ON SYSTEMS OF TELEGRAPH EQUATIONS

Juha Berkovits and Vesa Mustonen

1. Introduction

In this note we shall consider the existence of 2π-periodic solutions for systems of telegraph equations of the form

$$(1.1)\qquad \begin{cases} u_{tt} - u_{xx} - \beta u_t - g(t,x,u) = h(t,x), & (t,x) \in \Omega \\ u(t,0) = u(t,\pi) = 0, & t \in]0,2\pi[, \end{cases}$$

where h is a given function in $L^2(\Omega;\mathbb{R}^n)$ with $\Omega =]0,2\pi[\times]0,\pi[$ and n is a positive integer, the function $g(t,x,s)$ from $\Omega \times \mathbb{R}^n$ to $\mathbb{R}^n$ is 2π-periodic in t, measurable in (t,x) for each $s \in \mathbb{R}^n$, and continuous in s for almost all $(t,x) \in \Omega$ and β is a real diagonal $n \times n$-matrix $\operatorname{diag}(\beta_1, \beta_2, \dots, \beta_n)$. We shall assume that g satisfies the usual growth condition

$$(1.2)\qquad |g(t,x,s)| \le c_0|s| + k_0(t,x)$$

for all $s \in \mathbb{R}^n$ and a.a. $(t,x) \in \Omega$ with some constant $c_0 > 0$ and $k_0 \in L^2(\Omega)$. We are interested in conditions on g such that a solution $u \in L^2(\Omega;\mathbb{R}^n)$ exists for any $h \in L^2(\Omega;\mathbb{R}^n)$ (nonresonance) and in conditions on g and h such that a solution exists for a given h (resonance). In fact, we shall study the existence of *weak solutions* and therefore it is relevant to transform the system (1.1) into the operator equation

$$(1.3)\qquad \mathcal{L}u - \mathcal{N}(u) = h \qquad u \in D(\mathcal{L})$$

in $\mathcal{H} = [L^2(\Omega)]^n$, where $\mathcal{N}$ is the Nemytskii operator generated by g and $\mathcal{L}\colon D(\mathcal{L}) \subset \mathcal{H} \to \mathcal{H}$ is the abstract realization of the telegraph operator $\partial_{tt} - \partial_{xx} - \beta\partial_t$ with periodic Dirichlet boundary conditions in $\mathcal{H}$. It is not hard to see that the linear operator $\mathcal{L}$ is densely defined, closed, and normal. Moreover, if $\det\beta = \beta_1\beta_2\cdots\beta_n \ne 0$, then $\mathcal{L}$ is invertible but otherwise $\operatorname{Ker}\mathcal{L}$ is infinite dimensional. It is essential that the (partial) inverse of $\mathcal{L}$ is compact.

The problem (1.3) has been studied mainly in the case of gradient type nonlinearities $\mathcal{N}$ with no damping, i.e., with $\beta = 0$. We refer the reader to the papers by Brézis and Nirenberg [BN], Bates and Castro [BC], Amann [Am] and Mawhin [Ma] and to the recent contribution by Fonda and Mawhin [FM], where also a survey of the recent results and relevant references can be found. In this note we shall use a different approach based on the degree theory introduced in our earlier paper [BM1]. In [BM3] we treated the case of nonresonance for systems of wave equations without damping terms. The goal of this note is to deal with more general systems with $\beta \neq 0$ and also to study the resonance case.

Our paper is organized as follows. In Section 2 we first recall the concept of complexification of a real Hilbert space and a linear operator. Essential to our study is the structure of the complex spectrum of normal operator. We shall introduce some classes of nonlinear mappings of monotone type and give a useful continuation theorem based on topological degree argument. This abstract result will be used frequently in the sequel. Section 3 is devoted to the linear case, that is, a system of equations with linear coupling. We obtain some injectivity results and estimates of the norm of inverse operator. In Section 4 we consider the nonresonance for systems of normal operators with linear coupling and with nonlinear perturbations. The abstract result is then applied to a system of telegraph and wave equations with a nonlinearity of special type. In our last section the resonance for systems is briefly studied. We derive abstract results which are then applied to the model problem (1.1).

2. Prerequisites

Let H be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\|\cdot\|$. We recall the standard procedure of complexification. Indeed, the complexification of H, $H_{\mathbb{C}} = H + iH$, has the usual linear structure and inner product induced by H, that is,

$$\langle u + iv, x + iy \rangle_{\mathbb{C}} = \langle u, x \rangle + \langle v, y \rangle + i(\langle x, v \rangle - \langle u, y \rangle)$$

for all $u + iv, x + iy \in H_{\mathbb{C}}$. Let $L\colon D(L) \subset H \to H$ be a densely defined closed, normal linear operator with $\operatorname{Im} L = (\operatorname{Ker} L)^{\perp}$. The inverse K of the restriction of L to $\operatorname{Im} L \cap D(L)$ is a bounded linear operator on $\operatorname{Im} L$. We shall assume that K is compact. Note that $\operatorname{Ker} L$ may be infinite dimensional. We are interested in the case when L is not self adjoint and therefore we include the complex spectrum of L into consideration (cf. [BN], [He]). Indeed, we define the complex linear operator $L_{\mathbb{C}}\colon D(L_{\mathbb{C}}) \subset H_{\mathbb{C}} \to H_{\mathbb{C}}$ by setting

