{p,\min(p,q)}(\Omega) \subset F^s_{pq}(\Omega) \subset B^s_{p,\max(p,q)}(\Omega)$$

it is enough to deal with the B spaces. We may also assume, without loss of generality, that $\Omega \subset Q_1$, where $Q_r = \{x \in \mathbb{R}^n \colon |x_j| \le r \text{ for } j = 1, \dots, n\}$. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be such that $\operatorname{supp}\psi \subset Q_2$ and $\psi(x) = 1$ for all $x \in Q_1$. Since

$$B^{s_1}_{p_1 q_1}(\Omega) \xrightarrow{E} \{f \colon f \in B^{s_1}_{p_1 q_1}(\mathbb{R}^n), \operatorname{supp} f \subset Q_1\} \subset B^{s_2}_{p_2 q_2}(\mathbb{R}^n) \xrightarrow{R} B^{s_2}_{p_2 q_2}(\Omega),$$

where E is an appropriate extension operator and R is the restriction operator, it is enough to prove the theorem for the middle embedding. In particular, $f = \psi f$ and hence, using the dyadic partition of unity, we use the splitting

$$f = \psi \sum_{j=1}^{N} (\phi_j \widehat{f})^\vee + \psi \sum_{j=N+1}^{\infty} (\phi_j \widehat{f})^\vee := f_N + f^N, \qquad N \in \mathbb{N}. \tag{3.21}$$

Now let $\|f|B^{s_1}_{p_1 q_1}(\mathbb{R}^n)\| \le 1$, $f \in \psi f$. By pointwise multiplier and Fourier multiplier properties, together with Nikolskij's inequality (see Triebel [62, (1.3.2/5)]) it can be shown that

$$\|f^N|B^{s_2}_{p_2 q_2}(\mathbb{R}^n)\| \le 2^{-N\delta} c. \tag{3.22}$$

To handle f_N in (3.21) we need a representation formula for $(\phi_j \widehat{f})^\vee$. Expand $\phi_j \widehat{f}$ in the cube $Q_{2^j\pi}$ in a trigonometric series (in the sense of periodic distributions):

$$(\phi_j \widehat{f})(\xi) = \sum_{m\in\mathbb{Z}^n} a_m e^{-2^{-j} im\cdot\xi} \qquad (\xi \in Q_{2^j\pi}), \tag{3.23}$$

where $m\cdot\xi$ is the scalar product of $m \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n$. Write $\psi_\lambda(\xi) = \psi(2^\lambda \xi)$, where $\lambda = \lambda(n)$ is a positive number so chosen that

$$(\psi - \psi_\lambda)(2^{-j-1}\xi)\phi_j(\xi) = \phi_j(\xi) \quad \text{if } \xi \in \mathbb{R}^n \quad \text{and} \quad j \in \mathbb{N}.$$

Then put, for an appropriate constant c,

$$f_N^j(x) = c\psi(x) \sum_{|m|\le \max(N^2\delta^2, 2^{j+n}\sqrt{n})} (\phi_j \widehat{f})^\vee(2^{-j}m)(\psi-\psi_\lambda)^\vee(2^{j+1}x - 2m) \tag{3.24}$$

so that the term f_N in (3.21) may be decomposed as

$$f_N = \sum_{j=0}^{N} f_N^j + f_{N,2} := f_{N,1} + f_{N,2}. \tag{3.25}$$

Pointwise multiplier properties enable us to estimate the norm of $f_{N,2}$ by

$$\left\| f_{N,2} | B^{s_2}_{p_2 q_2}(\mathbb{R}^n) \right\| \le c 2^{-N\delta}. \tag{3.26}$$

This implies that

$$\left\| f - f_{N,1} | B^{s_2}_{p_2 q_2}(\mathbb{R}^n) \right\| \le c 2^{-N\delta},$$

and since the map $f \longmapsto f_{N,1}$ has rank less than $c2^{nN}$ it follows quickly that $a_k(\mathrm{id}) \le ck^{-\delta/n}$. A more refined analysis gives the other upper bounds listed in the Theorem.

To obtain an upper estimate for $e_k(\mathrm{id})$ is a good deal more complicated: it involves a detailed analysis of the geometric structure of the so-called 'N-core', which is defined to be

$$\bigcup_{j=0}^{N} \left\{ f_N^j : f \in B^{s_1}_{p_1 q_1}(\mathbb{R}^n), \left\| f | B^{s_1}_{p_1 q_1}(\mathbb{R}^n) \right\| \le 1, \operatorname{supp} f \subset Q_1 \right\}. \tag{3.27}$$

Further decomposition of $f_{N,1}$ is needed:

$$f_{N,1} = \sum_{j=0}^{K} f_N^j + \sum_{j=K+1}^{L} f_N^j + \sum_{j=L+1}^{N} f_N^j;$$

and the various components of the N-core have to be covered in $B^{s_2}_{p_2 q_2}(\mathbb{R}^n)$ by balls of appropriate radii. We refer to [31], [32] for the details, and for the proof of the lower estimate for $e_k(\mathrm{id})$, which again depends upon a detailed examination of the N-core.

As for the lower estimates for $a_k(\mathrm{id})$, these rely upon sharp estimates for the approximation numbers of embeddings between finite-dimensional sequence spaces. To explain this, recall that given any $m \in \mathbb{N}$ and any $p \in (0,\infty]$, ℓ_p^m is the linear space of all complex m-tuples $y = (y_i)$, endowed with the quasi-norm

$$\|y|\ell_p^m\| = \Big(\sum_{j=1}^{m} |y_j|^p\Big)^{1/p}$$

if $p < \infty$, with the usual modification if $p = \infty$. The idea is to prove that there is a positive number c such that for all j, $k \in \mathbb{N}$,

$$a_k(\mathrm{id}) \geq c 2^{-j(s_1-s_2)+jn(1/p_1-1/p_2)} a_k(\mathrm{id}_\ell), \tag{3.28}$$

(where $\mathrm{id}_\ell \colon \ell_{p_1}^{N_j} \to \ell_{p_2}^{N_j}$ $(N_j = 2^{jn})$ is the identity map) and, as before, id is the natural embedding of $B^{s_1}_{p_1 q_1}(\Omega)$ in $B^{s_2}_{p_2 q_2}(\Omega)$. With the help of two further bounded operators S and T we construct a commutative diagram

$$\begin{array}{ccc} B^{s_1}_{p_1 q_1}(\Omega) & \xrightarrow{\mathrm{id}} & B^{s_2}_{p_2 q_2}(\Omega) \\ S \uparrow & & \downarrow T \\ \ell_{p_1}^{N_j} & \xrightarrow{\mathrm{id}_\ell} & \ell_{p_2}^{N_j} \end{array}$$

such that $\mathrm{id}_\ell = T \circ \mathrm{id} \circ S$. If such bounded operators can be found, then

$$a_k(\mathrm{id}) \geq \|S\|^{-1} \|T\|^{-1} a_k(\mathrm{id}_\ell),$$

and this will reduce the estimation of $a_k(\mathrm{id})$ to that of $a_k(\mathrm{id}_\ell)$, provided that $\|S\|$ and $\|T\|$ can be estimated in a useful way. To construct S, suppose without loss of generality that Ω contains the unit cube Q_1 in $\mathbb{R}^n$, divide Q_1 in the usual way into 2^{jn} congruent cubes of side length 2^{-j} and with centres x^r. Let $\Phi \in C^\infty(\mathbb{R}^n)$ be a standard bump function with support contained in some small cube centred at the origin, set $\phi(x) = \Phi(2^j x)$ and $\phi_r(x) = \phi(x - x^r)$. Then we define S by

$$S\{\lambda_r\} = \sum_{r=1}^{N_j} \lambda_r \phi_r(x),$$

and choose Φ in such a way that the $2^{-j(s_1 - n/p_1)}\phi_r$ are atoms in $B^{s_1}_{p_1 q_1}(\mathbb{R}^n)$ in the sense of Frazier and Jawerth (cf. [38] and [63, Theorem 1.9.2]). By the atomic characterisation of $B^{s_1}_{p_1 q_1}(\mathbb{R}^n)$ given there we have

$$\big\| S\{\lambda_r\} | B^{s_1}_{p_1 q_1}(\Omega) \big\| \le c \Big(\sum_{r=1}^{N_j} |\lambda_r|^{p_1} \Big)^{1/p_1} 2^{j(s_1 - n/p_1)},$$

where c is independent of j. As for T, we let ϕ_r be as above and define

$$Bf = \{ 2^{jn} (\Delta^M_{2^{-j}h} f, \phi)_{L_2} \},$$

where $M \in \mathbb{N}$ is to be chosen appropriately, $h \in \mathbb{R}^n$ with $|h| \sim 1$, and $\Delta^M_{2^{-j}h}$ is the usual difference (appropriately interpreted if f is a distribution) defined in 3.1. It turns out that

$$\big\| Bf | \ell^{N_j}_{p_2} \big\| \le c 2^{-j(s_2 - n/p_2)} \big\| f | B^{s_2}_{p_2 q_2}(\Omega) \big\|$$

where c is independent of j. These considerations give us the desired inequality (3.28).

The lower estimates for $a_k(\mathrm{id})$ now follow from the following lemma, in which α_k stands for the k^{th} approximation of the natural embedding of ℓ^m_p in ℓ^m_q, where $m \in \mathbb{N}$, $0 < p \le \infty$ and $0 < q \le \infty$.

