L^p-REGULARITY FOR SYSTEMS OF PDE'S, WITH COEFFICIENTS IN VMO

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1. Introduction

The purpose of these lectures is to review some recent work on the L^p regularity of the maximum order derivatives of the solutions to a certain class of linear elliptic systems both in divergence and non divergence form with *discontinuous* coefficients.

First we point out that the L^p regularity we discuss here is not of the kind of Meyers' result (i.e. valid only for p close to 2, see [41], and [31]). On the contrary our results hold for any value of p in the range $(1, +\infty)$. Obviously, such a result requires additional "smoothness" of the coefficients, see [41] again. Here we shall see that the relevant assumption is that the coefficients belong to what is generally known as the space VMO. Recall that VMO consists of BMO functions whose integral oscillation over balls shrinking to a point converge uniformly to zero, see Section 3 for precise definitions and references.

L^p estimates of the kind we will discuss are well known in the case of continuous coefficients. In order to introduce the topic we will pause in the next section to discuss the classical methods for obtaining L^p estimates when the coefficients are continuous. This will show what are the natural limits of those methods. Also it will be clear on what slight modification of some of those methods our work is based. We will only sketch the idea in the simple but rather representative case of one elliptic equation of the second order in divergence form. To begin with we assume the coefficients to be continuous and later we move to the VMO case.

In Section 3 we will give the precise definitions and the statements, with some detailed proof, of the real analysis tools we need in order to deduce the L^p regularity result.

Later we will discuss the L^p regularity for elliptic systems in non divergence form also touching the parabolic case. Finally we will mention some work still in progress.

We wish to conclude this introduction thanking the organizers for the kind invitation to take part in this meeting. Also we want to take this opportunity to thank many friends who gave various (both in size and nature) contributions to the research reviewed here. Especially we like to mention Carlos Kenig for pointing out to us the existence of the BMO commutator theorem at the very early stage of our research making it possible all the subsequent work. Also we are indebted to Eugene Fabes whose suggestions and incouragement have been, as usual, extremely valuable. We want to express our friendship and gratitude to Michele Frasca and Giuseppe Di Fazio who helped in the preparation of this note. Finally we thank Mario Marino and Tadeusz Iwaniec for pointing out many inaccuracies present in the first version of this paper.

2. A Simple Case

In this section we will analyze the simple case of one elliptic second order equation in divergence form. To be more specific let us consider in Ω, a bounded open subset of $\mathbb{R}^n$ $(n \geq 3)$, the equation

$$Lu \equiv -\left(a_{ij}u_{x_i}\right)_{x_j} = -(f_i)_{x_i} \equiv -\operatorname{div}\mathbf{f} \tag{2.1}$$

where we assume

$$\begin{aligned}
&\exists \nu > 0 : \nu^{-1}|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \nu|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega \\
&a_{ij} = a_{ji}, \qquad i,j = 1,\dots,n \\
&\exists p \in (1,+\infty) \ : \quad \mathbf{f} \equiv (f_1,\dots,f_n) \in L^p.
\end{aligned} \tag{2.2}$$

We also add, in this first part of the section, the following smoothness assumption

$$a_{ij} \in C^0(\overline{\Omega}) \qquad \forall i,j = 1,\dots,n. \tag{2.3}$$

Suppose we wish to study the well posedness of the Dirichlet problem for equation (2.1) in $W_0^{1,p}(\Omega)$. For what is known to the author this kind of result is obtained establishing first the same result for constant coefficients operators and then extending it to the variable coefficients case via a perturbation argument (*freezing, Korn's trick*) which can be summarized as follows. Assume that $u(x)$ is a solution of equation (2.1) supported in a *small* ball $B = B(x_0, r) \subset\subset \Omega$. We then transform equation (2.1) as follows

$$-\left(a_{ij}(x_0)u_{x_i}\right)_{x_j} = -\left[\left(a_{ij}(x_0) - a_{ij}(x)\right)u_{x_i}\right]_{x_j} - (f_i)_{x_i}\,.$$

If we suppose at this level to be able to estimate the L^p norm of the gradient of solutions to the *constant coefficients* equation in terms of the L^p norm of the right hand side we are done because we can write

$$\begin{aligned}\big\| \,|\nabla u|\, \big\|_p &\le c\left[\sum_{i,j=1}^n \big\|\big(a_{ij}(x_0) - a_{ij}(x)\big)u_{x_i}\big\|_p + \|\mathbf{f}\|_p\right] \\ &\le c\left[\sum_{i,j=1}^n \max_B |a_{ij}(x_0) - a_{ij}(x)|\, \big\| \,|\nabla u|\, \big\|_p + \|\mathbf{f}\|_p\right].\end{aligned} \tag{2.4}$$

Now if $\max_B |a_{ij}(x_0) - a_{ij}(x)|$, $i, j = 1, \ldots, n$, is small enough we can move the first term in the right hand side to the left obtaining (under all the supplementary assumptions we did!) the following a priori estimate:

$$\big\| \,|\nabla u|\, \big\|_p \le c\,\|\mathbf{f}\|_p\,.$$

As it is well known this is the basic step in obtaining the L^p estimates for solutions of the Dirichlet problem in all Ω. Some more, very well known, technicalities will clearly be needed (localization, flattening of the boundary, etc.). We do not dwell here on these details. What we want to stress now is the very elementary heart of this procedure.
Once we know that

i) $\max_B |a_{ij}(x_0) - a_{ij}(x)|$ *is small*

(e.g. in the continuous coefficients case if the radius r of the ball B is small) we are done if we also know

ii) *the result for the constant coefficients case.*

Before spending some more lines in a brief outline of the ways for obtaining ii) we express the hope that by now it will be even too much obvious to the reader why we call this procedure a *pointwise perturbation* about the constant coefficient case.

Also we wish to call the attention of the reader on requirement i).
It is clear that if we want the oscillation in condition i) arbitrarily small

this is equivalent to require the continuity of the coefficients. This is what happens if we want the L^p estimates with this method for all p's in $(1, +\infty)$ because the blowing up of the constant c in (2.4) when p diverges or approaches 1. We could ask for the L^p result only for some p's and this is exactly what one obtains with Korn's trick under assumptions of the type of Cordes (see e.g. [14]).

To prove ii), which is in any case a difficult step, there are essentially two methods at author's knowledge. The first, at least chronologically, is associated to the names of A. Calderón and A. Zygmund. This method uses explicit representation formulas for the derivatives of the solutions by means of singular integrals applied to the known term $\mathbf{f}$. While these formulas were well known and had been used by many authors (let us quote at least G. Giraud and C. Miranda) to study the regularity problem in the Hölder spaces it was the achievement of A. Calderón and A. Zygmund [9] as well as one of the main motivations in developing their theory (see e.g. [8]) to establish the boundedness in L^p of the relevant singular integrals.

The other method we wish to mention here is related to the work of C.B. Morrey and to the research of S. Campanato.

This method has been applied by a number of authors to a great variety of problems in PDE's. Excellent and comprehensive accounts of the method and its applications are given in Campanato [15], Giaquinta [31]. The basic tools in this method are the fact that solutions to constant coefficients equations are endowed with derivatives of any order (which can be estimated by the difference quotients method) and the exploitation in a very precise way of local energy estimates (Caccioppoli estimates). By these means growth estimates in various norms over balls for the derivatives of solutions are obtained.

In particular what is crucial here (see [17] for the case of Hölder continuous coefficients and, for the case of continuous coefficients, [16]) is to show the belonging of the highest order derivatives to the $\mathscr{L}^{1,n}$ Campanato space (which is BMO) whenever the known term is in the same space. The final result is obtained interpolating between some known L^p result (for equation (2.1) L^2 which is obtained for free in the divergence case) and the $\mathscr{L}^{1,n}$ result, by means of a well known theorem of Stampacchia ([53], [53′]; see also Campanato [16′], Fefferman and Stein [30]).

Indeed the original technique in [17] was not to derive the L^p estimates for the constant coefficients equation and then, by freezing, deducing them for the variable coefficients case. On the contrary the authors, exploited the $\mathscr{L}^{1,n}$ estimate obtained by Campanato in [13] where it is deduced in the case of Hölder continuous coefficients by means of some simple but delicate perturbation argument. We mention this because the $\mathscr{L}^{1,n}$ estimate is false for general continuous coefficients and especially because this approach was used in a very interesting recent paper by Acquistapace [1] in which for a second order linear elliptic divergence form system the L^p estimates are obtained using an approximation of the coefficients with Hölder continuous coefficients. A careful analysis of the dependence of the constant in Campanato's method allows the author to obtain the result for a class of systems with discontinuous coefficients.