$D(L_{\mathbb{C}}) = D(L) + iD(L)$ and $L_{\mathbb{C}}(u+iv) = Lu + iLv$ for all $u+iv \in D(L_{\mathbb{C}})$. It is clear that $\operatorname{Im} L_{\mathbb{C}} = (\operatorname{Ker} L_{\mathbb{C}})^{\perp}$, $L_{\mathbb{C}}$ is normal and its partial inverse $K_{\mathbb{C}}$ is compact. Consequently, $L_{\mathbb{C}}$ has a pure point spectrum $\sigma_{\mathbb{C}}(L) = \{\mu_j\}_{j\in\mathbb{Z}}$ with corresponding orthonormal basis $\{\phi_j\}_{j\in\mathbb{Z}}$ of $H_{\mathbb{C}}$ such that

$$L_{\mathbb{C}}\phi_j = \mu_j \phi_j \qquad \text{for all } j \in \mathbb{Z}.$$

Note that for any complex eigenvalue $\mu = \alpha + i\beta$ also its complex conjugate $\overline{\mu} = \alpha - i\beta$ is an eigenvalue. It is not hard to see that the spectrum of $L_{\mathbb{C}}$ can be rewritten in the form

$$\sigma_{\mathbb{C}}(L) = \{\mu_j \colon \mu_j = \alpha_j \in \mathbb{R}\} \cup \{\mu_j^{\pm} \colon \mu_j^{\pm} = \alpha_j \pm i\beta_j, \beta_j > 0\},$$

and there exists an orthonormal basis $\{\psi_j\} \cup \{\omega_j\}$ of the original space H such that

$$\{\phi_j\} = \{\psi_j \colon \mu_j \in \mathbb{R}\} \cup \{\frac{1}{\sqrt{2}}(\psi_j \pm i\omega_j) \colon \mu_j^{\pm} \notin \mathbb{R}\}.$$

For each $z = u + iv \in D(L_{\mathbb{C}})$ we have the spectral representation

$$L_{\mathbb{C}} z = \sum_{j\in\mathbb{Z}} \mu_j \langle z, \phi_j\rangle_{\mathbb{C}} \phi_j.$$

Note that $L_{\mathbb{C}} u = Lu$ for all $u \in D(L)$ and hence

$$\langle Lu, u\rangle = \sum_{j\in\mathbb{Z}} \operatorname{Re} \mu_j |\langle u, \phi_j\rangle_{\mathbb{C}}|^2.$$

Using the real basis we can also write

$$Lu = \sum_{j\in\mathbb{Z}} \alpha_j \langle u, \psi_j\rangle \psi_j + \sum_{\mu_j^{\pm}\notin\mathbb{R}} \{\alpha_j \langle u, \omega_j\rangle \omega_j + \beta_j[\langle u, \omega_j\rangle \psi_j - \langle u, \psi_j\rangle \omega_j]\},$$

where one can easily see the representations of the symmetric and antisymmetric parts of the operator L.

Let $\mathcal{H} = H^n$, $n \geq 2$, and $\mathcal{L} \colon D(\mathcal{L}) \subset \mathcal{H} \to \mathcal{H}$ be a diagonal operator defined by

$$\mathcal{L}u = (L_1 u_1, L_2 u_2, \ldots, L_n u_n), \quad u = (u_1, u_2, \ldots, u_n) \in D(\mathcal{L}),$$

where the linear operators L_k, $k \in \{1, 2, \ldots, n\}$ satisfy the properties imposed on the operator L above. Note that the complexification $\mathcal{H}_{\mathbb{C}} := \mathcal{H} + i\mathcal{H} = (H_{\mathbb{C}})^n$. Obviously $\mathcal{L}$ and $\mathcal{L}_{\mathbb{C}}$ inherit the properties of the component operators. We shall use the notations $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ for the inner product and norm in a real Hilbert space and the subscript '$\mathbb{C}$' whenever the space is complex. For simplicity we shall frequently use the same symbol for an operator and its complexification.

Let $\mathcal{N} \colon \mathcal{H} \to \mathcal{H}$ be a possibly nonlinear map. We assume that $\mathcal{N}$ is bounded in the sense that it takes any bounded set into a bounded set, and demicontinuous, i.e., $u_k \to u$ (norm convergence) implies $\mathcal{N}(u_k) \rightharpoonup \mathcal{N}(u)$ (weak convergence). We recall that $\mathcal{N}$ is *monotone* if

$$\langle \mathcal{N}(u) - \mathcal{N}(v), u - v \rangle \geq 0 \quad \text{for all } u, v \in \mathcal{H}.$$

If there exists a constant $\mu > 0$ such that for all $u, v \in \mathcal{H}$

$$\langle \mathcal{N}(u) - \mathcal{N}(v), u - v \rangle \geq \mu \|u - v\|^2,$$

we say that $\mathcal{N}$ is *strongly monotone*. Such mappings belong to a wider *class* (S_+), defined by the property: $u_k \rightharpoonup u$ and $\limsup \langle \mathcal{N}(u_k), u_k - u \rangle \leq 0$ imply that $u_k \to u$. Note that any demicontinuous monotone map is *pseudomonotone*, i.e., the conditions $u_k \rightharpoonup u$, $\limsup \langle \mathcal{N}(u_k), u_k - u \rangle \leq 0$ imply $\mathcal{N}(u_k) \rightharpoonup \mathcal{N}(u)$ and $\langle \mathcal{N}(u_k), u_k - u \rangle \to 0$. In our previous paper [BM1] we introduced an extension of the classical Leray-Schauder degree suitable for our present purposes. Indeed, if $\mathcal{N}$ is demicontinuous, bounded, and of class (S_+), $G \subset \mathcal{H}$ is an open bounded set, and $h \notin (\mathcal{L} - \mathcal{N})(\partial G \cap D(\mathcal{L}))$, then the degree $\deg(\mathcal{L} - \mathcal{N}, G, h)$ is well defined and satisfies the properties of the classical topological degree. Let $\mathcal{P} \colon \mathcal{H} \to \operatorname{Ker} \mathcal{L}$ be an orthogonal projection. The injection $\mathcal{L} - \mathcal{P}$ is important because it is the *normalizing map*, that is,

$$\deg(\mathcal{L} - \mathcal{P}, G, h) = \begin{cases} 1, & \text{if } h \in (\mathcal{L} - \mathcal{P})(G \cap D(\mathcal{L})) \\ 0, & \text{if } h \notin (\mathcal{L} - \mathcal{P})(G \cap D(\mathcal{L}). \end{cases}$$

By a *reference map* we mean in the sequel any injection $\mathcal{L} - \mathcal{N}$ for which $\deg(\mathcal{L} - \mathcal{N}, G, h) \neq 0$ whenever $h \in (\mathcal{L} - \mathcal{N})(G \cap D(\mathcal{L}))$. For a given nonlinear problem it is crucial to find an appropriate reference map which reflects the structure of the original equation and enables to apply the continuation principle. We shall deal with reference maps of the form $\mathcal{L} - \mathcal{A}$, where $\mathcal{A}$ is linear. The following existence theorem is a direct consequence of the homotopy invariance of topological degree (see [BM1]).