Lemma ([32]). *Let $m \in \mathbb{N}$ be even.*

(i) *Let $0 < p \le q \le 2$ and put $k = m/2$. Then $\alpha_k \asymp 1$.*

(ii) *Let $0 < q \le p \le \infty$ and put $k = m/2$. Then*

$$\alpha_k \ge c m^{1/q - 1/p}$$

for some positive constant c which is independent of m, but may depend upon p and q.

(iii) *Let $0 < p < 2 \le q < p'$ and $k \le m/2$. Then*

$$\alpha_k \asymp \min(1, m^{1/q} k^{-1/2}).$$

The proof of the Lemma depends crucially on the fundamental work of Gluskin [41] on the α_k.

Remark 2. The results described in this section all require Ω to be a bounded domain with C^∞ boundary. For similar estimates of the entropy and aproximation numbers of embeddings between weighted B and F spaces on the whole of $\mathbb{R}^n$, and with weights with at most polynomial growth, we refer to the very recent work of Haroske [42] and Haroske and Triebel [43], [44].

4. Limiting embeddings

4.1. Orlicz embeddings

Throughout this section Ω will stand for a bounded domain in $\mathbb{R}^n$ with C^∞ boundary and, for the sake of simplicity, we shall deal with the (fractional) Sobolev spaces $H_p^s(\Omega)$ which, as was pointed out in 3.1, are just the spaces $F_{p2}^s(\Omega)$.

The results given in 3.3 show that if $1 < p < \infty$, then $H_p^{n/p}(\Omega)$ is compactly embedded in $\mathrm{L}_q(\Omega)$ for all $q \in (0,\infty)$. There is no embedding in the limiting space $L_\infty(\Omega)$, but one can come quite near to this by means of the Orlicz space $L_\infty(\log L)_{-a}(\Omega)$ $(a > 0)$, which we recall is the linear hull of the set of all (equivalence classes of) functions $f\colon \Omega \to \mathbb{C}$ such that

$$\int_\Omega \exp(|f(x)|^{1/a})\,dx < \infty;$$

endowed with the (Luxemburg) norm

$$\inf\Big\{\lambda > 0\colon \int_\Omega \exp(|f(x)/\lambda|^{1/a})\,dx\Big\}, \tag{4.1}$$

$L_\infty(\log L)_{-a}(\Omega)$ is a Banach space. It is well known that $H_p^{n/p}(\Omega)$ is continuously embedded in $L_\infty(\log L)_{-a}(\Omega)$ if, and only if, $a \geq 1/p'$, the embedding being compact if, and only if, $a > 1/p'$. This type of result goes back certainly to Trudinger [68] and Strichartz [59]; but in [1] references to earlier Russian work will be found. (For very recent variants of this result and for related work, see [26], [27], [28], [29] and [39].) After various upper estimates had been obtained for the approximation and entropy numbers of the embedding map I of the limiting space $H_p^{n/p}(\Omega)$ in $L_\infty(\log L)_{-a}(\Omega)$, Triebel [64] made a striking advance by showing that

$$e_k(I) \asymp k^{-1/p} \quad \text{if } a > 1+\frac{2}{p} \tag{4.2}$$

and

$$a_k(I) \asymp (\log k)^{1/p'-a} \quad \text{if } a > 1/p'. \tag{4.3}$$

For the remaining cases, there exist a positive number c and for any $\varepsilon > 0$ a positive number $c(\varepsilon)$ such that

$$ck^{-1/p} \le e_k(I) \le c(\varepsilon)k^{-\frac{1}{3}(a-1/p')+\varepsilon} \quad \text{if } 1 \le a \le 1 + \frac{2}{p}, \tag{4.4}$$

and

$$ck^{-(a-1/p')} \le e_k(I) \le c(\varepsilon)k^{-\frac{1}{3}(a-1/p')+\varepsilon} \quad \text{if } 1/p' < a \le 1. \tag{4.5}$$

The result contained in (4.3) establishes a long-standing conjecture, and (4.2), (4.4) and (4.5) disprove the conjecture that $e_k(I)$ behaves like $a_k(I)$, as had been expected from earlier work on this limiting case: cf. [64] for a brief account of the history of this. To obtain these results, Triebel reduced the proof to corresponding estimates for embeddings in non-limiting situations, and control of the constants in, for example,

$$e_k\big(\mathrm{id}\colon H_p^{n/p}(\Omega) \to L_q(\Omega)\big) \le cq^{1+2/p}k^{-1/p}.$$

He coupled these precise estimates with the observations that the space $L_\infty(\log L)_{-a}(\Omega)$ consists of all $f\colon \Omega \to \mathbb{C}$ such that

$$\sup_{0<\sigma<n} \sigma^a \big\| f | L_{n/\sigma}(\Omega) \big\| < \infty, \tag{4.6}$$

and that the expression in (4.6) gives a norm on $L_\infty(\log L)_{-a}(\Omega)$ equivalent to the Luxemburg norm (4.1).

4.2. The spaces $L_p(\log L)_a(\Omega)$

The idea advanced here is that the results described in 4.1 and involving $L_\infty(\log L)_{-a}(\Omega)$ should be extended from this 'L_∞-situation' to an 'L_p-situation' in which there should be analogues of the estimates (4.2)–(4.5) obtained by using an L_p analogue of the norm (4.6). We begin with a formal definition of the well-known spaces $L_p(\log L)_a(\Omega)$.

Definition. Let $0 < p < \infty$ and $a \in \mathbb{R}$. Then $L_p(\log L)_a(\Omega)$ is the set of all measurable functions $f\colon \Omega \to \mathbb{C}$ such that

$$\int_\Omega |f(x)|^p \log^{ap}(2 + |f(x)|)\, dx < \infty. \tag{4.7}$$

Remark 1. These spaces can be characterised by the non-increasing rearrangement f^* of a function: we recall that this is defined by

$$f^*(t) = \inf\{\tau > 0 \colon |\{x \in \Omega \colon |f(x)| > \tau\}| \le t\},$$

where $|\Omega_0|$ denotes the Lebesgue n-measure of a measurable subset Ω_0 of Ω. It is shown in [5] that $f \in L_p(\log L)_a(\Omega)$ if, and only if,

$$\Big(\int_0^{|\Omega|} [(1 + |\log t|)^a f^*(t)]^p\Big)\, dt < \infty \tag{4.8}$$

The expression in (4.8) is, in general, only a quasi-norm but it is shown in [5] that if $1 < p < \infty$ and $a \in \mathbb{R}$, the analogue of (4.8) with $f^*(t)$ replaced by $f^{**}(t) = t^{-1}\int_0^t f^*(s)\,ds$ defines a norm on $L_p(\log L)_a(\Omega)$ which is equivalent to the quasi-norm (4.8). Also, all these spaces are complete. Henceforth we shall assume that $L_p(\log L)_a(\Omega)$ is provided with the quasi-norm (4.8), and we may regard it as a Banach space if $1 < p < \infty$.

Remark 2. Let $0 < \varepsilon < p < \infty$, $-\infty < a_2 < a_1 < \infty$. Since Ω is bounded, we have the elementary embeddings

$$L_{p+\varepsilon}(\Omega) \subset L_p(\log L)_{a_1}(\Omega) \subset L_p(\log L)_{a_2}(\Omega) \subset L_{p-\varepsilon}(\Omega) \tag{4.9}$$

and

$$L_p(\log L)_{\varepsilon}(\Omega) \subset L_p(\Omega) \subset L_p(\log L)_{-\varepsilon}(\Omega). \tag{4.10}$$

Moreover, if $-\infty < b_1 < b_2 < 0$, then

$$L_\infty(\Omega) \subset L_\infty(\log L)_{b_2}(\Omega) \subset L_\infty(\log L)_{b_1}(\Omega). \tag{4.11}$$

This shows that the spaces $L_p(\log L)_a(\Omega)$ provide a refined tuning of the L_p scale.

Remark 3. It is also useful to have the Lorentz space version of $L_p(\log L)_a(\Omega)$. Let $0 < p < \infty$, $0 < q \le \infty$ and $a \in \mathbb{R}$. Then $L_{p,q}(\log L)_a(\Omega)$ is defined to be the set of all measurable functions $f \colon \Omega \to \mathbb{C}$ such that

$$\Big(\int_0^{|\Omega|} [t^{1/p}(1 + |\log t|)^a f^*(t)]^q \frac{dt}{t}\Big)^{1/q} < \infty \tag{4.12}$$

(with the obvious modification if $q = \infty$). It can be shown that this is a quasi-Banach space with quasi-norm given by (4.12), and that for $1 < p < \infty$, $1 < q \leq \infty$ and $a \in \mathbb{R}$, the functional obtained from (4.7) by replacing f^* by f^{**} is a norm on $L_{p,q}(\log L)_a(\Omega)$ equivalent to the original quasi-norm. Of course, $L_{p,p}(\log L)_a(\Omega) = L_p(\log L)_a(\Omega)$ and $L_{p,q}(\log L)_0(\Omega)$ is the usual Lorentz space $L_{p,q}(\Omega)$.