Precisely the coefficients belong to $\mathscr{L}_\phi$ with $\phi(r) = 1/|\log r|$. (See the next section for a definition of $\mathscr{L}_\phi$ and some comments).

$\mathscr{L}_\phi$ is contained in VMO properly and Acquistapace shows that the $\mathscr{L}^{1,n}$ estimate doesn't hold if one takes as coefficients functions in VMO which are not in $\mathscr{L}_\phi$ with the above mentioned ϕ (see [1], sect. 5).

Before giving the essential of the procedure to deal with general VMO coefficients it is better to recall that applications of the Calderón-Zygmund and Korn method to second order elliptic equations with continuous coefficients can be found in [40], [33] for non divergence form equations (see also [32], [21]) and an extremely farreaching extension has been given in the papers [2], [3], [27] where are considered non divergence form systems of a very general kind (see also [44]).

For the divergence form it is difficult to give very precise references. Let us quote at least the books [44] and [50] where higher order divergence form equations are studied.

Let us now outline the method of proving the L^p result for equation (2.1) with VMO coefficients. Our starting point, as in what we called the Calderón-Zygmund procedure, is to establish representation formulas for the solution directly for the case of variable coefficients. This is done by means of a parametrix leading to the expression of the derivatives of the solution in terms of a singular integral acting on the known term **f** plus an *error term* expressed by another singular integral acting on the very same derivatives one wants to estimate. Luckily these derivatives appear in a *singular commutator* whose norm can be made small if the coefficients have a small *integral oscillation* (i.e. if they belong to VMO).

Then we can consider the essence of "our" method as an *integral perturbation* about the constant coefficient case. We stress that the technique we used is not "ours"! It goes back at least to Eugenio Elia Levi and has been extensively used by the authors working with spaces of Hölder continuous functions (see Miranda [43] once more). This is an interesting point because we feel that in order to obtain the L^p estimates the only tool the "classical" authors needed (not considering their possible lack of interest in L^p) was some piece of real analysis machinery that was developed and became familiar in the middle 70's (of this century!).

We will now give an outline of the proof of the L^p result for the simple equation (2.1). The following is taken from Di Fazio [24] which in turn depends much on the papers [18], [19] by M. Frasca, P. Longo and the author.

We want to prove the following theorem

Theorem 2.1 ([24]). *Suppose condition* (2.2) *holds and $\partial\Omega$ is smooth (say $C^{1,1}$). If, moreover, $a_{ij} \in \mathrm{VMO}$ for $i, j = 1, \dots, n$ then the Dirichlet problem*

$$Lu = \operatorname{div} \mathbf{f}, \qquad u \in W_0^{1,p}(\Omega)$$

has a unique solution. In addition, we have

$$\big\| \, |\nabla u| \, \big\|_{L^p(\Omega)} \leq c \, \|\mathbf{f}\|_{L^p(\Omega)} \, .$$

We notice first that such an estimate can be proved with a constant c *independent* of the smoothness of the coefficients, assuming the coefficients and the solution to be smooth. This is possible because our coefficients are in VMO (see Theorem 3.7 and the following remarks). Also it is clearly enough to consider $p > 2$ (duality). Localization and flattening then reduce the result to proving the following two theorems.

Theorem 2.2 (interior estimate). *Assume that* (2.2) *holds and a_{ij} are smooth for $i, j = 1, \dots, n$. Then there exists $\sigma > 0$ such that for every ball $B_\sigma \subset\subset \Omega$ with radius σ and every smooth solution of*

$$Lu = f_0 - \operatorname{div} \mathbf{f}$$

with f_0, $\mathbf{f}$ and u compactly supported in B_σ, we have the estimate

$$\big\| \, |\nabla u| \, \big\|_{L^p(B_\sigma)} \leq c \left(\|\mathbf{f}\|_{L^p(B_\sigma)} + \|f_0\|_{L^{p_*}(B_\sigma)} \right)$$

with $p_* = np/(n+p)$.

Theorem 2.3 (boundary estimate). *Assume that* (2.2) *holds and* a_{ij} *are smooth for* $i, j = 1, \ldots, n$. *Given a ball* B_σ *with center at the origin we call* $B_\sigma^+ = \{x \in B_\sigma : x_n > 0\}$. *Then there exists* $\sigma > 0$ *such that, for every smooth solution of* (2.1) *in* B_σ^+ *which vanishes on* $\{x_n = 0\} \cap B_\sigma$ *and is compactly supported in* B_σ, *we have*

$$\| \, |\nabla u| \, \|_{L^p(B_\sigma^+)} \leq c \, \|\mathbf{f}\|_{L^p(B_\sigma^+)} \, .$$

Proof of Theorem 2.2. We start proving the representation formula we mentioned above for $u \in C_0^\infty(B_\sigma)$. Fix any point x_0 in B_σ ($\sigma > 0$ to be fixed later). We have

$$\begin{aligned} -\left(a_{ij}(x_0) u_{x_i}(x)\right)_{x_j} &= -\left(\left(a_{ij}(x_0) - a_{ij}(x)\right) u_{x_i}(x) + f_j(x)\right)_{x_j} + f_0(x) \\ &\equiv -\left(\lambda_j^{x_0}(x)\right)_{x_j} + f_0(x). \end{aligned}$$

If we consider the "fundamental solution" Γ for the *constant coefficients* operator

$$-\left(a_{ij}(x_0) u_{x_i}(x)\right)_{x_j}$$

we obtain the representation

$$u(x) = -\int_{B_\sigma} \Gamma_j(x_0, x-y) \lambda_j^{x_0}(y) \, dy - \int_{B_\sigma} \Gamma(x_0, x-y) f_0(y) \, dy \, . \tag{2.5}$$

To be more explicit we set

$$\Gamma(x_0, t) = \frac{1}{(n-2)\omega_n \left(\det a_{ij}(x_0)\right)^{1/2}} \left(\sum_{i,j=1}^n A_{ij}(x_0) t_i t_j \right)^{(2-n)/2}$$

for a.e. x_0 in Ω, $t \in \mathbb{R}^n$, $t \neq 0$, where $A_{ij}(x_0)$ is the cofactor of $a_{ij}(x_0)$ in the matrix $\left(a_{ij}(x_0)\right)_{i,j}$ and ω_n is the surface area of the unit ball. Then we denote by

$$\Gamma_i(x_0, t) = \frac{\partial}{\partial t_i} \Gamma(x_0, t)$$

and

$$\Gamma_{ij}(x_0, t) = \frac{\partial^2}{\partial t_i \partial t_j} \Gamma(x_0, t)$$

(compare e.g. [43] or [35]). Differentiating (2.5), which is a delicate though standard business because of the bad singularity appearing in Γ_{ij}, one has

$$
\begin{aligned}
u_{x_i}(x) = &-\mathrm{P.V.}\int_{B_\sigma} \Gamma_{ij}(x_0, x-y)\left\{[a_{hj}(x_0) - a_{hj}(y)]\, u_{x_h}(y) - f_j(y)\right\} dy \\
&- \int_{B_\sigma} \Gamma_i(x_0, x-y) f_0(y)\, dy + c_{ij}(x_0)\lambda_j^{x_0}(x)
\end{aligned} \tag{2.6}
$$

where P.V. in front of the first integral means that the integral is taken as a principal value integral, and

$$
c_{ij}(x_0) = \int_{|t|=1} \Gamma_i(x_0, t) t_j \, d\sigma.
$$

We now take in (2.6) $x = x_0$ obtaining

$$
\begin{aligned}
u_{x_i}(x) = &-\mathrm{P.V.}\int_{B_\sigma} \Gamma_{ij}(x, x-y)\left\{[a_{hj}(x) - a_{hj}(y)]\, u_{x_h}(y) - f_j(y)\right\} dy \\
&- \int_{B_\sigma} \Gamma_i(x, x-y) f_0(y)\, dy + c_{ij}(x) f_j(x) \qquad \forall x \in B_\sigma .
\end{aligned}
$$

Once we have an explicit representation formula for u_{x_i} in order to obtain the desired estimates we have only to evaluate the $L^p(B_\sigma)$ norm of the right hand side. The last term on the right side is good because $c_{ij}(x)$ are bounded functions whose L^∞ norm can be estimated in terms of ν in (2.2). The middle term is pointwise majorized by a Riesz type fractional integral

$$
\int_{B_\sigma} \frac{f_0(y)}{|x-y|^{n-1}}\, dy
$$

which obviously is a bounded operator from $L^{p_*}(B_\sigma)$ in $L^p(B_\sigma)$. The first term is more conveniently written as the sum of terms of the form

$$
K f_j \equiv \mathrm{P.V.}\int_{B_\sigma} \Gamma_{ij}(x, x-y) f_j(y)\, dy \tag{2.7}
$$

and of the form

$$
C[a_{hj}, K]\, u_{x_h} \equiv \mathrm{P.V.}\int_{B_\sigma} \Gamma_{ij}(x, x-y)\left\{[a_{hj}(x) - a_{hj}(y)]\, u_{x_h}(y)\right\} dy. \tag{2.8}
$$