Continuation Theorem. *Let G be an open bounded convex subset in $\mathcal{H}$ with $0 \in G$, $\mathcal{N}\colon \mathcal{H} \to \mathcal{H}$ a bounded demicontinuous pseudomonotone map, $h \in \mathcal{H}$, and let $\mathcal{A}\colon \mathcal{H} \to \mathcal{H}$ be a linear operator such that $\mathcal{L}-\mathcal{A}$ is a reference map. If*

$$\mathcal{L}u - t(\mathcal{N}(u) + h) - (1-t)\mathcal{A}u \neq 0$$

for all $t \in [0,1]$ and $u \in \partial G \cap D(\mathcal{L})$, then the equation

$$\mathcal{L}u - \mathcal{N}(u) = h, \qquad u \in G \cap D(\mathcal{L}),$$

admits at least one solution.

Note that the Continuation Theorem remains valid also in the following cases. First, the set G need not be convex if the map $\mathcal{N}$ is of class (S_+). Secondly, if $\operatorname{Ker}\mathcal{L}$ is finite dimensional, then the classical Leray-Schauder degree can be applied, and we can remove the convexity of G and also any monotonicity condition on $\mathcal{N}$. Naturally, this observation holds for the reference map, too. Hence we shall in the sequel assume that $\mathcal{A}$ is strongly monotone in the case $\dim \operatorname{Ker}\mathcal{L} = \infty$.

3. Linear case

Let H be a real separable Hilbert space and $\mathcal{H} = H^n$. We consider a real $n\times n$-matrix $A = (a_{lk})$ and the *constant multiplication operator* $\mathcal{A}\colon \mathcal{H} \to \mathcal{H}$ induced by A, that is, for any $u = (u_1, u_2, \dots, u_n) \in \mathcal{H}$

$$\mathcal{A}u = w = (w_1, w_2, \dots, w_n),$$

with $w_l = \sum_{k=1}^n a_{lk}u_k$, $l = 1, 2, \dots, n$. In the sequel A also denotes the linear operator in $\mathbb{R}^n$ and $\mathbb{C}^n$ corresponding to the matrix (a_{lk}). Clearly $\sigma(\mathcal{A}) = \sigma(A)$, a point spectrum which may be empty if n is even. Similarly $\sigma_{\mathbb{C}}(\mathcal{A}) = \sigma_{\mathbb{C}}(A)$. If the matrix A is strictly positive, it is not hard to prove that

$$\langle \mathcal{A}u, u\rangle \geq \alpha\|u\|^2 \qquad \text{for all } u \in \mathcal{H},$$

where $\alpha = \min\{(Ax,x)_{\mathbb{R}^n} : |x|_{\mathbb{R}^n} = 1\}$ is positive. Let $\mathcal{L}$ be the diagonal operator $\mathcal{L} = \operatorname{diag}(L_1, L_2, \dots, L_n)$, defined on a dense subset $D(\mathcal{L}) = D(L_1) \times D(L_2) \times \cdots \times D(L_n)$ of $\mathcal{H}$. We assume that each L_k is a closed normal linear operator with $\operatorname{Im} L_k = (\operatorname{Ker} L_k)^\perp$ and that K_k, the partial inverse of L_k, is compact. Futhermore we shall impose a strong assumption that each L_k has the same complex eigenbasis $\{\phi_j\}_{j\in\mathbb{Z}}$ with corresponding complex eigenvalues $\{\mu_j^{(k)}\}_{j\in\mathbb{Z}}$. This condition holds in particular for systems of telegraph equations with different damping terms. From now on we always assume that the matrix A is strictly positive whenever $\operatorname{Ker}\mathcal{L}$ is infinite dimensional. This is needed in order to employ the Continuation Theorem. For any $z = (z_1, z_2, \ldots, z_n) \in \mathcal{H}_{\mathbb{C}}$ we denote $\vec{z}_j = \sum_{k=1}^{n} \langle z_k, \phi_j \rangle_{\mathbb{C}} e_k$, where $\{e_k\}$ is the standard basis of $\mathbb{R}^n$. Hence we can write

$$z = \sum_{j\in\mathbb{Z}} \vec{z}_j \phi_j \quad \text{and} \quad \|z\|^2 = \sum_{j\in\mathbb{Z}} |\vec{z}_j|^2_{\mathbb{C}^n}.$$

It is easy to see that for any $z = u + iv \in D(\mathcal{L}) + iD(\mathcal{L})$ we have

$$(\mathcal{L} - \mathcal{A})(z) = \sum_{j\in\mathbb{Z}} [(M_j - A)\vec{z}_j]\phi_j, \tag{3.1}$$

where $M_j = \operatorname{diag}(\mu_j^{(1)}, \mu_j^{(2)}, \ldots, \mu_j^{(n)})$. By the above formula we get

Lemma 3.1. *The operator $\mathcal{L} - \mathcal{A}$ is injective if and only if $\det(M_j - A) \neq 0$ for all $j \in \mathbb{Z}$.*

Note that in the case $L_k = L$ for all $k = 1, 2, \ldots, n$, the injectivity condition in Lemma 3.1 can be written as $\sigma_{\mathbb{C}}(L) \cap \sigma_{\mathbb{C}}(A) = \emptyset$. We also get a simple formula for the norm of $(\mathcal{L} - \mathcal{A})^{-1}$.