We can now give the promised alternative characterisation of the space $L_p(\log L)_a(\Omega)$, confining attention to the case in which $1 < p < \infty$. Following the notation introduced in 3.2 we write

$$\frac{1}{p^\sigma} = \frac{1}{p} + \frac{\sigma}{n} \tag{4.13}$$

where

$$1 < p < \infty, \quad \sigma \in \mathbb{R} \quad \text{and} \quad 1 < p^\sigma < \infty. \tag{4.14}$$

For notional convenience we put

$$\sigma_j = 2^{-j}, \quad \lambda_j = -2^{-j} \quad (j \in \mathbb{N}). \tag{4.15}$$

The phrase 'small $\varepsilon > 0$' will mean that p^ε satisfies (4.14) with ε instead of σ; the phrase 'large $J \in \mathbb{N}$' is to be similarly interpreted.

Theorem [34]**.** (i) *Let $1 < p < \infty$ and $a \in \mathbb{R}$. Then $L_p(\log L)_a(\Omega)$ is a reflexive Banach space and $C_0^\infty(\Omega)$ is a dense subset of it.*

(ii) *Let $1 < p < \infty$ and $a < 0$. Then $L_p(\log L)_a(\Omega)$ is the set of all measurable functions $f \colon \Omega \to \mathbb{C}$ such that*

$$\Big(\int_0^\varepsilon \big[\sigma^{-a} \big\| f | L_{p^\sigma}(\Omega) \big\| \big]^p \frac{d\sigma}{\sigma} \Big)^{1/p} < \infty \tag{4.16}$$

for small $\varepsilon > 0$, and (4.16) defines a norm on $L_p(\log L)_a(\Omega)$ equivalent to the standard one. In addition, (4.16) can be replaced by the equivalent norm

$$\Big(\sum_{j=J}^\infty 2^{jap} \big\| f | L_{p^{\sigma_j}}(\Omega) \big\|^p \Big)^{1/p} \tag{4.17}$$

for large $J \in \mathbb{N}$.

(iii) *Let $1 < p < \infty$ and $a > 0$. Then $L_p(\log L)_a(\Omega)$ is the set of all measurable functions $g\colon \Omega \to \mathbb{C}$ which can be represented as*

$$g = \sum_{j=J}^{\infty} g_j, \quad g_j \in L_{p^{\lambda_j}}(\Omega) \tag{4.18}$$

for large J, with

$$\Big(\sum_{j=J}^{\infty} 2^{jap} \big\| g_j | L_{p^{\lambda_j}}(\Omega) \big\|^p \Big)^{1/p} < \infty. \tag{4.19}$$

The infimum of the expression in (4.19) *taken over all admissible representations* (4.18) *is a norm on $L_p(\log L)_a(\Omega)$ equivalent to the standard one.*

The proof of this theorem is given in [34]. Here we merely observe that there is not much to prove in (i) in view of the characterisation of the dual of $L_p(\log L)_a(\Omega)$ as $L_{p'}(\log L)_{-a}(\Omega)$ given in [4, Theorem 8.4]. Moreover, (iii) follows from (ii) by use of the Banach space $\ell_p(L_{p^{\sigma_j}}(\Omega))$ of all sequences $F = (F_J, F_{J+1}, \dots)$ with $F_j \in L_{p^{\sigma_j}}(\Omega)$, normed in the natural way; consideration of the subspace $\ell_p^{\mathrm{sym}}\big(L_{p^{\sigma_j}}(\Omega)\big)$ which consists of all elements $F = (F_J, F_{J+1}, \dots)$ with $F_j = 2^{-ja} f$, $f \in L_p(\log L)_{-a}(\Omega)$; and use of the Hahn–Banach theorem. The heart of the matter is the proof of (ii). Since $\mathrm{C}_0^\infty(\Omega)$ is dense in $L_p(\log L)_a(\Omega)$, we must show that (4.16) is an equivalent norm on $L_p(\log L)_a(\Omega)$ for $f \in C_0^\infty(\Omega)$, when $1 < p < \infty$ and $a < 0$. First we prove the desired equivalence for the modified norm

$$\|f\|_{p,a} := \Big(\int_0^{\varepsilon} \big[\sigma^{-a} \big\| f | L_{p^\sigma,p}(\Omega) \big\|\big]^p \frac{d\sigma}{\sigma} \Big)^{1/p}, \tag{4.20}$$

where $L_{p^\sigma,p}(\Omega)$ is the Lorentz space introduced in Remark 3. Since

$$\frac{p}{p^\sigma} - 1 = \frac{\sigma}{n} p$$

we see that, from (4.12),

$$\big\| f | L_{p^\sigma,p}(\Omega) \big\| = \Big(\int_0^{|\Omega|} t^{\sigma p/n} \big(f^*(t)\big)^p \, dt \Big)^{1/p}.$$

Thus

$$\|f\|_{p,a}^p = \int_0^{|\Omega|} \left(f^*(t)\right)^p \int_0^{\varepsilon} \sigma^{-ap-1} t^{\sigma p/n}\, d\sigma\, dt.$$

For small t we write the inner integral as

$$\int_0^{\varepsilon} \sigma^{-ap-1} \exp\left(\frac{-\sigma p}{n} |\log t|\right) d\sigma = \left(\frac{p}{n}|\log t|\right)^{ap} \int_0^{\varepsilon p|\log t|/n} \tau^{-ap-1} e^{-\tau}\, d\tau$$

and note that the final integral tends to $\Gamma(-ap)$ as $t \downarrow 0$. Hence

$$\|f\|_{p,a}^p \sim \int_0^{|\Omega|} \left(f^*(t)\right)^p (1 + |\log t|)^{ap}\, dt$$

which gives the equivalence needed. It is now comparatively easy to show that $\|\cdot\|_{p,a}$ is equivalent to (4.16).

Remark 4. The advantages of the norms (4.16), (4.17) and (4.19) over (4.7) and (4.8) are quite plain. They enable assertions which hold for $L_p(\Omega)$, such as mapping properties of integral operators or pseudodifferential operators, to be carried over immediately to $L_p(\log L)_a(\Omega)$, as long as accurate information about any constants involved is available.

4.3. Embeddings

Let $-\infty < s_2 < s_1 < \infty$, $0 < p_1 < \infty$ and $0 < p_2 < \infty$. We recall that (cf. [53], [62], [63], [66]), when Ω is a bounded domain in $\mathbb{R}^n$ with C^∞ boundary,

(4.21)

$$\mathrm{id}\colon H^{s_1}_{p_1}(\Omega) \to H^{s_2}_{p_2}(\Omega) \text{ is continuous if, and only if, } s_1 - \frac{n}{p_1} \geq s_2 - \frac{n}{p_2},$$

and

$$\mathrm{id}\colon H^{s_1}_{p_1}(\Omega) \to H^{s_2}_{p_2}(\Omega) \text{ is compact if, and only if, } s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2}. \tag{4.22}$$

There is no embedding if $s_1 - n/p_1 < s_2 - n/p_2$; and in the limiting situation

$$s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}, \tag{4.23}$$

the embedding id is continuous but not compact.Taking $s_2 = 0$ we see that

$$\mathrm{id}\colon H^s_{p_1}(\Omega) \to L_{p_2}(\Omega), \tag{4.24}$$

with

$$s > 0, \quad s - \frac{n}{p_1} = -\frac{n}{p_2}, \quad 1 < p_1 < p_2 < \infty, \tag{4.25}$$

is continuous but not compact. We use this to form the L_p counterpart of the embedding

$$H^{n/p}_p(\Omega) \to L_\infty(\log L)_{-1/p'}(\Omega) \tag{4.26}$$

discussed in 4.1. The counterpart of the embedding

$$H^{n/p}_p(\Omega) \to L_\infty(\log L)_{-a}(\Omega), \quad a > 1/p',$$

is given by

$$\mathrm{id}\colon H^s_{p_1}(\Omega) \to L_{p_2}(\log L)_a(\Omega) \tag{4.27}$$

with $a < 0$ and with conditions (4.25) holding. It turns out that this limiting embedding is compact and so we seek to determine the behaviour of its entropy numbers. With the strategy employed by Triebel in his work on embeddings in exponential Orlicz spaces in mind, we first look for accurate estimates in non-limiting situations. These are provided by the following

Proposition [34]**.** *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^∞ boundary and suppose that $1 < p < \infty$ and $s > 0$. Then given any $\varepsilon > 0$, there is a constant $c_\varepsilon > 0$ such that for all $\sigma > 0$ with $p^\sigma > 1$ (recall that $1/p^\sigma = 1/p + \sigma/n$),*

$$e_k\big(\,\mathrm{id}\colon H^s_{p^s}(\Omega) \to L_{p^\sigma}(\Omega)\big) \le c_\varepsilon \sigma^{-(2s/n)-\varepsilon} k^{-s/n} \qquad (k \in \mathbb{N}). \tag{4.28}$$

The proof is a 'battle against the constants', employing the same general techniques as in [31], [32] and [64], plus an additional interpolation argument; we refer to [34] for the details.

Now we can consider the limiting embedding

$$\mathrm{id}\colon H^s_{p^s}(\Omega) \to L_p(\log L)_a(\Omega), \quad a \le 0 \tag{4.29}$$

where $1 < p < \infty$ and $s > 0$. Of course, this is just a rewritten version of (4.26). If $a = 0$, id is continuous but not compact; if $a \le 0$, id is continuous, by (4.10). Let e_k be the k^{th} entropy number of id.