Both the (variable kernel) singular integrals are bounded operators in $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$, as it will be shown in the next section. Moreover the *commutator* $C[a_{hj}, K]\, u_{x_h}$ has a bound of the form

$$\|C[a_{hj}, K]\, u_{x_h}\|_{L^p(B_\sigma)} \le c(n,p)\, \|a_{hj}\|_* \,\big\|\, |\nabla u|\, \big\|_{L^p(B_\sigma)}$$

where $\|a_{hj}\|_*$ is the BMO "norm" (= the integral oscillation) of the relevant coefficient a_{hj} appearing inside (for the definition of $\|a_{hj}\|_*$ see Definition 3.1). The nice feature of this operator, which is the fundamental point of our estimate, is that, taking a_{hj} in VMO, the $\|a_{hj}\|_*$ can be made small as we like taking the radius σ of B_σ small enough. In other words we can fix σ so small to have the estimate

$$\begin{aligned} \big\|\, |\nabla u|\, \big\|_{L^p(B_\sigma)} \le & \frac{1}{2} \big\|\, |\nabla u|\, \big\|_{L^p(B_\sigma)} + \\ & + c(n,p,\nu)\, \left(\|\mathbf{f}\|_{L^p(B_\sigma)} + \|f_0\|_{L^{p_*}(B_\sigma)} \right). \end{aligned} \tag{2.9}$$

This proves Theorem 2.2.

Proof of Theorem 2.3. We now argue similarly to prove Theorem 2.3. In order to obtain a boundary representation formula it is better to recall the definition of the half space Green function for the constant coefficients operator

$$-\left(a_{ij}(x_0) u_{x_i}(x)\right)_{x_j}.$$

Such a Green function is easily obtained subtracting to the fundamental solution with pole at x in the upper half space the fundamental solution with pole at an appropriate point $T(x)$ in the lower half space ($T(x)$ would be a reflection when dealing with the Laplacian).

To be more precise let $\Gamma(x,t)$ have the same meaning as above. Set

$$a(x) = \{a_{in}(x)\}_{i=1,\dots,n}, \qquad T(x;y) = x - \frac{2x_n}{a_{nn}(y)} a(y), \qquad T(x) = T(x;x).$$

Finally set $A(y) = T(e_n; y)$ where $e_n = (0, \dots, 0, 1)$ and denote $A_i(y)$ the i-th component of $A(y)$. Consider now any $x_0 \in B_\sigma^+$ and, as in the previous proof, write the equation as

$$\begin{aligned} -\left(a_{ij}(x_0) u_{x_i}(x)\right)_{x_j} &= -\left(\left(a_{ij}(x_0) - a_{ij}(x)\right) u_{x_i}(x) + f_j(x)\right)_{x_j} \\ &\equiv -\left(\lambda_j^{x_0}(x)\right)_{x_j}. \end{aligned}$$

Then using the fact that the Green function for the constant coefficients operator

$$(a_{ij}(x_0)u_{x_i}(x))_{x_j}$$

in the half space is

$$\Gamma(x_0, x-y) - \Gamma(x_0, T(x;x_0) - y)$$

we see that a solution in B_σ^+ can be written as

$$u(x) = -\int_{B_\sigma^+} [\Gamma_j(x_0, x-y) - \Gamma_j(x_0, T(x;x_0) - y)]\, \lambda_j^{x_0}(y)\, dy. \tag{2.10}$$

Taking the derivatives, keeping in due regard the singularities of the first term inside (the second being never singular in B_σ^+), and finally setting $x_0 = x$ we obtain

$$\begin{aligned} u_{x_i}(x) = -\text{P.V.} \int_{B_\sigma^+} \Gamma_{ij}(x, x-y) \left\{[a_{hj}(x) - a_{hj}(y)]\, u_{x_h}(y) - f_j(y)\right\} dy \\ + c_{ij}(x) f_j(x) + I_i(x) \end{aligned} \tag{2.11}$$

where

$$c_{ij}(x) = \int_{|t|=1} \Gamma_i(x,t) t_j \, d\sigma_t,$$

as before, and

$$I_i(x) = \int_{B_\sigma^+} \Gamma_{ij}(x, T(x) - y) \left\{[a_{hj}(x) - a_{hj}(y)]\, u_{x_h}(y) - f_j(y)\right\} dy$$

for $i = 1, \dots, n-1$ while

$$I_n(x) = \int_{B_\sigma^+} A_h(x)\Gamma_{hj}(x, T(x) - y) \left\{[a_{hj}(x) - a_{hj}(y)]\, u_{x_h}(y) - f_j(y)\right\} dy.$$

(2.11) is very similar to (2.6) except for the term I_i. The I_i integral can be written as a sum of integrals of the form

$$\widetilde{K} f_j \equiv \int_{B_\sigma^+} \Gamma_{ij}(x, T(x) - y) f_j(y)\, dy \tag{2.12}$$

and of the form

$$\widetilde{C}[a_{hj}, K]\, u_{x_h} \equiv \int_{B_\sigma^+} \Gamma_{ij}(x, T(x) - y) \left\{[a_{hj}(x) - a_{hj}(y)]\, u_{x_h}(y)\right\} dy \tag{2.13}$$

(the $A_h(x)$ terms are not relevant because they are bounded functions whose L^∞ norm is estimated in terms of the ellipticity constant ν). Then, as in the interior case, we are reduced to study the L^p boundedness of the integral operators defined by (2.12) and (2.13). This will be done in Section 3 giving a result similar to what we obtained in the interior case. Precisely a positive σ can be fixed in such a way to obtain

$$\| \, |\nabla u| \, \|_{L^p(B_\sigma^+)} \le \frac{1}{2} \| \, |\nabla u| \, \|_{L^p(B_\sigma^+)} + c(n,p,\nu) \, \|\mathbf{f}\|_{L^p(B_\sigma^+)} \, .$$

This concludes the proof of Theorem 2.3.

As mentioned after the statement of Theorem 2.1 by localization and flattening one can prove (by means of Theorems 2.2 and 2.3) an estimate like

$$\| \, |\nabla u| \, \|_{L^p(\Omega)} \le c \left(\| \, |\nabla u| \, \|_{L^2(\Omega)} + \|\mathbf{f}\|_{L^p(\Omega)} \right) \le c \, \|\mathbf{f}\|_{L^p(\Omega)} \tag{2.14}$$

for the solution of the Dirichlet problem. In the last majorization we used the fact that $p > 2$ and the L^2 bound for such a solution in terms of $\|\mathbf{f}\|_{L^2(\Omega)}$ is known (Lax-Milgram). (2.14) gives the conclusion at least for smooth solutions. The general result is obtained by approximation.

3. Real Analysis Tools

In this section we collect the definitions and theorems concerning VMO and the action on L^p of the singular integral operators which are used in the proof of the L^p estimates.

Definition 3.1 (John-Nirenberg [39]). We say that a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is in the space BMO if

$$\sup_B \frac{1}{|B|} \int_B |f(x) - f_B| \, dx \equiv \|f\|_* < +\infty$$

where B ranges in the class of the balls of $\mathbb{R}^n$, and

$$f_B \equiv \frac{1}{|B|} \int_B f(x) \, dx$$

is the average of f in B.

$\|f\|_*$ is a norm in BMO modulo constant functions under which BMO is a Banach space (see Neri [47]).

Definition 3.2 (Sarason [49]). For $f \in \mathrm{BMO}$ and $r > 0$, we set

$$\sup_{\varrho \le r} \frac{1}{|B|} \int_B |f(x) - f_B| \, dx = \eta(r)$$

where B ranges in the class of the balls of $\mathbb{R}^n$ with radius ϱ less than or equal to r.

According to [49], $f \in \mathrm{BMO}$ is in VMO if

$$\lim_{r \to 0} \eta(r) = 0\,.$$

We will call η the VMO modulus of f.

Bounded uniformly continuous (BUC) functions belong to VMO. Functions in $W^{1,n}$ belong to VMO. Indeed by Poincaré's inequality,

$$\frac{1}{|B|} \int_B |f(x) - f_B| \, dx \le c(n) \left(\int_B |\nabla f(x)|^n \, dx \right)^{1/n}.$$

To give another example of the same kind we recall the definition of the Morrey space $L^{p,\lambda}$.