Lemma 3.2. *If $\det(M_j - A) \neq 0$ for all $j \in \mathbb{Z}$, then*

$$\|(\mathcal{L} - \mathcal{A})^{-1}\| = \sup_j \|(M_j - A)^{-1}\| < \infty.$$

Proof. The image of $\mathcal{L} - \mathcal{A}$ is closed and hence the inverse operator is bounded. By (3.1) we see that

$$z = (\mathcal{L} - \mathcal{A})^{-1} w = \sum_{j\in\mathbb{Z}} [(M_j - A)^{-1} \vec{w}_j]\phi_j$$

is the unique solution of the equation $\mathcal{L}z - \mathcal{A}z = w$. Hence

$$\|(\mathcal{L} - \mathcal{A})^{-1}\| \leq \sup_j \|(M_j - A)^{-1}\|.$$

On the other hand, if $\vec{x} \in \mathbb{C}^n$ and $|\vec{x}|_{\mathbb{C}^n} = 1$, then clearly $(\mathcal{L} - \mathcal{A})^{-1}(\vec{x}\phi_j) = [(M_j - A)^{-1}\vec{x}]\phi_j$ for any $j \in \mathbb{Z}$ and thus

$$\|(\mathcal{L} - \mathcal{A})^{-1}\| \geq \sup_j \|(M_j - A)^{-1}\|$$

completing the proof. □

In what follows we shall consider two special cases. If $\mathcal{A} = \tau I$, where $\tau > 0$ and $\tau \notin \sigma(\mathcal{L})$, then $\sigma_{\mathbb{C}}(\mathcal{L} - \tau I) = \{\mu_j^{(k)} - \tau \colon j \in \mathbb{Z}, k = 1, 2, \dots, n\}$ and

$$\|(\mathcal{L} - \tau I)^{-1}\| = \frac{1}{\min\limits_{j,k} |\tau - \mu_j^{(k)}|}. \tag{3.2}$$

If $L_1 = L_2 = \dots = L_n = L$ and A is normal, then $\mathcal{L} - \mathcal{A}$ can be diagonalized and $\sigma_{\mathbb{C}}(\mathcal{L} - \mathcal{A}) = \{\mu_j - q_k \colon j \in \mathbb{Z}, k = 1, 2, \dots, n\}$, where the complex eigenvalues of A are denoted by $\{q_k\}$ and $\mu_j^{(k)} \equiv \mu_j$. In the case $\mathcal{L} - \mathcal{A}$ is invertible we have

$$\|(\mathcal{L} - \mathcal{A})^{-1}\| = \frac{1}{\min\limits_{j,k} |q_k - \mu_j|}. \tag{3.3}$$

Now any invertible $\mathcal{L} - \mathcal{A}$ is a reference map. As a direct consequence of the results in [BM1] we get

Lemma 3.3. *Assume that $A = (a_{lk})$ is a real $n \times n-$matrix such that $\mathcal{L} - \mathcal{A}$ is invertible. For any open bounded set G in $\mathcal{H}$ and any $h \in \mathcal{H}$ such that $h \in (\mathcal{L} - \mathcal{A})(G \cap D(\mathcal{L}))$ we have*

$$\deg(\mathcal{L} - \mathcal{A}, G, h) \neq 0.$$

In the study of resonance we shall need further estimates of the type

$$-\frac{1}{\theta_-}\|\mathcal{L}u - \mathcal{A}u\|^2 \leq \langle \mathcal{L}u - \mathcal{A}u, u\rangle \leq \frac{1}{\theta_+}\|\mathcal{L}u - \mathcal{A}u\|^2 \text{ for all } u \in D(\mathcal{L}), \tag{3.4}$$

where $\mathcal{L} - \mathcal{A}$ may not be injective. In the special case $\mathcal{A} = \tau I, \tau \in \mathbb{R}$, and possibly $\tau \in \sigma(\mathcal{L})$, we have in an analogy with [BM2]

$$\theta_+ = \inf_{\operatorname{Re}\mu_j^{(k)} > \tau} \frac{|\mu_j^{(k)} - \tau|^2}{\operatorname{Re}(\mu_j^{(k)} - \tau)}, \qquad \theta_- = \inf_{\operatorname{Re}\mu_j^{(k)} < \tau} \frac{|\mu_j^{(k)} - \tau|^2}{\operatorname{Re}(\tau - \mu_j^{(k)})}. \tag{3.5}$$

If $L_k = L$ for all $k = 1, 2, \dots, n$ and the matrix A is normal, then

$$\theta_+ = \inf_{\operatorname{Re}(\mu_j - q_k) > 0} \frac{|\mu_j - q_k|^2}{\operatorname{Re}(\mu_j - q_k)}, \qquad \theta_- = \inf_{\operatorname{Re}(\mu_j - q_k) < 0} \frac{|\mu_j - q_k|^2}{\operatorname{Re}(\mu_j - q_k)}. \tag{3.6}$$

In [BM2] the values of θ_+ and θ_- are calculated in the case of a single telegraph equation.

4. Nonresonance for systems

Let $\mathcal{L} = \operatorname{diag}(L_1, L_2, \dots, L_n)$ be as defined in section 2 and let $\mathcal{N} \colon \mathcal{H} \to \mathcal{H}$ be a demicontinuous bounded operator. We assume that $\mathcal{N}$ is monotone or, more generally, pseudomonotone. Note that this assumption is not needed if $\operatorname{Ker} \mathcal{L}$ is finite dimensional. By nonresonance we mean that the equation

$$\mathcal{L}u - \mathcal{N}(u) = h, \qquad u \in D(\mathcal{L}), \tag{4.1}$$

admits a solution for any given h in $\mathcal{H}$. From the Continuation Theorem we derive the following abstract nonresonance result.

Theorem 4.1. *Assume that $\mathcal{L} - \mathcal{A}$ is a reference map and $\mathcal{N}$ is pseudomonotone if* $\operatorname{Ker} \mathcal{L}$ *is infinite dimensional. If there exist constants $\eta > 0$ and $C \geq 0$ such that*

$$\eta < \|(\mathcal{L} - \mathcal{A})^{-1}\|^{-1}, \quad \|\mathcal{N}(u) - \mathcal{A}u\| \leq \eta \|u\| + C \quad \textit{for all } u \in \mathcal{H},$$

then the equation (4.1) *admits at least one solution for any $h \in \mathcal{H}$.*

Proof. It is sufficient to show that there exists $R > 0$ such that

$$\mathcal{L}u - t\big(\mathcal{N}(u) + h\big) - (1 - t)\mathcal{A}u \neq 0$$

for all $t \in [0, 1]$ and $u \in D(\mathcal{L})$ with $\|u\| = R$. Indeed, if

$$\mathcal{L}u - \mathcal{A}u = t[\mathcal{N}(u) - \mathcal{A}u + h],$$

then the fact that $\mathcal{L} - \mathcal{A}$ is a reference map together with the other assumptions implies

$$\|u\| \leq \|(\mathcal{L} - \mathcal{A})^{-1}\| \big[\eta \|u\| + C + \|h\|\big].$$