Theorem. ([34]) *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^∞ boundary and suppose that*

$$1 < p < \infty, \quad s > 0, \quad a < 0. \tag{4.30}$$

Then the embedding (4.29) *is compact.*

(i) *In addition, suppose that $a < -2s/n$. Then*

$$e_k \asymp k^{-s/n}. \tag{4.31}$$

(ii) *Suppose that $-2s/n \le a < -s/n$ and $\varepsilon > 0$. Then there are positive numbers c and c_ε such that*

$$ck^{-s/n} \le e_k \le c_\varepsilon k^{\frac{a}{2}+\varepsilon} \qquad (k \in \mathbb{N}). \tag{4.32}$$

(iii) *Suppose that $-s/n \le a < 0$ and $\varepsilon > 0$. Then there are positive numbers c and c_ε such that*

$$ck^{a} \le e_k \le c_\varepsilon k^{\frac{a}{2}+\varepsilon} \qquad (k \in \mathbb{N}). \tag{4.33}$$

This theorem may be considered as the counterpart of Theorem 3.2.3 of Triebel [64]; see (4.2), (4.4), (4.5). Its proof follows a strategy similar to that of Triebel's theorem.

4.4. The spaces $H^s_p(\log H)_a$

Roughly speaking, these spaces are lifted versions of $L_p(\log L)_a(\Omega)$ which are natural counterparts in an 'H^s_p-situation' of the spaces $L_\infty(\log L)_{-a}(\Omega)$ in an 'L_∞-situation'. Throughout this section Ω will again stand for a bounded domain in $\mathbb{R}^n$ with C^∞ boundary. Let $1 < p < \infty$ and let $1/p^\sigma = 1/p + \sigma/n$, $\sigma_j = 2^{-j}$, $\lambda_j = -2^{-j}$, as before. The phrases 'small $\varepsilon > 0$' and 'large $J \in \mathbb{N}$'are assumed to have the same meaning as that given just before Theorem 4.2. To exhibit further connections with the $L_p(\log L)_a(\Omega)$ spaces we introduce the space

$$\widetilde{H}^s_p(\Omega) := \{f \in H^s_p(\mathbb{R}^n)\colon \operatorname{supp} f \subset \overline{\Omega}\}. \tag{4.34}$$

Definition 1. Let $1 < p < \infty$ and $s \in \mathbb{R}$.

(i) Let $a < 0$. Then $H^s_p(\log H)_a(\Omega)$ is the set of all (complex) distributions $f \in \mathcal{D}'(\Omega)$ such that

$$\Big(\sum_{j=J}^{\infty} 2^{jap} \big\| f | H^s_{p^{\sigma_j}}(\Omega)\big\|^p\Big)^{1/p} < \infty \tag{4.35}$$

for large $J \in \mathbb{N}$. It is normed by the expression in (4.35).

(ii) Let $a > 0$. Then $H^s_p(\log H)_a(\Omega)$ is the set of all (complex) distributions $g \in \mathcal{D}'(\Omega)$ which can be represented as

$$g = \sum_{j=J}^{\infty} g_j, \quad g_j \in H^s_{p^{\lambda_j}}(\Omega) \quad \text{for large } J \in \mathbb{N}, \tag{4.36}$$

with

$$\Big(\sum_{j=J}^{\infty} 2^{jap} \big\| g_j | H^s_{p^{\lambda_j}}(\Omega) \big\|^p \Big)^{1/p} < \infty. \tag{4.37}$$

It is normed by the infimum of all expressions in (4.37) over all admissible representations (4.36).

(iii) Let $a \in \mathbb{R}$, $a \neq 0$. Then $\widetilde{H}^s_p(\log \widetilde{H})_a(\Omega)$ is defined as in (i) and (ii) with $\widetilde{H}$ instead of H.

Note that in view of the monotonicity properties of the spaces $H^s_p(\Omega)$, the norm (4.35) can be replaced by the (equivalent) norm

$$\Big(\int_0^{\varepsilon} \big[\sigma^{-a} \big\| f | H^s_{p^\sigma}(\Omega) \big\| \big]^p \frac{d\sigma}{\sigma} \Big)^{1/p} \tag{4.38}$$

for small $\varepsilon > 0$. This corresponds to (4.16). Note also that

$$H^0_p(\log H)_a(\Omega) = L_p(\log L)_a(\Omega).$$

Put

$$A_m f = (-\Delta + \mathrm{id})^m f \qquad (m \in \mathbb{N}), \tag{4.39}$$

where Δ is the Laplace operator. Then $A_{m,D}$ and $A_{m,N}$, defined by

$$\begin{aligned} &A_{m,D} f = A_m f, \\ &\operatorname{dom} A_{m,D} = \Big\{ f \in H^{2m}_p(\Omega) \colon \frac{\partial^j f}{\partial \nu^j}\Big|_{\partial\Omega} = 0 \text{ for } j = 0, \ldots, m-1 \Big\} \end{aligned} \tag{4.40}$$

and

$$A_{m,N} f = A_m f,$$

(4.41)
$$\operatorname{dom} A_{m,N} = \Big\{ f \in H_p^{2m}(\Omega)\colon \frac{\partial^{j+m}}{\partial\nu^{j+m}}\big|_{\partial\Omega} = 0 \text{ for } j = 0, \ldots, m-1, \Big\}$$

are the Dirichlet and Neumann realisations, respectively of A_m. Following the arguments given in [61, 4.9.2], we have for the fractional powers $A^{\tau}_{m,D}$ and $A^{\tau}_{m,N}$,

$$\operatorname{dom} A^{\tau}_{m,D} = \widetilde{H}_p^{2m\tau}(\Omega), \quad \operatorname{dom} A^{\tau}_{m,N} = H_p^{2m\tau}(\Omega) \quad \text{for } 0 \le \tau \le \frac{1}{2}, \tag{4.42}$$

and with appropriate interpretations we have, up to isomorphisms,

$$A^{\tau}_{m,N} H_p^{s+2m\tau}(\Omega) = H_p^s(\Omega), \quad 0 \le s \le s + 2m\tau \le m \tag{4.43}$$

and
(4.44)
$$A^{\tau}_{m,D} \overline{H}_p^{s+2m\tau}(\Omega) = \overline{H}_p^s(\Omega), \quad -m \le s \le s + 2m\tau \le m, \quad 0 \le \tau \le \frac{1}{2},$$

where $\overline{H}_p^{\kappa}(\Omega) = \widetilde{H}_p^{\kappa}(\Omega)$ if $\kappa \ge 0$ and $\overline{H}_p^{\kappa}(\Omega) = H_p^{\kappa}(\Omega)$ if $\kappa \le 0$. In this sense $A^{\tau}_{m,N}$ and $A^{\tau}_{m,D}$, now even with $|\tau| \le 1/2$, provide isomorphic maps in the way indicated. Further details will be found in [61, 4.9.2].

The connections mentioned above are those given in the following theorem:

Theorem 1 ([34])**.** *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^∞ boundary and let $1 < p < \infty$, $a \in \mathbb{R}$.*

(i) *Let $0 \le \tau \le \frac{1}{2}$. Then*

$$A^{-\tau}_{m,N} L_p(\log L)_a(\Omega) = H_p^{2m\tau}(\log H)_a(\Omega), \tag{4.45}$$

$$A^{-\tau}_{m,D} L_p(\log L)_a(\Omega) = \widetilde{H}_p^{2m\tau}(\log \widetilde{H})_a(\Omega), \tag{4.46}$$

and

$$A^{\tau}_{m,D} L_p(\log L)_a(\Omega) = H_p^{-2m\tau}(\log H)_a(\Omega). \tag{4.47}$$

(ii) *For all $s \in \mathbb{R}$, $C_0^\infty(\Omega)$ is dense in $\widetilde{H}_p^s(\log \widetilde{H})_a(\Omega)$.*

(iii) *If* $s \geq 0$, *then in the sense of the dual pairing between* $C_0^\infty(\Omega)$ *and* $D'(\Omega)$,

$$[\widetilde{H}_p^s(\log \widetilde{H})_a(\Omega)]' = H_{p'}^{-s}(\log H)_{-a}(\Omega). \tag{4.48}$$

(iv) *If* $s \in \mathbb{N}_0$, *then*

$$H_p^s(\log H)_a(\Omega) = \{f \in D'(\Omega) \colon D^\alpha f \in L_p(\log L)_a(\Omega) \text{ if } |\alpha| \leq s\}, \tag{4.49}$$

with the equivalent norm

$$\sum_{|\alpha| \leq s} \big\| D^\alpha f | L_p(\log L)_a(\Omega) \big\|. \tag{4.50}$$

Remark 1. The equivalent norm (4.50) and the practice of omitting the word 'fractional' from the phrase 'fractional Sobolev spaces H_p^s' lead us to call the spaces $H_p^s(\log H)_a(\Omega)$ *logarithmic Sobolev spaces* despite the competition offered by logarithmic Sobolev inequalities.

Remark 2. By (i) of the theorem, the embeddings in Remark 2 in Section 4.2 can be extended. Let $1 < p < \infty$ and $s \in \mathbb{R}$. Then if $\varepsilon > 0$ and $0 < \eta < p - 1$ we have

$$H_{p+\eta}^s(\Omega) \subset H_p^s(\log H)_\varepsilon(\Omega) \subset H_p^s(\Omega) \subset H_p^s(\log H)_{-\varepsilon}(\Omega) \subset H_{p-\eta}^s(\Omega) \tag{4.51}$$

together with similar embeddings with $\widetilde{H}$ instead of H.

We can now extend the results of 4.3 to the setting of logarithmic Sobolev spaces.