Definition 3.3. Let $p \in [1, +\infty)$, $\lambda \in (0, n)$. We say that a function $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ is in $L^{p,\lambda}(\mathbb{R}^n)$ if

$$\sup_r \left(\frac{1}{r^\lambda} \int_{B_r} |f(x)|^p \, dx \right)^{1/p} \equiv \|f\|_{p,\lambda} < +\infty.$$

We also define $\mathrm{VL}^{p,\lambda}(\mathbb{R}^n)$, p, λ as above, as the subspace of $L^{p,\lambda}(\mathbb{R}^n)$ of the functions f such that

$$\sup_{\varrho \le r} \left(\frac{1}{\varrho^\lambda} \int_{B_\varrho} |f(x)|^p dx \right)^{1/p} \equiv \vartheta(r)$$

vanishes as r approaches zero.

Then it is immediately seen that functions whose gradient is in $L^{1,n-1}$ are in BMO and functions whose gradient is in $\mathrm{VL}^{1,n-1}$ are in VMO. This shows easily that there are VMO functions which are discontinuous. E.g. one can take $f(x) = |\log |x||^\alpha$, $0 < \alpha < 1$.

Similarly one can see that $W^{\theta,n/\theta}(\mathbb{R}^n)$ $(0<\theta<1)$ is contained in VMO. Indeed

$$\begin{aligned}
\frac{1}{|B|}\int_B |f(x)-f_B|\,dx &\le \left(\frac{1}{|B|}\int_B |f(x)-f_B|^{n/\theta}\,dx\right)^{\theta/n} \\
&= \left(\frac{1}{|B|}\int_B \left|\frac{1}{|B|}\int_B (f(x)-f(y))\,dy\right|^{n/\theta} dx\right)^{\theta/n} \\
&\le \left(\frac{1}{|B|}\int_B \frac{1}{|B|}\int_B |f(x)-f(y)|^{n/\theta}\,dxdy\right)^{\theta/n} \\
&\le c_n \left(\int_B\int_B \frac{|f(x)-f(y)|^{n/\theta}}{|x-y|^{2n}}\,dxdy\right)^{\theta/n}.
\end{aligned}$$

Because $f\in W^{\theta,n/\theta}(\mathbb{R}^n)$ implies that

$$\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^{n/\theta}}{|x-y|^{2n}}\,dxdy\right)^{\theta/n} < +\infty\,,$$

we get the conclusion by the absolute continuity of the integral.

Definition 3.4 (e.g. Campanato [13]). Let $p\in[1,+\infty)$, $\lambda\in[0,n+p]$. We say that a function $f\in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ is in the space $\mathscr{L}^{p,\lambda}(\mathbb{R}^n)$ if

$$\sup_r \left(\frac{1}{r^\lambda}\int_{B_r} |f(x)-f_{B_r}|^p dx\right)^{1/p} \equiv \|f\|_{p,\lambda} < +\infty.$$

This family of spaces contains BMO (which is the same as $\mathscr{L}^{1,n}(\mathbb{R}^n)$) and also $C^{0,\alpha}$ as can be seen by the following theorem.

Theorem 3.5 (Campanato [12], Meyers [42]). *Let $p\in[1,+\infty)$, $\lambda\in(n,n+p)$. Then $\mathscr{L}^{p,\lambda}$ coincides with the space of the Hölder continuous functions $C^{0,\alpha}$ $(\alpha=(\lambda-n)/p)$.*

One immediately realizes that functions in VMO are Hölder continuous if $\eta(r)\le cr^\alpha$.
The following question may then be posed naturally:

when (depending on $\eta(r)$) functions in VMO *are continuous?*

The answer has been given by S. Spanne, who perhaps really "invented" VMO about ten years before Sarason. Indeed while studying some generalizations of the $\mathscr{L}^{1,\lambda}$ spaces Spanne introduced in [52] the $\mathscr{L}_\phi$ spaces. Precisely:

Definition 3.6 ([52]). Let $\phi : [0,+\infty) \to [0,+\infty)$ non-decreasing. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We say that f is in the space $\mathscr{L}_\phi$ if

$$\sup \frac{1}{\phi(|B|^{\frac{1}{n}})} \frac{1}{|B|} \int_B |f(x) - f_B|\, dx = \|f\|_{\mathscr{L}_\phi} < +\infty.$$

Clearly for $\phi(t) = t^{\lambda-n}$ $(\lambda > n)$ this gives back the Campanato spaces $\mathscr{L}^{1,\lambda}(\mathbb{R}^n)$. Also, considering $f \in \mathscr{L}_\phi$, $B_\varrho \subseteq B_r$, we have

$$\frac{1}{|B_\varrho|} \int_{B_\varrho} |f(x) - f_{B_\varrho}|\, dx \le \|f\|_{\mathscr{L}_\phi} \cdot \phi(c_n \varrho) \le \|f\|_{\mathscr{L}_\phi} \cdot \phi(c_n r)\, .$$

This shows that $\mathscr{L}_\phi \subseteq \mathrm{VMO}$ if $\phi(r)$ vanishes as r approaches zero and that as VMO modulus for $f \in \mathscr{L}_\phi$ we can take

$$\eta(r) = \|f\|_{\mathscr{L}_\phi} \cdot \phi(c_n r)\, .$$

Spanne proved (see [52] p. 601) that, if $\phi(t)$ is Dini continuous i.e.,

$$\exists \delta > 0 : \quad \int_0^\delta \frac{\phi(t)}{t}\, dt < +\infty\, ,$$

then $\mathscr{L}_\phi \subset C^0$. This is “almost” sharp as can be seen by Corollary 2 in [52].

Another slightly different definition of VMO (called there c.m.o.) was given by U. Neri (see [46], [47]) where some interesting examples can be found. Let us also mention the paper [36], mainly concerning the problem of the characterization of pointwise multipliers for BMO, where others remarks about VMO and c.m.o. can be found (pp. 195-196).

After this introduction of the spaces we now quote two results on which it rests the real analysis tools we shall develop later. The first is part of the Sarason's characterization of VMO.

Theorem 3.7 ([49]). *For $f \in \mathrm{BMO}$ the following conditions are equivalent*

i) *f is in* VMO*;*
ii) *f is in the* BMO *closure of* BUC*;*
iii) $\lim_{h\to 0} \|f(\cdot - h) - f(\cdot)\|_* = 0.$

We wish to observe explicitely that iii) implies the good behaviour (in the sense of the BMO convergence) of the mollifiers of VMO functions (see [19], p. 843).

To state the most important result for the following development we need to give one more definition.

Definition 3.8. Let $k : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$. We say that k is a Calderón-Zygmund kernel (CZ) if:

i) $k \in C^\infty(\mathbb{R}^n \setminus \{0\})$;
ii) $k(x)$ is homogeneous of degree $-n$;
iii) $\int_\Sigma k(x)\, d\sigma_x = 0 \quad where \quad \Sigma \equiv \{x : |x| = 1\}$.

It is well known that to a CZ kernel it is possible to associate a bounded operator in L^p. Precisely:

Theorem 3.9 ([9]). *Let $k(x)$ be a CZ kernel and ε a positive number. For $f \in L^p(\mathbb{R}^n)$, $p \in (1, +\infty)$, set*

$$K_\varepsilon f(x) = \int_{|x-y|>\varepsilon} k(x-y) f(y)\, dy.$$

Then, for any $f \in L^p(\mathbb{R}^n)$, there exists $Kf \in L^p(\mathbb{R}^n)$ such that

$$\lim_{\varepsilon \to 0} \|K_\varepsilon f - Kf\|_p = 0.$$

Also there exists $c = c(n,p)$ such that

$$\|Kf\|_p \le c \left(\int_\Sigma k^2 d\sigma \right)^{1/2} \|f\|_p\ , \qquad \forall f \in L^p(\mathbb{R}^n).$$

We shall call K a Calderón-Zygmund singular integral operator and we shall use the notation

$$Kf(x) = \text{P.V.} k * f(x) = \text{P.V.} \int_{\mathbb{R}^n} k(x-y) f(y)\, dy.$$

We are now in a position to state the second result (and most important for our future development) we need to recall.

Definition 3.10 (Coifman, Rochberg and Weiss [22]). Let $\varphi \in$ BMO, k be a CZ kernel and K the associated singular integral operator. We define the commutator $Tf = C[\varphi, K] f$ as the principal value

$$\varphi\, \text{P.V.} k * f - \text{P.V.} k * (\varphi f).$$

Then we have

Theorem 3.11 (Coifman, Rochberg and Weiss [22]). *Let φ, k be as above. Then $C[\varphi,K]\,f$ is well defined for $f \in L^p(\mathbb{R}^n)$ $p \in (1,+\infty)$. Moreover $C[\varphi,K]\,f$ is a bounded operator in $L^p(\mathbb{R}^n)$, i.e. there exists a constant $c = c(n,p,\|k\|_{L^2(\Sigma)})$ such that*

$$\|C[\varphi,K]\,f\|_p \le c\,\|\varphi\|_* \|f\|_p \ .$$

For the sake of completeness let us quote one more theorem from the literature which is an important complement to Theorem 3.11.