Since $\|(\mathcal{L} - \mathcal{A})^{-1}\| \eta < 1$ it suffices to take any

$$R > \big(1 - \eta \|(\mathcal{L} - \mathcal{A})^{-1}\|\big)^{-1} \|(\mathcal{L} - \mathcal{A})^{-1}\| (C + \|h\|)$$

to get the desired result. □

Instead of (4.1) it is often more natural and more convenient to consider the equation

$$\mathcal{L}u - \mathcal{A}u - \widetilde{\mathcal{N}}(u) = h, \qquad u \in D(\mathcal{L}), \tag{4.2}$$

where $\widetilde{\mathcal{N}} = \mathcal{N} - \mathcal{A}$ and $\widetilde{\mathcal{N}}$ satisfies the growth condition $\|\widetilde{\mathcal{N}}(u)\| \leq \eta\|u\| + C$ in the previous theorem.

Example. In order to illustrate the application of Theorem 4.1 we shall consider a system of a telegraph equation and a wave equation with linear coupling

(4.3)

$$\begin{cases} \frac{\partial^2 u_1}{\partial t^2} - \frac{\partial^2 u_1}{\partial x^2} - \beta_1 \frac{\partial u_1}{\partial t} - a_{11}u_1 - a_{12}u_2 - q_1(t,x,u_1) = h_1(t,x) & \text{in } \Omega \\ \frac{\partial^2 u_2}{\partial t^2} - \frac{\partial^2 u_2}{\partial x^2} - a_{21}u_1 - a_{22}u_2 - q_2(t,x,u_2) \qquad\qquad = h_2(t,x) & \text{in } \Omega \\ u_i(t,0) = u_i(t,\pi) = 0, \qquad t \in]0,2\pi[, \quad i = 1,2 \\ u_i(\cdot,x) \text{ is } 2\pi\text{-periodic}, \qquad x \in]0,\pi[, \quad i = 1,2, \end{cases}$$

where $\Omega =]0,2\pi[\times]0,\pi[$. We assume that the coupling matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is strictly positive, i.e., $a_{11} > 0$, $a_{22} > 0$, $|a_{12} + a_{21}| < 2\sqrt{a_{11}a_{22}}$, and that $q_1, q_2 \colon \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying the usual growth condition of the form (1.2) and are 2π-periodic in t.

Denoting $H = L^2(\Omega)$ and $\phi_{jk}(t,x) = \frac{1}{\pi}\sin(jx)\exp(ikt)$, $(t,x) \in \Omega$, $j \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, the set $\{\phi_{jk}\}$ forms an orthonormal basis in $H_{\mathbb{C}}$. Observe the natural indexing used here. The telegraph operator $L_1 = \partial_{tt} - \partial_{xx} - \beta_1\partial_t$ of (4.3) has in $H_{\mathbb{C}}$ the representation

$$L_1 v = \sum_{j,k} (l_{jk} - i\beta_1 k)\langle v, \phi_{jk}\rangle_{\mathbb{C}} \phi_{jk}$$

for all $v \in D(L_1)$ with $l_{jk} = j^2 - k^2$ and

$$D(L_1) = \Big\{ v \in H_{\mathbb{C}} \colon \sum_{j,k} |l_{jk} - i\beta_1 k|^2 |\langle v, \phi_{jk}\rangle_{\mathbb{C}}|^2 < \infty \Big\}.$$

Obviously, $\operatorname{Ker} L_1 = \{0\}$ and $\sigma_{\mathbb{C}}(L_1) = \{\mu_{jk}^{(1)} = l_{jk} - i\beta_1 k \colon j \in \mathbb{Z}_+, k \in \mathbb{Z}\}$. Note that $\sigma(L_1) = \sigma_{\mathbb{C}}(L_1) \cap \mathbb{R} = \{\mu_{j0}^{(1)} = j^2 \colon j \in \mathbb{Z}_+\}$. For the wave operator $L_2 = \partial_{tt} - \partial_{xx}$ we respectively have an analogous representation, but now $\operatorname{Ker} L_2$ is infinite dimensional and $\sigma_{\mathbb{C}}(L_2) = \sigma(L_2) = \{l_{jk} = j^2 - k^2 \colon j \in \mathbb{Z}_+, k \in \mathbb{Z}\}$.

The diagonal operator $\mathcal{L} = \operatorname{diag}(L_1, L_2)$ is defined on $D(\mathcal{L}) = D(L_1) \times D(L_2) \subset \mathcal{H}_{\mathbb{C}} = H^2_{\mathbb{C}}$. Since L_1 is normal and L_2 is self adjoint both having compact (partial) inverses, also $\mathcal{L}$ is normal with compact partial inverse. The system (4.3) corresponds to the equation

$$\mathcal{L}u - \mathcal{A}u - \widetilde{\mathcal{N}}(u) = h, \qquad u \in D(\mathcal{L}),$$

where $h = (h_1, h_2) \in \mathcal{H}$, $\widetilde{\mathcal{N}}(u_1, u_2) = (q_1(\cdot,\cdot,u_1), q_2(\cdot,\cdot,u_2))$ and $\mathcal{A}$ is the constant multiplication operator induced by the matrix A. In view of Lemma 3.1 the linear part $\mathcal{L} - \mathcal{A}$ is injective if and only if

$$\det(M_{jk} - A) \equiv \det \begin{pmatrix} \lambda_{jk} + i\beta_1 k - a_{11} & -a_{12} \\ -a_{21} & \lambda_{jk} - a_{22} \end{pmatrix} \neq 0$$

for all $j \in \mathbb{Z}_+$, $k \in \mathbb{Z}$. Denote $A_{jk} = M_{jk} - A$, $D_{jk} = \det A_{jk}$ and let $\|\cdot\|_E$ stand for the Frobenius norm of a matrix, that is, the Euclidean norm in $\mathbb{R}^4$. Then it is not hard to see that

$$\|A_{jk}^{-1}\| = \frac{1}{2|D_{jk}|^2}(\|A_{jk}\|_E^2 + \sqrt{\|A_{jk}\|_E^4 - 4|D_{jk}|^2}).$$