Proposition 1 ([34]). *Let*

$$-\infty < s_2 < s_1 < \infty, \quad 1 < p_1 < p_2 < \infty, \quad s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}, \tag{4.52}$$

and suppose that $a_1, a_2 \in \mathbb{R}$. *Then*

$$\mathrm{id} \colon H_{p_1}^{s_1}(\log H)_{a_1}(\Omega) \to H_{p_2}^{s_2}(\log H)_{a_2}(\Omega) \tag{4.53}$$

is continuous if, and only if, $a_2 \leq a_1$; *it is compact if, and only if* $a_2 < a_1$.

For the proof we refer to [34].

Corollary 1. *Suppose that* (4.52) *holds, let* $a_2 \in \mathbb{R}$ *be such that* $a_2 < -\frac{2}{n}(s_1 - s_2)$ *and let* e_k *be the* k^{th} *entropy number of*

$$\mathrm{id}\colon H^{s_1}_{p_1}(\Omega) \to H^{s_2}_{p_2}(\log H)_{a_2}(\Omega). \tag{4.54}$$

Then

$$e_k \asymp k^{-(s_1-s_2)/n}. \tag{4.55}$$

Proof. We use the liftings of the Theorem. Since Ω has the extension property for all the spaces in question (cf. [63, 5.1.3] and its obvious generalisation to these spaces) it does not matter whether we use the H or $\widetilde{H}$ spaces. Thus (4.55) follows from lifting and Theorem 4.3(i). $\square$

Corollary 2. *Suppose that* (4.52) *holds, let* $a_1 \in \mathbb{R}$ *satisfy* $0 < a_1 - \frac{2}{n}(s_1 - s_2)$ *and let* e_k *be the* k^{th} *entropy number of*

$$\mathrm{id}\colon H^{s_1}_{p_1}(\log H)_{a_1}(\Omega) \to H^{s_2}_{p_2}(\Omega). \tag{4.56}$$

Then

$$e_k \asymp k^{-(s_1-s_2)/n}. \tag{4.57}$$

Proof. The idea is to use Theorem 1 of [13] which implies that if $T \in L(X,Y)$ is compact and X is uniformly convex, then $e_k(T) \asymp k^{-\lambda}$ $(\lambda > 0)$ implies that $e_k(T^*) \asymp k^{-\lambda}$ also. Now if $1 < p < \infty$ and $a \leq 0$, then $\widetilde{H}^s_p(\log \widetilde{H})_a(\Omega)$ is uniformly convex (see [18] for related statements). We apply Theorem 1 of [13] to (4.53) with $a_1 = 0$, $\mathrm{s}_2 = 0$ and $\widetilde{H}$ instead of H. Then by (4.48) and lifting we obtain the desired result (4.57). $\square$

So far we have restricted consideration of the spaces $H^s_p(\log H)_a(\Omega)$ to the situation in which $1 < p < \infty$. We now remove this limitation.

Definition 2. Let $0 < p < \infty$, $a \in \mathbb{R}$, $s \in \mathbb{R}$. Then $H^s_p(\log H)_a(\Omega)$ is defined just as in Definition 1.

If $0 < p \leq 1$ there is no direct connection with $L_p(\log L)_a(\Omega)$, no duality and no immediate isomorphism properties such as those in Theorem 1. Despite this the spaces, especially when $a < 0$, are useful. We give some partial extensions of Proposition 4.3 and Theorem 4.3.

Proposition 2 ([34]). *Suppose that*

$$-\infty < s_2 < s_1 < \infty, \quad 0 < p_1 < p_2 < \infty, \quad s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}. \tag{4.58}$$

Then given any $\varepsilon > 0$, there is a constant $c_\varepsilon > 0$ such that for all $\sigma > 0$,

$$e_k\big(\mathrm{id}\colon H^{s_1}_{p_1^\sigma}(\Omega) \to H^{s_2}_{p_2^{2\sigma}}(\Omega)\big) \le c_\varepsilon \sigma^{-2(s_1-s_2)/n-\varepsilon}\, k^{-(s_1-s_2)/n} \qquad (k \in \mathbb{N}). \tag{4.59}$$

The proof is essentially the same as that of Proposition 4.3.

Theorem 2 ([34]). (i) *Let*

$$-\infty < s_2 < s_1 < \infty, \quad 0 < p_1 < p_2 < \infty, \quad s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}, \tag{4.60}$$

$$a_1 \le 0, \quad a_2 < a_1 - \frac{2}{n}(s_1 - s_2). \tag{4.61}$$

Then

$$e_k\big(\mathrm{id}\colon H^{s_1}_{p_1}(\log H)_{a_1}(\Omega) \to H^{s_2}_{p_2}(\log H)_{a_2}(\Omega)\big) \asymp k^{-(s_1-s_2)/n}. \tag{4.62}$$

(ii) *Suppose that*

$$-\infty < s_2 < s_1 < \infty, \quad 1 < p_1 < p_2 < \infty, \quad s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2}, \tag{4.63}$$

$$a_1 > 0, \quad a_2 < a_1 - \frac{2}{n}(s_1 - s_2). \tag{4.64}$$

Then (4.62) *holds.*

For the proof we refer to [34]. However, note that the restriction $p_1 > 1$ arises from duality arguments used in the proof; presumably this restriction can be removed.

5. Applications to eigenvalue problems

Our object here is to apply the results described in the earlier sections, and particularly those related to entropy numbers of embeddings, to study the distribution of eigenvalues of degenerate elliptic (pseudo-) differential operators. The operators studied typically have the structure

$$B = b_2 C b_1,$$

where b_1 and b_2 are singular functions belonging to some function space such as H_p^s, and C may be the inverse of a regular elliptic differential operator or a fractional power of it, or an (exotic) pseudodifferential operator. Much work has been done on the symmetric case in which $b_1 = b_2 = b$ and C is symmetric with respect to, say, the L_2 inner product; in this case, provided that b and C are compatible with the techniques used, excellent results have been obtained about the distribution of eigenvalues, now counted with respect to their geometric multiplicity. The work of Birman and Solomyak [8]–[12], Rosenbljum [50, 51], Solomyak [57] and Tashkian [60] is especially noteworthy in this connection. These results extend the classical theory of the distribution of eigenvalues of self-adjoint elliptic differential operators with smooth coefficients. Moreover, mapping properties of operators of type $b(X)a(D)$, including the distribution of eigenvalues, have been thoroughly examined in a Hilbert space setting, mostly $L_2(\mathbb{R}^n)$: see Birman, Karadhzov and Solomyak [6], Cwikel [21], Lieb [46] and Simon [54], together with the references contained in these works. In these specific situations the deep Hilbert space techniques employed by these authors often give better results than those provided by the simple arguments to be given here; on the other hand, these arguments are not confined to Hilbert spaces or symmetric operators. We thus present our method largely (but not exclusively – see 5.3) for more general non-symmetric operators in Banach or quasi-Banach spaces.

5.1. Regular elliptic differential operators

Throughout this section Ω will be a bounded domain in $\mathbb{R}^n$ with C^∞ boundary. Our object is to collect various mapping properties which originated in the work of Agmon [2] and Agmon, Douglis and Nirenberg [3], so that they may be used in conjunction with our work on entropy numbers. For details of these properties we refer to [51] and [52].

Let A be a properly elliptic operator,

$$Af = \sum_{|\alpha| \le 2m} a_\alpha(x) D^\alpha f, \quad \text{where each } a_\alpha \in C^\infty(\overline{\Omega}), \tag{5.1}$$

with boundary operators

$$B_j f = \sum_{|\alpha| \le \ell_j} b_{j,\alpha}(x) D^\alpha f, \quad \text{where each } b_{j,\alpha} \in C^\infty(\partial\Omega), \tag{5.2}$$

with $j = 1, \ldots, m$ and $0 \le \ell_1 < \cdots < \ell_m \le 2m-1$, which form a normal system satisfying the complementarity condition. Under these conditions $\{A; B_1, \ldots, B_m\}$ is called a regular elliptic system: see [62, 4.1.2] for details. We assume that

$$\begin{cases} Af = 0 & \text{in } \Omega, \\ B_j f = 0 & \text{on } \partial\Omega \text{ for } j = 1, \ldots, m \end{cases} \tag{5.3}$$

has only the trivial C^∞ solution. Let

$$0 < p < \infty, \quad s \ge n\Big(\frac{1}{p} - 1\Big)_+. \tag{5.4}$$

Then A maps

$$\{f \in H_p^{s+2m}(\Omega) \colon B_j f = 0 \text{ on } \partial\Omega \text{ for } j = 1, \ldots, m\} \tag{5.5}$$

isomorphically onto $H_p^s(\Omega)$.

If we denote by $A_{s,p}$ the operator A with domain of definition (5.5), then (5.3) ensures that 0 is not in the spectrum $\sigma(A_{s,p})$ of $A_{s,p}$, and thus $\sigma(A_{s,p})$ consists of isolated eigenvalues of finite algebraic multiplicity. When $1 < p < \infty$ all this is contained in Agmon [2]; Triebel [62] gives a partial extension to $p \le 1$ and the full proof is provided by Franke and Runst [40].