Theorem 3.12 (Uchiyama [55]). *The commutator in Definition 3.10 is a compact operator from $L^p(\mathbb{R}^n)$ in itself if and only if φ is in the* BMO *closure of $C_0^\infty(\mathbb{R}^n)$.*

Finally concerning the relevance of Theorem 3.11 we wish to mention the recent and deep paper [23].

A note of warning: the *commutator* in Definition 3.10 above is very different from the famous A. Calderón commutator. Theorems 3.9 and 3.11 are what is needed to study the slightly more involved operators which appears in our representation formulas for solutions. We have the following theorem which is crucial in the proof of the interior estimates.

Theorem 3.13 ([18]). *Let $k : \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}$ be such that:*

i) *$k(x,\cdot)$ is a Calderón-Zygmund kernel for a.a. $x \in \mathbb{R}^n$;*

ii) $\displaystyle \max_{|j|\le 2n} \left\| \frac{\partial^j}{\partial y^j} k(x,y) \right\|_{L^\infty(\mathbb{R}^n\times\Sigma)} \equiv M < +\infty.$

If $f \in L^p(\mathbb{R}^n)$, $1 < p < +\infty$, $\varphi \in L^\infty(\mathbb{R}^n)$, set

$$K_\varepsilon f(x) = \int_{|x-y|>\varepsilon} k(x,x-y) f(y)\,dy$$

and

$$C[\varphi,K_\varepsilon]\,f = \varphi K_\varepsilon f - K_\varepsilon(\varphi f) = \int_{|x-y|>\varepsilon} k(x,x-y)[\varphi(x)-\varphi(y)] f(y)\,dy \, .$$

Then, for any $f \in L^p(\mathbb{R}^n)$ there exist Kf, $C[\varphi,K]\,f \in L^p(\mathbb{R}^n)$ such that

$$\lim_{\varepsilon\to 0} \|K_\varepsilon f - Kf\|_p = 0 \quad , \quad \lim_{\varepsilon\to 0} \|C[\varphi,K_\varepsilon]\,f - C[\varphi,K]\,f\|_p = 0\, .$$

Moreover there exists a constant $c \equiv c(n,p,M)$ *such that*

$$\|Kf\|_p \le c\,\|f\|_p \qquad \text{and} \qquad \|C[\varphi,K]\,f\|_p \le c\,\|\varphi\|_*\|f\|_p \ .$$

The proof of the previous theorem seems to be at this time classical using techniques which go back to the work of A. Calderón and A. Zygmund in the late 50's. We essentially took it from [6]. The interested reader can also see [10], [11], [45]. An outstanding reference for the further development of this topic is Stein [54].

From the commutator bound given in Theorem 3.13 one easily deduces the following localized estimate.

Corollary 3.14 ([18])**.** *Let* k *be as in Theorem 3.13 and* $\varphi \in \mathrm{VMO} \cap L^\infty(\mathbb{R}^n)$ *and denote by* η *the* VMO *modulus of* φ*. Then for any* $\varepsilon > 0$ *there exists* $\varrho_0 = \varrho_0(\varepsilon,\eta)$ *such that for* $r \in (0,\varrho_0)$ *we have*

$$\|C[\varphi,K]\,f\|_{L^p(B_r)} \le c\,\varepsilon\|f\|_{L^p(B_r)} \qquad \forall f \in L^p(B_r)\ .$$

Proof. We start approximating φ with a BUC $\tilde\varphi$ such that $\|\varphi-\tilde\varphi\|_* < \varepsilon/2$. Then we fix ϱ_0 so small that the modulus of continuity of $\tilde\varphi$, $\omega_{\tilde\varphi}(r)$, is less than $\varepsilon/2$ when evaluated at ϱ_0. Finally we extend the restriction of $\tilde\varphi$ to B_r $(r < r_0)$ to all $\mathbb{R}^n$ preserving the modulus of continuity in B_r.

Using Theorem 3.13 one can deal with the interior estimates as we have already shown when considering the simple case above and as we will see below when considering elliptic systems. The reason for this is that when deducing a representation formula for the maximum order derivatives we always find a CZ variable kernel operator and commutator like in Theorem 3.13.

The study of the boundary estimates is different and it leads to the appearing of singular integral operators and commutators which are singular in a less severe way than those in Theorem 3.13 and can be sometimes treated much more simply. The kind of singularity appearing is much as in the Hardy operator (see [34] p. 226 and ff.). We shall study now in detail both the operators (singular integral and commutator) in the following theorems.

Let $T(x)$ be the map introduced during the construction of the boundary representation formula in Section 2. We first observe the following simple inequality.

Lemma 3.15 ([19], [4]). *Let* $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, x_n > 0\}$, $\tilde{x} = (x', -x_n)$. *Then there exist positive constants* c_1, c_2 *such that, for any* $y \in \mathbb{R}^n_+$ *and all* $x \in \mathbb{R}^n_+$ *for which* $T(x)$ *is defined, we have*

$$c_1 |\tilde{x} - y| \le |T(x) - y| \le c_2 |\tilde{x} - y| \ .$$

This lemma clearly reduces the study of operator $\widetilde{K} f$ in (2.12) to that of another operator of the same kind with $T(x)$ replaced by $\tilde{x}$ which we still call $\widetilde{K} f$. We have

Theorem 3.16 ([19]). *Let* $\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, x_n > 0\}$, $\tilde{x} = (x', -x_n)$. *Set*

$$\widetilde{K} f(x) = \int_{\mathbb{R}^n_+} \frac{f(y)}{|\tilde{x} - y|^n} \, dy \ .$$

Then there exists a constant $c = c(n,p)$ *such that*

$$\|\widetilde{K} f\|_{L^p(\mathbb{R}^n_+)} \le c \, \|f\|_{L^p(\mathbb{R}^n_+)} \ .$$

Proof. For $x \in \mathbb{R}^n_+$ let

$$\begin{aligned} I(x_n) &= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}^n_+} \frac{|f(y)|}{(|x' - y'|^2 + (x_n + y_n)^2)^{n/2}} \, dy \right)^p dx' \\ &= \int_{\mathbb{R}^{n-1}} \left(\int_0^{+\infty} \int_{\mathbb{R}^{n-1}} \frac{|f(y)|}{(|x' - y'|^2 + (x_n + y_n)^2)^{n/2}} \, dy' dy_n \right)^p dx'. \end{aligned}$$

Using the Minkowski integral inequality we have

$$\begin{aligned} & I(x_n) \\ & \le \left[\int_0^{+\infty} \left(\int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}^{n-1}} \frac{|f(y)|}{(|x' - y'|^2 + (x_n + y_n)^2)^{n/2}} \, dy' \right)^p dx' \right)^{1/p} dy_n \right]^p . \end{aligned}$$

The inner integral is a convolution in the $'$ variables. Then using the Young inequality for convolutions we obtain

$$
I(x_n) \le \left[\int_0^{+\infty} \left(\int_{\mathbb{R}^{n-1}} |f(y)|^p \, dy' \right)^{1/p} \left(\int_{\mathbb{R}^{n-1}} \frac{dy'}{(|y'|^2 + (x_n + y_n)^2)^{n/2}} \right) dy_n \right]^p .
$$

Now

$$
\int_{\mathbb{R}^{n-1}} \frac{dy'}{(|y'|^2 + (x_n + y_n)^2)^{n/2}} = \frac{1}{x_n + y_n} \int_{\mathbb{R}^{n-1}} \frac{dy'}{(x_n + y_n)^{n-1} \left(\frac{|y'|^2}{(x_n+y_n)^2} + 1 \right)^{n/2}}
$$

or, setting $t' = \dfrac{y'}{x_n + y_n}$,

$$
\int_{\mathbb{R}^{n-1}} \frac{dy'}{(|y'|^2 + (x_n + y_n)^2)^{n/2}} = \frac{1}{x_n + y_n} \int_{\mathbb{R}^{n-1}} \frac{dt'}{(|t'|^2 + 1)^{n/2}} .
$$

Substituting in the majorization we found above we get

$$
I(x_n) \le \left(\int_{\mathbb{R}^{n-1}} \frac{dt'}{(|t'|^2 + 1)^{n/2}} \right)^p \left(\int_0^{+\infty} \frac{\|f(\cdot, y_n)\|_{L^p(\mathbb{R}^{n-1})}}{x_n + y_n} \, dy_n \right)^p .
$$