Note that for any matrix B we have $\|B\| \leq \|B\|_E$, and the Frobenius norm is very easy to calculate. In the two dimensional case as above also $\|B\| \geq \frac{1}{\sqrt{2}}\|B\|_E$. We have

$$D_{jk} = \det(M_{jk} - A) = \det(\lambda_{jk} I - A) + i\beta_1 k(\lambda_{jk} - a_{22}).$$

Hence, if $a_{12}a_{21} \neq 0$, the operator $\mathcal{L} - \mathcal{A}$ is injective if and only if $j^2 \notin \sigma(A)$ for all $j \in \mathbb{Z}_+$, that is, $\sigma(L_1) \cap \sigma(A) = \emptyset$. Similarly, if $a_{12}a_{21} = 0$, then $\mathcal{L} - \mathcal{A}$ is injective if and only if $\sigma(L_2) \cap \sigma(A) = \emptyset$.

Before introducing further conditions for the nonlinearity $\widetilde{\mathcal{N}}$ required in Theorem 4.1 we need an estimate of the norm $\|(\mathcal{L} - \mathcal{A})^{-1}\| = \sup_{jk} \|A_{jk}^{-1}\|$. It is not hard to see that

$$\|A_{jk}^{-1}\|^2 \leq \frac{2d_{jk} + \beta_1^2 k^2 + R^2}{d_{jk}^2 + \beta_1^2 k^2 (l_{jk} - a_{22})^2},$$

where $d_{jk} = \det(l_{jk} I - A)$ and $R^2 = (a_{11} - a_{22})^2 + (a_{12} + a_{21})^2$. Denoting

$$\kappa^2 = \sup_{j,k} \frac{2d_{jk} + \beta_1^2 k^2 + R^2}{d_{jk}^2 + \beta_1^2 k^2 (l_{jk} - a_{22})^2}$$

we obtain an upper bound for $\|(\mathcal{L}-\mathcal{A})^{-1}\|$ which is easy to calculate in each specific case when $\mathcal{L}-\mathcal{A}$ is injective. Since $\operatorname{Ker}\mathcal{L}$ is infinite dimensional, we shall require that the map $\mathcal{A}+\widetilde{\mathcal{N}}$ is monotone. Indeed, this will be true if we assume that

$$\frac{q_i(t,x,s)-q_i(t,x,\widehat{s})}{s-\widehat{s}} \geq -\alpha, \qquad i=1,2$$

for a.a. $(t,x)\in\Omega$ and all real $s\neq\widehat{s}$, where $\alpha=\min\{(Ax,x)_{\mathbb{R}^2}\colon |x|=1\}$. Therefore, assuming further that there exists $\eta>0$ such that

$$\limsup_{|s|\to\infty}\left|\frac{q_i(t,x,s)}{s}\right| \leq \eta < \frac{1}{\kappa}, \qquad i=1,2,$$

uniformly in $(t,x)\in\Omega$, we can conclude by Theorem 4.1 that the system (4.3) admits a weak solution $u=(u_1,u_2)\in D(\mathcal{L})$ for any given $h=(h_1,h_2)\in[L^2(\Omega)]^2$.

5. Resonance for systems

Let $\mathcal{L}=\operatorname{diag}(L_1,L_2,\dots,L_n)$ be a linear operator in $\mathcal{H}=H^n$ satisfying the properties given in Section 2. Let $\mathcal{N}\colon\mathcal{H}\to\mathcal{H}$ be a bounded demicontinuous map. Assume that there exist a constant $\gamma>0$ and a real valued function $C(\cdot)$ such that

$$\langle\mathcal{N}(u)-\mathcal{N}(z),u-z\rangle \geq \frac{1}{\gamma}\|\mathcal{N}(u)\|^2 - C(z) \quad \text{for all } u,z\in\mathcal{H}. \tag{5.1}$$

We shall consider the solvability of the abstract operator equation

$$\mathcal{L}u - lu - \mathcal{N}(u) = h, \qquad u\in D(\mathcal{L}), \tag{5.2}$$

where $l\in\sigma(\mathcal{L})$ and $h\in\mathcal{H}$. We assume that $lI+\mathcal{N}$ is monotone if $\operatorname{Ker}\mathcal{L}$ is infinite dimensional. We denote by $\operatorname{conv}S$ the set of convex combinations of elements of any set S and by $\mathcal{R}(\mathcal{N})$ the range of $\mathcal{N}$. In the sequel we shall use the constant θ_+ given in (3.5). We start with

Theorem 5.1. *Assume that the condition* (5.1) *holds with* $0<\gamma<\theta_+$ *and* $l\geq 0$. *Then*

$$\overline{\operatorname{Im}(\mathcal{L}-lI)-\operatorname{conv}\mathcal{R}(\mathcal{N})} = \overline{\mathcal{R}(\mathcal{L}-lI-\mathcal{N})}$$

if $l\in\sigma(\mathcal{L})$ *and* $\mathcal{R}(\mathcal{L}-lI-\mathcal{N})=\mathcal{H}$ *if* $l\notin\sigma(\mathcal{L})$.

Proof (cf. [BM2] for a more detailed proof). It is sufficient to show that $\operatorname{Im}\mathcal{L}_l - \operatorname{conv}\mathcal{R}(\mathcal{N})$ is contained in the closure of $\mathcal{R}(\mathcal{L}_l - \mathcal{N})$, where $\mathcal{L}_l = \mathcal{L} - lI$. Each $h \in \operatorname{Im}(\mathcal{L}_l) - \operatorname{conv}\mathcal{R}(\mathcal{N})$ has a representation $h = \mathcal{L}_l v_0 - p$, where $p = \sum_{i=1}^m \alpha_i \mathcal{N}(v_i)$ with $\alpha_i \geq 0$, $\sum_{i=1}^m \alpha_i = 1$. Let $\mathcal{P}_l$ stand for the orthogonal projection to $\operatorname{Ker}\mathcal{L}_l$ and $\mathcal{Q}_l = I - \mathcal{P}_l$. Note that $\mathcal{P}_l = 0$ if $l \notin \sigma(\mathcal{L})$. By [BM1] the mapping $\mathcal{L}_l - \mathcal{P}_l$ is a reference map. If the equation (5.3) now has a solution, the proof is complete. Hence we assume that the equation (5.3) has no solution. In view of the Continuation Theorem this implies the existence of sequences $\{u_j\}$ and $\{t_j\}$ such that $\|u_j\| \to \infty$, $0 < t_j < 1$, and