Fractional powers will be needed later on. We consider the ground level $s = 0$ and $1 < p < \infty$; write $A_p = A_{0,p}$. The basic theory of fractional powers is given in Triebel [61, 1.15] and its application to A_p is covered by [61, 4.9.1]: the fractional powers A_p^κ can be constructed for every $\kappa \in \mathbb{R}$ (cf. [61, 1.15]). If $-1 \le \kappa < 0$, then A_p^κ is compact in $L_p(\Omega)$; and indeed it is not hard to show (cf. [33]) that in this case,

$$e_k(A_p^\kappa) \asymp k^{2m\kappa/n}. \tag{5.6}$$

We shall also need the inclusion

$$\operatorname{dom}(A_p^{|\kappa|}) \subset H_p^{2m|\kappa|}(\Omega), \quad |\kappa| \le 1, \tag{5.7}$$

where with the exception of those κ such that $2m|\kappa| - 1/p = \ell_j$ for some $j \in \{1, \ldots, m\}$, $\operatorname{dom}(A_p^{|\kappa|})$ is even a closed subspace of $H_p^{2m|\kappa|}(\Omega)$. See [33, 2.5] for further discussion of this point.

5.2. Eigenvalue distributions

We shall consider the fractional powers A_p^κ, where $A_p = A_{0,p}$ as in 5.1, and shall write A^κ for A_p^κ as it will be clear from the context between which spaces A^κ acts. Our goal is to study

$$Bf = b_2 A^{-\kappa} b_1 f, \quad 0 < \kappa \leq 1, \tag{5.8}$$

where b_1 and b_2 belong to certain spaces $L_{r_1}(\log L)_{a_1}(\Omega)$ and $L_{r_2}(\log L)_{a_2}(\Omega)$ or to $H_{r_1^s}(\log H)_{a_1}(\Omega)$ and $H_{r_2^s}(\log H)_{a_2}(\Omega)$, respectively. Initially we shall confine discussion to the ground level $s = 0$, $1 < p < \infty$. In the theorem to be given below B will be a compact operator in a certain $L_p(\Omega)$: we denote by μ_k the eigenvalues of B, ordered by decreasing modulus and repeated according to algebraic multiplicity. The object is to estimate $|\mu_k|$.

Theorem 1 ([34]). *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^∞ boundary, let*

$$r_1, r_2 \in (1, \infty], \qquad 0 < \kappa \leq 1, \qquad m \in \mathbb{N} \tag{5.9}$$

with

$$1 > \frac{1}{r_1} + \frac{1}{r_2} = \frac{2m\kappa}{n}, \tag{5.10}$$

and let a_1, $a_2 \in \mathbb{R}$ be such that

$$\frac{1}{r_2} < \frac{1}{p} < \frac{1}{r_1'} \quad \text{and} \quad a_1 + a_2 > \frac{4m\kappa}{n}. \tag{5.11}$$

Suppose that

$$b_1 \in L_{r_1}(\log L)_{a_1}(\Omega) \quad \text{and} \quad b_2 \in L_{r_2}(\log L)_{a_2}(\Omega). \tag{5.12}$$

Then the map B given by (5.8) *is compact in $L_p(\Omega)$ and there exists $c > 0$ such that*

$$|\mu_k| \leq c\big\|b_1|L_{r_1}(\log L)_{a_1}(\Omega)\big\| \ \big\|b_2|L_{r_2}(\log L)_{a_2}(\Omega)\big\| \ k^{-2m\kappa/n} \qquad (k \in \mathbb{N}). \tag{5.13}$$

Sketch of proof. First suppose that $a_2 = 0$, $r_2 = \infty$. Then $b_2 \in L_\infty(\Omega)$; assume without loss of generality that $b_2 = 1$. We use the decomposition

$$B = b_2 \circ \mathrm{id} \circ A^{-\kappa} \circ b_1$$

with (see Fig. 2)

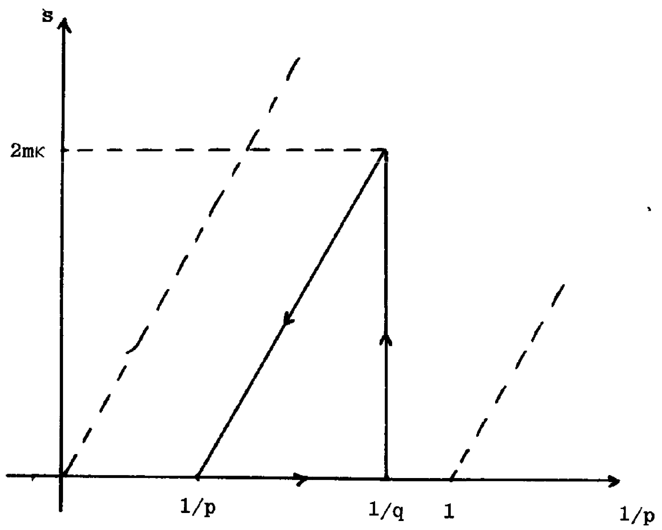

Fig. 2

$$(5.14)\qquad \begin{cases} b_1 : L_p \to L_q(\log L)_{a_1}, \; 1/q = 1/p + 1/r_1, \\ A^{-\kappa} : L_q(\log L)_{a_1} \to H_q^{2m\kappa}(\log H)_{a_1}, \\ \mathrm{id} : H_q^{2m\kappa}(\log H)_{a_1} \to L_p. \end{cases}$$

Here, and subsequently in the proof, we omit the symbol 'Ω' from the spaces. The first embedding follows immediately from Theorem 4.2 (iii) and Hölder's inequality; the second is covered by (5.7) and Definition 4.4. The last embedding is compact: by Corollary 4.4/2, if we observe that

$$(5.15)\qquad 2m\kappa - \frac{n}{q} = -\frac{n}{p}, \quad a_1 > \frac{4m\kappa}{n},$$

it follows that

$$(5.16)\qquad e_k(\mathrm{id}) \asymp k^{-2m\kappa/n}.$$

Use of Carl's inequality (2.4) now gives (5.13).

Now suppose that $a_1 = 0$, $r_1 = \infty$, so that $b_1 \in L_\infty(\Omega)$; again without loss of generality we assume that $b_1 = 1$. This time we use the decomposition

$$B = b_2 \circ \mathrm{id} \circ A^{-\kappa},$$

with

$$\text{(5.17)} \qquad \begin{cases} A^{-\kappa}\colon L_p \to H_p^{2m\kappa}, \\ \mathrm{id}\colon H_p^{2m\kappa} \to L_t(\log L)_{-a_2}, \ 1/t = 1/p - 1/r_2, \\ b_2\colon L_t(\log L)_{-a_2} \to L_p. \end{cases}$$

The first mapping is covered by (5.7), and the last mapping essentially comes from Hölder's inequality, extended to the spaces $L_u(\log L)_v$: see Theorem 4.2. To handle id we apply Theorem 4.3. Again (5.13) folows.

Next, let $a_1 > 0$ and $a_2 > 0$. Choose κ_1, κ_2 such that

$$\text{(5.18)} \qquad a_1 > \frac{4m\kappa_1}{n}, \quad a_2 > \frac{4m\kappa_2}{n}, \quad \kappa_1 + \kappa_2 = \kappa,$$

use the decomposition

$$\text{(5.19)} \qquad B = (b_2 A^{-\kappa_2}) \circ (A^{-\kappa_1} b_1)$$

plus the first two stages of the proof and the multiplicative properties of entropy numbers and obtain

$$\text{(5.20)} \qquad e_k(B) \le ck^{-2m\kappa/n},$$

from which (5.13) follows.

Finally, we have to deal with the cases $a_1 < 0$, $a_2 > 0$ or $a_1 > 0$, $a_2 < 0$. Then $a_1(-a_2) > 0$, and we use the decomposition

$$\text{(5.21)} \qquad \begin{cases} b_1\colon L_p \to L_q(\log L)_{a_1}, \quad 1/q = 1/p + 1/r_1, \\ A^{-\kappa}\colon L_q(\log L)_{a_1} \to H_q^{2m\kappa}(\log H)_{a_1}, \\ \mathrm{id}\colon H_q^{2m\kappa}(\log H)_{a_1} \to L_t(\log L)_{-a_2}, \quad 1/t = 1/p - 1/r_2, \\ b_2\colon L_t(\log L)_{-a_2} \to L_p. \end{cases}$$

This time we use Theorem 2 of Section 4.4, coupled with the observation that $2m\kappa/n - 1/q = -1/t$, to show that

$$\text{(5.22)} \qquad e_k(\mathrm{id}) \le ck^{-2m\kappa/n},$$

and the rest follows much as before. $\square$

Remark 1. It is plain from this proof that all we need of the fractional power $A^{-\kappa}$ is that it should have certain mapping properties; the rest is taken care of by the entropy number estimates for id and the mapping properties of b_1 and b_2. Then $A^{-\kappa}$ could be replaced by any operator with the right mapping behaviour: in particular, it can be replaced by a pseudodifferential operator in the Hörmander class $S_{1,\delta}^{-2m\kappa}(\Omega)$ (see [63] for details of this class) with $0 \le \delta \le 1$. This idea is pursued in [34]; it is especially remarkable that even the so-called exotic case $\delta = 1$ is included. In [34] there is also an extension of the theorem to allow, say b_2, to belong to a space of type $H_p^s(\log H)_a(\Omega)$.