Now we integrate $I(x_n)$ over $(0, +\infty)$ obtaining

$$
\|\widetilde{K} f\|_p^p \le c(n,p) \int_0^{+\infty} \left(\int_0^{+\infty} \frac{\|f(\cdot, \lambda x_n)\|_{L^p(\mathbb{R}^{n-1})}}{1 + \lambda} \, d\lambda \right)^p dx_n
$$

and, by Minkowski again

$$
\begin{aligned}
\|\widetilde{K} f\|_p^p &\le c(n,p) \left(\int_0^{+\infty} \left(\int_0^{+\infty} \frac{\|f(\cdot, \lambda x_n)\|^p_{L^p(\mathbb{R}^{n-1})}}{(1+\lambda)^p} \, dx_n \right)^{1/p} d\lambda \right)^p \\
&= c(n,p) \left(\int_0^{+\infty} \frac{1}{(1+\lambda)\lambda^{1/p}} \, d\lambda \right)^p \|f\|_p^p .
\end{aligned}
$$

Our final theorems concern the boundary commutator $\widetilde{C}[a, K]$ defined in (2.13). Its study seems at this moment more delicate and it is worth to be done in some detail. In the paper [19] the study of $\widetilde{C}$ is done in a different (and simpler) way from what we shall see below. Unfortunately the approach of [19] is wrong as it was kindly pointed to us by M. Bramanti and M.C. Cerutti. The proof given below is essentially the same as in [4]. We collect here the definition and some properties of spherical harmonics which are needed for the proof of the following theorem (as well as in the proof of Theorem 3.13). Indeed in all these theorems the idea is to reduce the "variable kernel" case to the "constant kernel" case. This is done by expanding the kernel in a series of spherical harmonics each term then defining a constant kernel operator that one knows how to treat. Then, having the proper control on the decay of each term, one can see that the series of operators absolutely converges in L^p. The technique seems to have been employed for the first time by Calderón and Zygmund in [10].

Definition 3.17. A homogeneous polynomial $p(x)$ of degree m, which solves the equation $\Delta u = 0$, will be called a solid spherical harmonic of degree m. Its restriction to the unit sphere Σ will be called a spherical harmonic of degree m.

Then we have

Lemma 3.18 ([10], [11], [45]). *The space of n-dimensional spherical harmonics of degree m has dimension*

$$g_m = \binom{m+n+1}{n-1} - \binom{m+n+3}{n-1} \le c(n)\, m^{n-2}\,.$$

If $Y_m(x)$ is any n-dimensional spherical harmonic of degree m, then

$$\left|\frac{\partial^\alpha}{\partial x^\alpha} Y_m(x)\right| \le c\, m^{|\alpha|+(n-2)/2} \qquad \text{for } x \in \Sigma\,.$$

Let $\{Y_{km}(x)\}_{\substack{k=1,\dots,g_m\\ m=0,1,\dots}}$ be a complete (in $L^2(\Sigma)$) orthonormal system of spherical harmonics.

Let $k : \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}$ be such that

i) *$k(x,\cdot)$ is a CZ kernel for a.a. $x \in \mathbb{R}^n$;*

ii) $\displaystyle\max_{|j|\le 2n} \left\| \frac{\partial^j}{\partial y^j} k(x,y) \right\|_{L^\infty(\mathbb{R}^n\times\Sigma)} \equiv M < +\infty.$

Setting, for $m \in \mathbb{N}$ and $k = 1,\dots,g_m$,

$$a_{km}(x) = \int_\Sigma k(x,z) Y_{km}(z)\, d\sigma_z$$

by the completeness of $\{Y_{km}(x)\}$ *in* $L^2(\Sigma)$ *we have*

$$k(x,z) = \sum_{m=1}^{+\infty} \sum_{k=1}^{g_m} a_{km}(x) Y_{km}(z), \qquad x \in \mathbb{R}^n, \quad z \in \Sigma$$

and

$$\|a_{km}\|_\infty \le c(n)\, M\, m^{-2n}.$$

We need one more result concerning the "constant kernel" boundary commutator before we prove the main theorem.

Theorem 3.19. *Let* $k(x)$ *be a* CZ *kernel. Let* $T(x)$ *have the same meaning as in Lemma 3.15, and suppose* $\varphi \in L^\infty(\mathbb{R}^n)$. *Then*

$$\widetilde{C}[\varphi, K]\, f(x) = \int_{\mathbb{R}^n_+} k(T(x) - y)\, (\varphi(x) - \varphi(y))\, f(y)\, dy$$

is bounded in $L^p(\mathbb{R}^n_+)$ $(1 < p < \infty)$ *and we have*

$$\|\widetilde{C}[\varphi, K]\, f\|_{L^p(\mathbb{R}^n_+)} \le c\, \|\varphi\|_*\, \|f\|_{L^p(\mathbb{R}^n_+)}\ .$$

Proof. Because of lack of regularity of the function $T(x)$ our operator is no longer the commutator of φ with a CZ kernel. Hence we cannot apply the Coifman, Rochberg and Weiss Theorem 3.11 but have to give a new proof of the boundedness. The proof which we give below (which is taken from [37] where it is quoted by Janson as J.O. Strömberg's) cannot be applied directly to our operator again for lack of regularity in the x variable. This obstacle is easily removed taking into consideration the adjoint operator which we study below (we are indebted for this suggestion to E. Fabes).

Let

$$\widetilde{C}^* f(x) = \int_{\mathbb{R}^n_+} k(x - T(y))\, (\varphi(x) - \varphi(y))\, f(y)\, dy\, .$$

To show that $\widetilde{C}^* f$ satisfies a bound of the same kind of the one we want for $\widetilde{C}[\varphi, K]\, f$ we estimate its sharp function. Precisely, keeping in mind where we are working, we define, for $f \in L^p(\mathbb{R}^n_+)$,

$$f^\#_+(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q|\, dy\ ,$$

where Q is a cube contained in $\mathbb{R}^n_+$ and

$$M_+f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy\,.$$

It is easily seen that these operators satisfy

$$\|M_+f\|_{L^p(\mathbb{R}^n_+)} \le c\, \|f\|_{L^p(\mathbb{R}^n_+)}\,, \quad \|f\|_{L^p(\mathbb{R}^n_+)} \le c\, \|f^{\#}_+\|_{L^p(\mathbb{R}^n_+)}.$$

Then we have, for any Q as above,

$$\begin{aligned}
\widetilde{C}^* f(x) &= \int_{\mathbb{R}^n_+} k(x - T(y))\,(\varphi(x) - \varphi_Q)\, f(y)\, dy \\
&\quad + \int_{\mathbb{R}^n_+} k(x - T(y))\,(\varphi_Q - \varphi(y))\, f(y)\, dy \\
&= \int_{\mathbb{R}^n_+} k(x - T(y))\,(\varphi(x) - \varphi_Q)\, f(y)\, dy \\
&\quad + \int_{2Q} k(x - T(y))\,(\varphi_Q - \varphi(y))\, f(y)\, dy \\
&\quad + \int_{\mathbb{R}^n_+ \setminus 2Q} k(x - T(y))\,(\varphi_Q - \varphi(y))\, f(y)\, dy \\
&\equiv I + J + L\,.
\end{aligned}$$

I and J are easily estimated as in [19] p. 845.
We have indeed

$$\frac{1}{|Q|} \int_Q |I(y)|\, dy \le c\, \|\varphi\|_* \left(M_+ |\widetilde{K}^* f|^r(x)\right)^{1/r}\,, \qquad 1 < r < p$$

where $\widetilde{K}^* f$ is the adjoint operator of $\widetilde{K}$ in Theorem 3.16 which clearly majorizes the adjoint of our operator because of Lemma 3.15. Also

$$\frac{1}{|Q|} \int_Q |J(y)|\, dy \le c\, \|\varphi\|_* \,(M_+|f|^r(x))^{1/r}\,, \qquad 1 < r < p\,.$$

We come to L. We call x_Q the center of the cube Q and observe that

$$\big|k(x - T(y)) - k(x_Q - T(y))\big| \le c\, \frac{|x - x_Q|}{|x_Q - \tilde{y}|^{n+1}} \qquad \text{for } x \in Q,\ y \notin 2Q\,.$$

This because of the regularity the adjoint kernel has in the x variable and again Lemma 3.15. Then we have

$$|L(x) - L(x_Q)| \leq c \int_{\mathbb{R}^n_+} \frac{|x - x_Q|}{|x_Q - \tilde{y}|^{n+1}} \, |f(y)||\varphi(y) - \varphi_Q| \, dy \, .$$

Calculations identical to those in [19] p. 846 then imply

$$\frac{1}{|Q|} \int_Q |L(y) - L_Q| \, dy \leq c \, \|\varphi\|_* \left(M_+ |f|^r(x)\right)^{1/r} \, , \qquad 1 < r < p \, .$$

This gives the conclusion because what we obtained is the pointwise estimate

$$|\widetilde{C}^* f(x)|^\sharp_+ \leq c \, \|\varphi\|_* \left[\left(M_+ |\widetilde{K}^* f|^r(x) \right)^{1/r} + \left(M_+ |f|^r(x)\right)^{1/r} \right] ,$$

which raised to the p-th power owing to the boundedness properties of the relevant operators in the right hand side gives the conclusion.