$$\mathcal{L}_l u_j - (1-t_j)\mathcal{P}_l u_j - t_j \mathcal{N}(u_j) = \mathcal{L}_l v_0 + (1-t_j)\mathcal{P}_l v_0 - t_j p \tag{5.3}$$

for all $j \in \mathbb{Z}_+$. Taking $\mathcal{Q}_l$-projection we get

$$\|\mathcal{L}_l(u_j - v_0)\| \leq t_j\big(\|\mathcal{N}(u_j)\| + \|p\|\big). \tag{5.4}$$

Using the representation of p we have

$$\begin{aligned}\langle \mathcal{L}(u_j - v_0), u_j - v_0\rangle = t_j \sum_{i=1}^m \big\{\alpha_i \langle \mathcal{N}(u_j) - \mathcal{N}(v_i), u_j - v_0\rangle\big\} \\ + (1-t_j)\|\mathcal{P}(u_j - v_0)\|^2\end{aligned}$$

and by (5.1), (5.4) and (3.5) together with the fact that $0 < t_j < t_j^2 < 1$ we get further

$$\frac{t_j^2}{\theta_+}\big(\|\mathcal{N}(u_j)\| + \|p\|\big)^2 \geq t_j^2\Big(\frac{1}{\gamma}\|\mathcal{N}(u_j)\|^2 - C(h)\Big) + (1-t_j)\|\mathcal{P}_l(u_j - v_0)\|^2,$$

where $C(h)$ is a constant depending on h only. Since $0 < \gamma < \theta_+$ we conclude that the sequence $\{t_j\mathcal{N}(u_j)\}$ remains bounded which also implies that $\{\mathcal{L}_l u_j\}$, $\{\mathcal{Q}_l u_j\}$ and $\{(1-t_j)\|\mathcal{P}_l u_j\|^2\}$ remain bounded. Note that in the case $l \notin \sigma(\mathcal{L})$ we have $\mathcal{Q}_l u_j = u_j$, which gives a contradiction showing that $h \in \mathcal{R}(\mathcal{L}_l - \mathcal{N})$. Moreover, in this case $\operatorname{Im}\mathcal{L}_l = \mathcal{H}$, proving our second assertion $\mathcal{R}(\mathcal{L}_l - \mathcal{N}) = \mathcal{H}$. If $l \in \sigma(\mathcal{L})$, then $\|\mathcal{P}_l u_j\| \to \infty$ and hence $(1-t_j)\mathcal{P}_l u_j \to 0$ and $t_j \to 1$. Thus we get from (5.3) that

$$\mathcal{L}_l u_j - \mathcal{N}(u_j) \to \mathcal{L}_l v_0 - p = h$$

which shows that $h \in \overline{\mathcal{R}(\mathcal{L}_l - \mathcal{N})}$. □

Note that in the above proof we obtained a sequence $\{u_j\}$ such that $\|u_j\| \to \infty$ but $\{\mathcal{N}(u_j)\}$ is bounded as a consequence of the assumption that the equation (5.2) has no solution. Excluding this possibility we obtain the following surjectivity result.

Corollary 1. *Assume that $\mathcal{N}$ is monotone,* (5.1) *holds with* $0 < \gamma < \theta_+$ *and* $l \in \sigma(\mathcal{L})$ *with* $l \geq 0$. *If* $\|\mathcal{N}(u)\| \to \infty$ *whenever* $\|u\| \to \infty$, *then the equation* (5.2) *admits a solution for any* $h \in \mathcal{H}$.

Proof. Let $h \in \mathcal{H}$ be arbitrary. Since $\mathcal{R}(\mathcal{N}) = \mathcal{H}$, we have $h \in \operatorname{Im} \mathcal{L}_l - \mathcal{R}(\mathcal{N})$ and the assertion follows from the proof of Theorem 5.1. □

To tackle the equation (5.2) in the resonance case we recall some notation and useful facts from [BN]. The support function of $\mathcal{R}(\mathcal{N})$ is defined by

$$I^*_{\mathcal{R}(\mathcal{N})}(v) = \sup_{u \in \mathcal{H}} \langle \mathcal{N}(u), v \rangle, \qquad v \in \mathcal{H},$$

and the recession function for $\mathcal{N}$ by

$$J_{\mathcal{N}}(v) = \liminf_{t \to \infty, u \to v} \langle \mathcal{N}(tu), u \rangle, \qquad v \in \mathcal{H}.$$

Clearly we have

$$J_{\mathcal{N}}(v) \leq I^*_{\mathcal{R}(\mathcal{N})}(v) \qquad \text{for all } v \in \mathcal{H}.$$

The following result is a direct consequence of Propositions II.5 and II.6 of [BN] and the fact that $\operatorname{Im} \mathcal{L}_l = (\operatorname{Ker} \mathcal{L}_l)^{\perp}$.

Proposition A. *Let* $S \colon \mathcal{H} \to \mathcal{H}$ *be any nonlinear map and* $h \in \mathcal{H}$.