To give some impression of the scope of the theorem an example may be helpful. Let $\Omega = \Big\{ y \in \mathbb{R}^n \colon |y| < 1/2 \Big\}$, $0 < r < \infty$ and $\lambda \in \mathbb{R}$; put

$$b(x) = |x|^{-n/r} |\log|x||^{-\lambda}. \tag{5.23}$$

It is easy to see that $b \in L_r(\log L)_a(\Omega)$ if, and only if,

$$\lambda > \frac{1}{r} + a. \tag{5.24}$$

Theorem 2 ([34])**.** *Let Ω be as above, let*

$$r_1, r_2 \in (1, \infty), \quad 0 < \kappa \le 1, \qquad m \in \mathbb{N}$$

with

$$1 > \frac{1}{r_1} + \frac{1}{r_2} = \frac{2m\kappa}{n}, \tag{5.25}$$

and let λ_1, $\lambda_2 \in \mathbb{R}$ be such that

$$\frac{1}{r_2} < \frac{1}{p} < \frac{1}{r_1'} \quad \text{and} \quad \lambda_1 + \lambda_2 > \frac{1}{r_1} + \frac{1}{r_2} + \frac{4m\kappa}{n}. \tag{5.26}$$

Suppose that

$$b_j(x) = |x|^{-n/r_j} |\log|x||^{-\lambda_j} \qquad (j = 1, 2)$$

and let A be a regular elliptic differential operator of order $2m$. Then the map B defined by

$$B = b_2 A^{-\kappa} b_1$$

is compact in $L_p(\Omega)$ and there is a constant c such that

$$|\mu_k| \leq ck^{-2m\kappa/n} \qquad (k \in \mathbb{N}).$$

Proof. The proof is immediate from Theorem 1 and the observation (5.24) above. $\square$

Remark 2. If $b_1(x) \neq 0$ and $b_2(x) \neq 0$, a.e. and $\kappa = 1$, then B is invertible in $L_p(\Omega)$ and, at least formally, $D = B^{-1}$, $D = b_1^{-1} A b_2^{-1}$ is a degenerate elliptic differential operator, considered as an unbounded operator in $L_p(\Omega)$. Let $\{\lambda_k\}$ be the sequence of its eigenvalues, counted according to algebraic multiplicity and ordered by increasing modulus. Then $\lambda_k = \mu_k^{-1}$ and so

$$|\lambda_k| \geq c\big\|b_1|L_{r_1}(\log L)_{a_1}(\Omega)\big\|^{-1} \ \big\|b_2|L_{r_2}(\log L)_{a_2}(\Omega)\big\|^{-1} k^{2m/n}.$$

For elliptic operators of order $2m$ and with smooth coefficients it is well known that the k^{th} eigenvalue behaves like a multiple of $k^{2m/n}$, so that we have the expected behaviour. If $p = 2$ and D is symmetric, with $b_1 = b_2 = b$, then mild additional conditions are enough to ensure lower bounds of the same form: thus if $1/b \in C^2$ in some subdomain of Ω, then by Courant's principle on the monotonicity of eigenvalues with respect to domain variations, $|\lambda_k| \leq c_1 k^{2m/n}$.

Remark 3. Note in particular that λ_1 might be negative; of course, this must be compensated by λ_2 so that (5.26) holds. For other examples see [33] and [34]: [34] deals with a case in which

$$B = b_2 A^{-\kappa} \sum_{j=1}^{n} b_{1,j} \frac{\partial}{\partial x_j},$$

where b_2 belongs to a space of the form $H_r^1(\log H)_a(\Omega)$ and the $b_{1,j}$ to spaces $L_s(\log L)_t(\Omega)$.

5.3. The negative spectrum

Here we propose to estimate the number of negative eigenvalues which certain differential operators may have. The key idea is to use the Birman–Schwinger principle, which we now recall.

Let A be a positive self-adjoint operator acting in a Hilbert space $\mathcal{H}$, let V be a closable operator acting in $\mathcal{H}$, let $K\colon \mathcal{H} \to \mathcal{H}$ be a compact linear operator such that $Ku = VA^{-1}V^*u$ for all u in $\operatorname{dom}(VA^{-1}V^*)$, and suppose that $\mathrm{dom}(A) \cap dom(V^*V)$ is dense in $\mathcal{H}$. The Birman–Schwinger principle (cf. [52, Theorem 5.3]) states that under these conditions, $A - V^*V$ has a self-adjoint extension $\mathcal{H}$, with spectrum $\sigma(H)$, and that

$$\#\{\sigma(H) \cap (-\infty, 0]\} \le \#\{\sigma(K) \cap [1, \|K\|]\}.$$

In view of Carl's inequality (2.4) this can be immediately reformulated as

Theorem 1. *Under the above assumptions,*

$$\#\{\sigma(H) \cap (-\infty, 0]\} \le \#\{k \in \mathbb{N}\colon \sqrt{2}e_k(K) \ge 1\}.$$

We shall now obtain upper estimates of the number of non-positive eigenvalues of the operator H_α (acting in $L_2(\Omega)$), where

$$H_\alpha f = Gf - \alpha V f, \tag{5.27}$$

and $G = gA^\kappa g$, $g(x) > 0$ a.e. in Ω, $0 < \kappa \le 1$, $V > 0$ a.e. in Ω, $\alpha \ge 0$ and A is a regular elliptic, positive self-adjoint operator of order $2m$. As before, it is assumed that Ω is a bounded domain in $\mathbb{R}^n$ with C^∞ boundary.

Theorem 2. *Let $2 < r < \infty$ and $rm\kappa = n$, and suppose that*

$$V^{1/2}g^{-1} \in L_r(\log L)_a(\Omega) \text{ with } ra > 2. \tag{5.28}$$

Then

$$\#\{\sigma(H_\alpha) \cap (-\infty, 0]\} \le c\alpha^{r/2} \big\| V^{1/2}g^{-1} | L_r(\log L)_a(\Omega) \big\|^r \tag{5.29}$$

for some $c > 0$ which is independent of V, g and α.

Proof. We apply Theorem 1 of Section 5.2, with $a_1 = a_2 = a$ and $r_1 = r_2 = r$, to

$$K_\alpha f = \alpha V^{1/2} g^{-1} A^{-\kappa} g^{-1} V^{1/2} f.$$

Naturally $e_k(K_\alpha) = \alpha e_k(K_1)$. Then by Theorem 1 of Section 5.2 and the Birman-Schwinger principle we simply have to count the $k \in \mathbb{N}$ such that

$$1 \le c\alpha \big\| V^{1/2}g^{-1} | L_r(\log L)_a(\Omega) \big\|^2 k^{-2/r},$$

and (5.29) follows. □

Other results of this nature will be found in [33].

By way of generalisation of Theorem 1 we mention the following result, contained in [34]: it is designed to illustrate how the $H^s_r(\log H)_a(\Omega)$ spaces may be used.

Theorem 3. *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^∞ boundary, let $r_1, r_2 \in (1,\infty)$, $m \in \mathbb{N}$, with*

$$1 > \frac{1}{r_1} + \frac{1}{r_2} = \frac{2m}{n},$$

suppose that

$$\frac{1}{r_2} < \frac{1}{p} < 1 - \frac{1}{r_1} \quad \text{and} \quad a > \frac{4m}{n},$$

that $s > 0$ and that

$$b_1 \in H^s_{r_1^s}(\Omega) \quad \text{and} \quad b_2 \in H^s_{r_2^s}(\log H)_a(\Omega),$$

and let A be a regular elliptic differential operator of order $2m$. Then

$$B = b_2 A^{-1} b_1$$

is compact in $H^s_{p^s}(\Omega)$ and its eigenvalues μ_k satisfy

$$|\mu_k| \le c \big\| b_1 | H^s_{r_1^s}(\Omega) \big\| \, \big\| b_2 | H^s_{r_2^s}(\log H)_a(\Omega) \big\| \, k^{-2m/n} \qquad (m \in \mathbb{N}).$$

The idea of the proof is to use the decomposition

$$B = b_2 \circ \mathrm{id} \circ A^{-1} \circ b_1,$$

where

$$\begin{aligned} b_1 &: H^s_{p^s} \to H^s_{q^s} \text{ with } \frac{1}{q} = \frac{1}{p} + \frac{1}{r_1}, \\ A^{-1} &: H^s_{q^s} \to H^{s+2m}_{q^s}, \\ \mathrm{id} &: H^{s+2m}_{q^s} \to H^s_{t^s}(\log H)_{-a+\varepsilon}, \text{ with } a - \varepsilon > 4m/n, \\ b_2 &: H^s_{t^s}(\log H)_{-a+\varepsilon} \to H^s_{p^s}. \end{aligned}$$

For details we refer to [34]: here we mention only that the first embedding is a consequence of the Hölder inequality (3.8).

References

1. D.R. Adams, *A sharp inequality of J. Moser for higher order derivatives*, Ann. of Math. **128** (1988), 385–398.

2. S. Agmon, *On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems*, Comm. Pure Appl. Math. **15** (1962), 119–147.

3. S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions*, Comm. Pure Appl. Math. **12** (1959), 623–727.

4. C. Bennett and K. Rudnick, *On Lorentz-Zygmund spaces*, Dissertationes Math. **175** (1980), 1–72.

5. C. Bennett and R. Sharpley, *Interpolation of operators*, Academic Press, New York, 1988.

6. M.S. Birman, G.E. Karadzhov and M.Z. Solomyak, *Boundedness conditions and spectrum estimates for the operators $b(X)a(D)$ and their analogs*, Adv. in Soviet Math. **7** (1991), 85–106.

7. M.S. Birman and M.Z. Solomyak, *Piecewise polynomial approximations of functions of the classes W_p^α*, Mat. Sb. **73** (1967), 331–355.

8. M.S. Birman and M.Z. Solomyak, *Spectral asymptotics of non-smooth elliptic operators, I*, Trans. Moscow Math. Soc. **27** (1972), 1–52.