Coming back to the variable kernel boundary commutator we have

Theorem 3.20. *Let* $k : \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}$ *satisfy the assumption* i) *and* ii) *of Theorem 3.13 and let* $\varphi \in L^\infty(\mathbb{R}^n)$. *Define*

$$\widetilde{C}[\varphi, K] \, f(x) = \int_{\mathbb{R}^n_+} k(x, T(x) - y) \, (\varphi(x) - \varphi(y)) \, f(y) \, dy$$

for $f \in L^p(\mathbb{R}^n_+)$, $1 < p < \infty$.

Then, we have the bound

$$\|\widetilde{C}[\varphi, K] \, f\|_{L^p(\mathbb{R}^n_+)} \leq c \, \|\varphi\|_* \, \|f\|_{L^p(\mathbb{R}^n_+)} \, .$$

Proof. Using Lemma 3.18 we can expand the variable kernel $k(x, T(x) - y)$ as

$$\sum_{m=1}^{+\infty} \sum_{k=1}^{g_m} a_{km}(x) \, \frac{Y_{km}(T(x) - y)}{|T(x) - y|^n} \ .$$

Then we are led to study the series

$$\sum_{m=1}^{+\infty} \sum_{k=1}^{g_m} \|a_{km}\|_{L^\infty(\mathbb{R}^n_+)} \, \|\widetilde{C}[\varphi, K_{km}] \, f\|_{L^p(\mathbb{R}^n_+)}$$

where

$$\widetilde{C}[\varphi, K_{km}]\, f = \int_{\mathbb{R}^n_+} \frac{Y_{km}(T(x)-y)}{|T(x)-y|^n}\, (\varphi(x)-\varphi(y))\, f(y)\, dy\,.$$

Because of Theorem 3.19 and Lemma 3.18 we see that the last series can be estimated by

$$\begin{aligned} c \sum_{m=1}^{+\infty} \sum_{k=1}^{g_m} m^{-2n} \cdot m^{(n-2)/2}\, \|\varphi\|_* \|f\|_{L^p(\mathbb{R}^n_+)} \\ \leq c \sum_{m=1}^{+\infty} m^{-2n} \cdot m^{(n-2)/2} \cdot m^{n-2}\, \|\varphi\|_* \|f\|_{L^p(\mathbb{R}^n_+)} \end{aligned}$$

which gives the desired estimate.

4. Elliptic Systems and Related Results

In this last section we briefly review all the results obtained up to now using the technique we outlined in Section 2. First of all we mention the already quoted papers [18], [19] by the author, M. Frasca and P. Longo with a proof of the existence and uniqueness of a strong solution to the Dirichlet problem for an elliptic nondivergence form second order equation with coefficients in VMO.

The technique of the proof is mainly along the lines we hinted at in the divergence form case with the extra problems arising because of lack of a priori estimates except for the Alexandrov-Pucci L^∞ bound. A generalization of this result has been given by C. Vitanza who considered the same equation with lower order terms under various sets of assumptions on these lower order coefficients ([56], [57]).

Also the same equation as in [18], [19] have been studied in the Morrey spaces by Di Fazio and Ragusa [26]. Precisely those authors show that interior estimates hold true in Morrey spaces by means of the same representation formula of [18]. This is enough because they were able to extend Theorem 3.13 to the framework of the Morrey spaces. This extension, which is of some interest in itself, is achieved using some L^p weighted estimates (with A_p weights).

The parabolic case of the same non divergence equation, obtaining similar estimates and results for the Cauchy-Dirichlet problem, has been considered by M. Bramanti and C. Cerutti in the interesting work ([4]) which has been already mentioned.The idea of the proof is essentially the same. They find explicit interior and boundary representation formulas for the solution in terms of parabolic singular integrals and commutators. Then they study these operators with techniques which are now very much close to the elliptic ones. This because everything is seen in the framework of the homogeneous spaces. Fabes and Rivière (see [28], [29]) and some other authors who were able in the middle sixties to extend to the parabolic case the work of Calderón and Zygmund, had a much harder life.

Elliptic systems of arbitrary order in non divergence form have been studied in the paper [20] proving local L^p estimates for the highest order derivatives of solutions. We want to give a brief outline of paper [20] to show the great closeness of the basic treatment with the simple case we discussed in Section 2.

Consider the differential operators in $\Omega \subset \mathbb{R}^n$

$$m_{ij}(x,D) = \sum_{|\alpha|=2r} a_{ij}^{(\alpha)}(x) D^\alpha \,, \qquad i,j = 1,\dots,N \,. \tag{4.1}$$

Given $f_i \in L^q_{\text{loc}}(\Omega)$, $1 < q < +\infty$, $i = 1,\dots,N$, consider the system

$$\sum_{j=1}^N m_{ij}(x,D) u_j(x) = f_i(x) \,, \qquad i = 1,\dots,N \,. \tag{4.2}$$

Assume

$$a_{ij}^{(\alpha)} \in \mathrm{VMO} \cap L^\infty(\mathbb{R}^n) \,, \qquad i,j = 1,\dots,N, \quad |\alpha| = 2r \tag{4.3}$$

and suppose system (4.2) to be elliptic in the following sense

$$\exists \lambda > 0 : L(x,\xi) \equiv \det\,(m_{ij}(x,\xi)) \geq \lambda\, |\xi|^{2rN} \;, \tag{4.4}$$

a.e. in Ω, for any $\xi \in \mathbb{R}^n$. We will now show how to obtain local interior representation formulas for strong solutions of system (4.2).

Fix any ball $B \subset \Omega$ and call $\widetilde{B}$ the subset of B where all the $a_{ij}^{(\alpha)}$ are defined and (4.4) holds. Also fix (for a moment) any $x_0 \in \widetilde{B}$. Consider the constant coefficient operator

$$L(x_0) \equiv \det\,(m_{ij}(x_0,D)) \tag{4.5}$$

obtained from $L(x,\xi)$ letting $x = x_0$ and substituting ξ with the correspondent derivative. $L(x_0)$ is clearly a linear elliptic differential operator of order $2rN$.

Let $\Gamma(x_0,t)$ be F. John's fundamental solution of (4.5). Suppose n to be odd, an assumption that one can always make (see [27]). It can be shown (see John [38]) that

$$\Gamma(x_0,t) = |t|^{2rN-n}\psi\left(x_0,\frac{t}{|t|}\right) ,$$

where $\psi(x_0,\cdot)$ is an analytic function.

This implies that

$$|D^\alpha\Gamma(x_0,t)| \le c\,|t|^{2rN-n-|\alpha|}.$$

Also it is possible to prove that $D^\alpha\Gamma(x_0,t)$ for $|\alpha| = 2rN$ is a homogeneous function of degree $-n$ with zero mean value on the sphere $|t| = 1$.
For this fundamental solution and for any $v \in C_0^\infty(B)$

$$v(x) = \int_\Omega \Gamma(x_0,x-y)\,L(x_0)\,v(y)\,dy\ . \tag{4.6}$$

Let $\big(m^{ij}(x_0,\xi)\big)$ be the cofactor of the element $m_{ij}(x_0,\xi)$ in the matrix $\big(m_{ij}(x_0,\xi)\big)_{i,j}$. The correspondent differential operators $m^{ij}(x_0,D)$ has order $2r(N-1)$, unless $m^{ij}(x_0,D) \equiv 0$.
Assume $u = (u_1,\dots,u_N)$ to be a $C_0^\infty(B)$ vector function. Then we can write

$$\begin{aligned} L(x_0)u_i = L(x_0)\left(\sum_{j=1}^N \delta_{ij}u_j\right) &= \sum_{j=1}^N \delta_{ij}L(x_0)u_j \\ &= \sum_{j=1}^N\left(\sum_{k=1}^N m^{ki}(x_0,D)\,m_{kj}(x_0,D)\right)u_j\ , \end{aligned}$$

where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$, $i,j = 1,\dots,N$. The above simple trick, of introducing the operator $L = \det m_{ij}$ and then coming back from the scalar equation $Lu = 0$ to the system as seen in the previous lines, is due to Bureau [5].