(a) *If* $l \in \sigma(\mathcal{L}), l > 0$, *then the following conditions are equivalent:*

$$I^*_{\mathcal{R}(S)}(v) > -\langle h, v \rangle \quad \text{for all } v \in \operatorname{Ker} \mathcal{L}_l, v \neq 0; \tag{5.5}$$

$$h \in \operatorname{int}(\operatorname{Im} \mathcal{L}_l - \operatorname{conv} \mathcal{R}(S)). \tag{5.6}$$

(b) *If* $l = 0$, *then the following conditions are equivalent:*

$$I^*_{\mathcal{R}(S)}(v) \geq -\langle h, v \rangle \quad \text{for all } v \in \operatorname{Ker} \mathcal{L}; \tag{5.7}$$

$$h \in \overline{\operatorname{Im} \mathcal{L} - \operatorname{conv} \mathcal{R}(S)}. \tag{5.8}$$

The difference between the cases $l > 0$, $l = 0$ above is due to the fact that $\operatorname{Ker} \mathcal{L}_l$ is always finite-dimensional if $l \neq 0$. From Theorem 5.1 and Proposition A we derive

Theorem 5.2. *Assume that $l \in \sigma(\mathcal{L})$, $l \geq 0$, and the condition* (5.1) *holds with $0 < \gamma < \theta_+$.*

(a) *If $l > 0$ and $J_{\mathcal{N}}(v) > -\langle h, v\rangle$ for all $v \in \operatorname{Ker} \mathcal{L}_l$, $v \neq 0$, then the equation* (5.2) *has at least one solution.*

(b) *If $l = 0$ and $J_{\mathcal{N}}(v) \geq -\langle h, v\rangle$ for all $v \in \operatorname{Ker} \mathcal{L}$, then the equation* (5.2) *is almost solvable in the sense that $h \in \overline{\mathcal{R}(\mathcal{L} - \mathcal{N})}$.*

Proof. The case (b) follows immediately from Theorem 5.1 and Proposition A. To prove (a) we observe first by Proposition A that $h \in \operatorname{Im} \mathcal{L}_l - \operatorname{conv} \mathcal{R}(\mathcal{N})$. Hence we can argue exactly in the same way as in the proof of Theorem 5.1. Thus we assume that the equation (5.2) does not have a solution. But this implies the existence of a sequence $\{u_j\} \subset D(\mathcal{L})$ such that

$$\mathcal{L}_l u_j - \mathcal{N}(u_j) \to h. \tag{5.9}$$

By the proof of Theorem 5.1 we also know that $\{L_l(u_j)\}$, $\{\mathcal{N}(u_j)\}$ and $\{\mathcal{Q}_l u_j\}$ remain bounded while $\|\mathcal{P}_l u_j\| \to \infty$. Denoting $v_j = \frac{u_j}{\|u_j\|}$ and $s_j = \|u_j\|$ we have $v_j \rightharpoonup v$ and $\mathcal{L}_l v_j \rightharpoonup w \in \operatorname{Im} \mathcal{L}_l$ for a subsequence. Since $\mathcal{Q}_l v_j \to 0$ and $\operatorname{Ker} \mathcal{L}_l$ is finite-dimensional, we have $v_j \to v = \mathcal{P}_l v$. Therefore $\langle \mathcal{L}_l v_j, v_j\rangle \to \langle w, v\rangle = 0$ and by (5.9) $\langle \mathcal{N}(u_j), v_j\rangle \to \langle -h, v\rangle$. Hence we get

$$-\langle h, v\rangle = \lim_{j\to\infty} \langle \mathcal{N}(s_j v_j), v_j\rangle \geq J_{\mathcal{N}}(v),$$

giving a contradiction. □

Note that the results presented in this section so far are purely abstract. In particular, they are independent of the fact that we consider systems. For this reason we assume that $\mathcal{H} = [L^2(\Omega)]^n$ and $\mathcal{L} = \operatorname{diag}(L_1, L_2, \ldots, L_n)$ is the telegraph operator, i.e., $\mathcal{L}u = u_{tt} - u_{xx} - \beta u_t$ as given in our model problem (1.1). Let $g\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function satisfying the usual growth condition of the form (1.2). Moreover, we assume that $l \geq 0$ is an eigenvalue of $\mathcal{L}$ and the map $lI + g$ is monotone if $\det \beta = 0$. We notice that in the one-dimensional case the condition (5.1) follows from the growth and monotonicity conditions (see [BN]) and the recession function can be estimated by

$$J_{\mathcal{N}}(v) \geq \int_{v>0} g_+ v + \int_{v<0} g_- v, \qquad v \in L^2(\Omega),$$

where $g_+(t,x) = \liminf_{s\to+\infty} g(t,x,s)$ and $g_-(t,x) = \limsup_{s\to-\infty} g(t,x,s)$. We assume that the function $g\colon \Omega\times\mathbb{R}^n \to \mathbb{R}^n$ satisfies condition

$$\langle g(t,x,s) - g(t,x,\widetilde{s}), s-\widetilde{s}\rangle_{\mathbb{R}^n} \geq \frac{1}{\gamma}|g(t,x,s)|^2 - C(\widetilde{s}) \tag{5.10}$$

for a.a. $(t,x)\in\Omega$ and all $s,\widetilde{s}\in\mathbb{R}^n$, where $\gamma>0$ and $C(\cdot)$ a real-valued function. Now it is easy to see that (5.1) holds. For a discussion on various types of functions satisfying (5.10) we refer to [BN]. In place of g_+ and g_- we consider now the recession function J_g of g defined by

$$J_g(t,x,s) = \liminf_{r\to\infty, z\to s} \langle g(t,x,rz), z\rangle.$$

By (5.10) and Fatou's lemma we obtain the estimate

$$J_{\mathcal{N}}(v) \geq \int_{v\neq 0} J_g(t,x,v) \quad \text{for all } v\in\mathcal{H}.$$

Hence we conclude from Theorem 5.2 the following Landesman-Lazer type result (cf. [LL]).

Corollary 2. *Assume that $l=j^2$ for some $j\in\mathbb{Z}_+$ and that the function g satisfies the condition* (5.10). *If $h\in\mathcal{H}$ and*

$$\int_{v\neq 0} J_g(t,x,v) > -\langle h,v\rangle \quad \textit{for all } v\in \operatorname{Ker}(\mathcal{L}-j^2 I), v\neq 0$$

holds, then the system

$$\begin{cases} u_{tt} - u_{xx} - \beta u_t - j^2 u - g(t,x,u) = h(t,x), & (t,x)\in\Omega \\ u(t,0) = u(t,\pi) = 0, \quad t\in[0,2\pi] \\ u(\cdot,x) \quad \textit{is } 2\pi\textit{-periodic in } t \end{cases}$$

admits at least one weak solution.

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ADDRESS OF J. BERKOVITS: Department of Mathematical Sciences
University of Oulu
Linnanmaa, P.O. Box 333
90570 Oulu
Finland