9. M.S. Birman and M.Z. Solomyak, *Spectral asymptotics of non-smooth elliptic operators, II*, Trans. Moscow Math. Soc. **28** (1973), 1–32.

10. M.S. Birman and M.Z. Solomyak, *Quantitative analysis in Sobolev imbedding theorems and applications to spectral theory*, 10th Math. School Kiev 1974, pp. 5–189; *Transl. Math. Monographs 114, Amer. Math. Soc.*, 1980, pp. 1–132.

11. M.S. Birman and M.Z. Solomyak, *The asymptotics of the spectrum of pseudodifferential operators with anisotropically homogeneous symbols, I, II*, Vestnik Leningrad Univ. Mat. **13** (1977), 13–21 (Russian); **13** (1979), 5–10.

12. M.S. Birman and M.Z. Solomyak, *Estimates for the number of negative eigenvalues of the Schrödinger operator and its generalizations*, Adv. in Soviet Math. **7** (1991), 1–55.

13. J. Bourgain, A. Pajor, S.J. Szarek and N. Tomczak-Jaegermann, *On the duality problem for entropy numbers of operators*, Geometric aspects of functional analysis (1987–88), Lecture Notes in Mathematics, Vol. 1376, Springer, Berlin, 1989, pp. 50–63.

14. B. Carl, *Entropy numbers, s-numbers and eigenvalue problems*, J. Funct. Anal. **41** (1981), 290–306.

15. B. Carl, *Entropy numbers of embedding maps between Besov spaces with an application to eigenvalue problems*, Proc. Roy. Soc. Edinburgh Sect. A **90** (1981), 63–70.

16. B. Carl and I. Stephani, *Entropy, compactness and the approximation of operators*, Cambridge University Press, Cambridge, 1990.

17. B. Carl and H. Triebel, *Inequalities between eigenvalues, entropy numbers and related quantities in Banach spaces*, Math. Ann. **251** (1980), 129–133.

18. F. Cobos and D.E. Edmunds, *Clarkson's inequalities, Besov spaces and Triebel–Sobolev spaces*, Z. Anal. Anwendungen **7** (1988), 229–232.

19. F. Cobos, D.E. Edmunds and A.J.B. Potter, *Real interpolation and compact linear operators*, J. Funct. Anal. **88** (1990), 351–365.

20. F. Cobos and D.L. Fernandez, *On interpolation of compact operators*, Ark. Mat. **27** (1989), 211–217.

21. M. Cwikel, *Weak type estimates for singular values and the number of bound states of Schrödinger operators*, Ann. of Math. **106** (1977), 93–100.

22. M. Cwikel, *Real and complex interpolation and extrapolation of compact operators*, Technion, Haifa, 1990, preprint.

23. D.E. Edmunds and W.D. Evans, *Spectral theory and differential operators*, Oxford University Press, Oxford, 1987.

24. D.E. Edmunds, W.D. Evans and D.J. Harris, *Approximation numbers of certain Volterra integral operators*, J. London Math. Soc. **37** (1988), no. 2, 471–489.

25. D.E. Edmunds, P. Gurka and L. Pick, *Compactness of Hardy-type integral operators in weighted Banach function spaces*, Studia Math. **109** (1994), 73–90.

26. D.E. Edmunds, P. Gurka and B. Opic, *Double exponential integrability of convolution operators in generalised Lorentz–Zygmund spaces*, to appear.

27. D.E. Edmunds and A.A. Ilyin, *On some multiplicative inequalities and approximation numbers*, Quart. J. Math. Oxford **45** (1994), 159–179.

28. D.E. Edmunds and A.A. Ilyin, *Approximation numbers of embeddings of Sobolev spaces*, to appear.

29. D.E. Edmunds and M. Krbec, *Two limiting cases of Sobolev imbeddings*, to appear.

30. D.E. Edmunds and V.D. Stepanov, *The measure of non-compactness and approximation numbers of certain Volterra integral operators*, Math. Ann. **298** (1994), 41–66.

31. D.E. Edmunds and H. Triebel, *Entropy numbers and approximation numbers in function spaces*, Proc. London Math. Soc. **58** (1989), no. 3, 137–152.

32. D.E. Edmunds and H. Triebel, *Entropy numbers and approximation numbers in function spaces, II*, Proc. London Math. Soc. **64** (1992), no. 3, 153–169.

33. D.E. Edmunds and H. Triebel, *Eigenvalue distributions of some degenerate elliptic operators: an approach via entropy numbers*, Math. Ann. **299** (1994), 311–340.

34. D.E. Edmunds and H. Triebel, *Logarithmic Sobolev spaces and their applications to spectral theory*, Proc. London Math. Soc., to appear.

35. D.E. Edmunds and H. Triebel, *Function spaces, entropy numbers, differential operators*, (forthcoming book).

36. D.E. Edmunds and H.-O. Tylli, *On the entropy numbers of an operator and its adjoint*, Math. Nachr. **126** (1986), 231–239.

37. D.E. Edmunds and H.-O. Tylli, *Entropy numbers of tensor products of operators*, Ark. Mat. **31** (1993), 247–274.

38. M. Frazier and B. Jawerth, *Decomposition of Besov spaces*, Indiana Univ. Math. J. **34** (1985), 777–799.

39. N. Fusco, P.L. Lions and C. Sbordone, *Some remarks on Sobolev imbeddings in borderline cases*, Preprint No. 25, Università degli Studi di Napoli "Federico II", 1993, pp. 7.

40. J. Franke and Th. Runst, *Regular elliptic boundary value problems in Besov–Triebel–Lizorkin spaces*, to appear.

41. E.D. Glushin, *Norms of random matrices and diameter of finite-dimensional sets*, Mat. Sb. **120** (1983), 180–189. (Russian)

42. D. Haroske, *Approximation numbers in weighted function spaces*, to appear.

43. D. Haroske and H. Triebel, *Entropy numbers in weighted function spaces and eigenvalue distributions of some degenerate pseudodifferential operators I*, Math. Nachr. **167** (1994), 131–156.

44. D. Haroske and H. Triebel, *Entropy numbers in weighted function spaces and eigenvalue distributions of some degenerate pseudodifferential operators II*, Math. Nachr. **168** (1994), 109–137.

45. H. König, *Eigenvalue distributions of compact operators*, Birkhäuser, Basel, 1986.

46. E.W. Lieb, *Bounds on the eigenvalues of the Laplace and Schrödinger operators*, Bull. Amer. Math. Soc. **82** (1976), 751–753.

47. R. Linde, *s-numbers of diagonal operators and Besov embeddings*, Proc. 13th Winter School, Suppl. Rend. Circ. Mat. Palermo **2** (1986), 83–110.

48. A. Pietsch, *Operator ideals*, North-Holland, Amsterdam, 1980.

49. A. Pietsch, *Eigenvalues and s-numbers*, Cambridge University Press, Cambridge, 1987.

50. G.V. Rosenbljum, *The distribution of the discrete spectrum for singular differential operators*, Soviet Math. Dokl. **13** (1972), 245–249.

51. G.V. Rosenbljum, *Distribution of the discrete spectrum of singular differential operators*, Izv. Vyssh. Uchebn. Zaved. Mat. **1** (1976), 75–86. (Russian)

52. M. Schechter,, *Spectra of partial differential operators (2nd edition)*, North Holland, Amsterdam, 1986.

53. W. Sickel and H. Triebel, *Hölder inequalities and sharp embeddings in function spaces of B^s_{pq} and F^s_{pq} type*, to appear.

54. B. Simon, *Analysis with weak trace ideals and the number of bound states of Schrödinger operators*, Trans. Amer. Math. Soc. **224** (1976), 367–380.

55. B. Simon, *Trace ideals and their applications*, Cambridge University Press, Cambridge, 1979.

56. M.Z. Solomyak, *Piecewise polynomial approximation of functions from $H^\ell((0,1)^d)$, $2\ell = d$, and applications to the spectral theory of the Schrödinger operator*, Israel J. Math. **86** (1994), 253–275.

57. M.Z. Solomyak, *Piecewise polynomial approximation of functions from Sobolev spaces, revisited*, to appear.

58. M.Z. Solomyak, *Two-sided estimates on singular values for a class of integral operators on the semi-axis*, to appear.

59. R.S. Strichartz, *A note on Trudinger's extension of Sobolev's inequalities*, Indiana Univ. Math. J. **21** (1972), 841–8.

60. G.M. Tashkian, *The classical formula of the asymptotics of the spectrum of elliptic equations, degenerate at the boundary of the domain*, Mat. Zametki **30** (1981), no. 6, 871–880. (Russian)

61. H. Triebel, *Interpolation theory, function spaces, differential operators*, North-Holland, Amsterdam, 1978.

62. H. Triebel, *Theory of function spaces*, Birkhäuser, Basel, 1983.

63. H. Triebel, *Theory of function spaces II*, Birkhäuser, Basel, 1992.

64. H. Triebel, *Approximation numbers and entropy numbers of embeddings of fractional Besov–Sobolev spaces in Orlicz spaces*, Proc. London Math. Soc. **66** (1993), 589–618.

65. H. Triebel, *Relations between approximation numbers and entropy numbers*, to appear in J. Approx. Theory.

66. H. Triebel, *A localization property of B^s_{pq} and F^s_{pq} spaces*, Studia Math. **109** (1994), 183–195.

67. H. Triebel and H. Winkelvoss, *Intrinsic atomic characterizations of function spaces on domains*, to appear.

68. N.S. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–483.