Then, recalling (4.6), we obtain

$$\begin{aligned} u_i(x) &= \int_B \Gamma(x_0,x-y)\sum_{j=1}^N\left(\sum_{k=1}^N m^{ki}(x_0,D)\,m_{kj}(x_0,D)\right)u_j(y)\,dy \\ &= \sum_{k=1}^N\int_B \Gamma(x_0,x-y)\,m^{ki}(x_0,D)\,\sum_{j=1}^N m_{kj}(x_0,D)\,u_j(y)\,dy\ . \end{aligned}$$

From this (integrating by parts) we understand that u_i is a linear combination of integrals of the form

$$\int_B D^\sigma \Gamma(x_0, x-y) \sum_{j=1}^{N} m_{kj}(x_0, D)\, u_j(y)\, dy\,,$$

where $|\sigma| = 2r(N-1)$, which may be rewritten as

$$\begin{aligned}
&\int_B D^\sigma \Gamma(x_0, x-y) \left[\sum_{j=1}^{N} \left(m_{kj}(x_0, D) - m_{kj}(y, D)\right) u_j(y) \right. \\
&\qquad\qquad \left. + \sum_{j=1}^{N} m_{kj}(y, D)\, u_j(y)\right] dy \\
&= \int_B D^\sigma \Gamma(x_0, x-y) \sum_{j=1}^{N} \sum_{|\alpha|=2r} \left[a_{kj}^{(\alpha)}(x_0) - a_{kj}^{(\alpha)}(y)\right] D^\alpha u_j(y)\, dy \\
&\qquad + \int_B D^\sigma \Gamma(x_0, x-y) \sum_{j=1}^{N} m_{kj}(y, D)\, u_j(y)\, dy\,.
\end{aligned}$$

Hence, with the usual care in differentiation, one can see that $D^\alpha u_i$, for $|\alpha| = 2r$, can be expressed as a linear combination of terms like

$$\begin{aligned}
&\text{P.V.} \int_B D^{\sigma+\alpha} \Gamma(x_0, x-y) \sum_{j=1}^{N} \sum_{|\alpha|=2r} \left[a_{kj}^{(\alpha)}(x_0) - a_{kj}^{(\alpha)}(y)\right] D^\alpha u_j(y)\, dy \\
&\qquad + \text{P.V.} \int_B D^{\sigma+\alpha} \Gamma(x_0, x-y) \sum_{j=1}^{N} m_{kj}(y, D)\, u_j(y)\, dy \\
&+ c_{\sigma+\alpha}(x_0) \left(\left[\sum_{j=1}^{N} \sum_{|\alpha|=2r} \left(a_{kj}^{(\alpha)}(x_0) - a_{kj}^{(\alpha)}(x)\right) D^\alpha u_j(x)\right] \right. \\
&\qquad\qquad\qquad \left. + \sum_{j=1}^{N} m_{kj}(x, D)\, u_j(x) \right),
\end{aligned}$$

with $|\sigma| = 2r(N-1)$, $|\alpha| = 2r$, $x \in B$, $x_0 \in \widetilde{B}$, where $c_{\sigma+\alpha}(x_0)$ is bounded uniformly in B.

As in Section 2, setting $x = x_0$, we obtain the representation formula for $D^\alpha u_i(x)$, $|\alpha| = 2r$ as a linear combination of terms of the form

$$\text{P.V.} \int_B D^{\sigma+\alpha} \Gamma(x, x-y) \sum_{j=1}^{N} \sum_{|\alpha|=2r} \left[a_{kj}^{(\alpha)}(x) - a_{kj}^{(\alpha)}(y)\right] D^\alpha u_j(y)\, dy\,,$$

$$\text{P.V.}\int_B D^{\sigma+\alpha}\Gamma(x,x-y)\sum_{j=1}^N m_{kj}(y,D)\,u_j(y)\,dy\ , \qquad \sum_{j=1}^N m_{kj}(x,D)\,u_j(x)\ .$$

This formula then allows us to establish by a simple contraction argument the following interior regularity result.

Theorem 4.1. *Assume* (4.3) *and* (4.4). *Let* $1 < q \le p < +\infty$ *and let* $u \in W^{2r,q}_{\mathrm{loc}}(\Omega)$ *be a solution of* (4.2) *with* $f_i \in L^p_{\mathrm{loc}}$, $i = 1,\dots,N$. *Then* $u \in W^{2r,p}_{\mathrm{loc}}(\Omega)$.

Also, given $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, *there is a constant* c *not depending on* u_i *and* f_i, $1 = 1,\dots,N$ *such that*

$$\sum_{\substack{j=1\\|\alpha|=2r}}^N \|D^\alpha u_j\|_{L^p(\Omega')} \le c\left(\sum_{j=1}^N\left(\|u_j\|_{L^p(\Omega'')} + \|f_j\|_{L^p(\Omega'')}\right)\right)\ .$$

Similar techniques certainly work in the case of elliptic divergence form systems. Di Fazio announced to have in preparation a paper ([25]) with a result like the above for divergence form systems (obtained using the Green function estimates given by Solonnikov in [51]).

Finally the linear estimates obtained in [18], [19] can be applied in the study of quasilinear equations. Recently D. Palagachev (Sofia University) announced to me the following result contained in the paper [48].

Consider the Dirichlet problem

$$\text{(4.7)}\qquad \begin{cases} a_{ij}(x,u)D_{ij}u + b(x,u,\nabla u) = 0 & \text{a.e. in } \Omega \\ u = \varphi \quad \text{on } \partial\Omega, & \varphi \in W^{2,n}(\Omega) \end{cases}$$

under the following assumptions.

Let $\Omega \subset \mathbb{R}^n$ $(n \ge 2)$, $\partial\Omega \in C^{1,1}$. Suppose the functions $a_{ij}(x,z)$ and $b(x,z,p)$ to be Carathéodory's functions, i.e. they are measurable in x for all $z \in \mathbb{R}$, $p \in \mathbb{R}^n$, and continuous with respect to the other variables for almost all $x \in \Omega$.

$$a_{ij}(x,z)\xi^i\xi^j \ge \lambda(|z|)|\xi|^2, \quad \forall\xi \in \mathbb{R}^n, \quad \text{a.e. in } \Omega, \quad a_{ij} = a_{ji}\,,$$

with $\lambda(t)$ positive and decreasing,

$$\left|a_{ij}(x,z) - a_{ij}(x,z')\right| \le a(x)\,\mu_M(|z-z'|), \quad \text{a.e. in } \Omega, \ \ \forall z,\ z' \in [-M,\, M]$$

where $a(x) \in L^\infty(\Omega)$, $\mu_M(t)$ is an increasing function vanishing as t approaches zero and $a_{ij}(x,0) \in L^\infty(\Omega)$. Also assume $a_{ij}(x,z)$ are VMO in x, locally uniformly in z, i.e.

$$\sup_{\rho\le r} \frac{1}{|B_\rho|}\int_{B_\rho}\left| a_{ij}(x,z) - \frac{1}{|B_\rho|}\int_{B_\rho} a_{ij}(y,z)\,dy\right| dx = \eta_M(r) \quad \forall z \in [-M,\, M]$$

and $\lim_{r\to 0} \eta_M(r) = 0$.

Concerning the function $b(x,z,p)$ one assumes

$$|b(x,z,p)| \le \nu(|z|)\big(b_1(x) + |p|^2\big), \qquad \text{a.a. } x \in \Omega,\ \ \forall (z,\, p) \in \mathbb{R}\times\mathbb{R}^n\,,$$

with $\nu(t)$ positive and increasing and $b_1(x) \in L^n(\Omega)$; furthermore

$$\operatorname{sign} z \cdot \frac{b(x,z,p)}{\lambda(|z|)} \le \nu_1(x)\,|p| + \nu_2(x), \qquad \text{a.a. } x \in \Omega,\ \ \forall (z,\, p) \in \mathbb{R}\times\mathbb{R}^n\,,$$

$\nu_1,\ \nu_2 \in L^n(\Omega)$, $\nu_1,\ \nu_2 \ge 0$.

Then one has

Theorem 4.2. *Suppose that all the above conditions are fulfilled. If $u \in W^{2,n}(\Omega)$ and $u - \varphi \in W_0^{1,n}(\Omega)$ is a strong solution of (4.7), then*

$$\|u\|_{W^{2,n}(\Omega)} \le C$$

where the constant C is independent of u.

Corollary 4.3 (existence). *Under assumptions of Theorem 4.2 there exists a strong solution u, ($u \in W^{2,n}(\Omega)$, $u - \varphi \in W_0^{1,n}(\Omega)$) of the problem (4.7).*

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