SPECTRAL INVARIANCE FOR PSEUDODIFFERENTIAL OPERATORS ON FUNCTION SPACES

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1. INTRODUCTION

Pseudodifferential operators with symbols in the Hörmander class $S^0_{1,\delta}$ induce bounded operators in Lebesgue and Sobolev spaces if $1 < p < \infty$ and $\delta < 1$ —see [Il], [Na] and [Bo1].

The same is true in the case of Besov-Triebel-Lizorkin spaces $B^s_{p,q}$ and $F^s_{p,q}$ without weights [Pä] and also if additionally admissible weight functions, roughly speaking, at most of polynomial growth, are added [HT2].

Of interest is the dependence of the spectrum of a pseudodifferential operator P on the underlying space. For that, in joint papers with Schrohe [LS2] and Triebel [LT] the relation between invertibility in the operator theoretical sense with respect to one of the underlying function spaces of Besov-Triebel-Lizorkin-type, and invertibility in the subalgebra of pseudodifferential operators $\Psi^0_{1,\delta}$ were discussed.

First it is obvious, that for a pseudodifferential operator which is invertible in $\Psi^0_{1,\delta}$, the inverse is again a pseudodifferential operator belonging to the same class, and consequently, this inverse induces a bounded operator in the operator algebra, too.

On the other hand, the results of Beals [Be] (the L_2-case and the operator class $\Psi^0_{\rho,\delta}$ with $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$) and Ueberberg [Ue] (the L_p-case and the operator class $\Psi^0_{1,\delta}$) imply in both cases: If a pseudodifferential operator belonging to $\Psi^0_{\rho,\delta}$ or $\Psi^0_{1,\delta}$, respectively, is invertible as an operator on $L_2(\mathbb{R}^n)$ or $L_p(\mathbb{R}^n)$, then its inverse is again a pseudodifferential operator of class $\Psi^0_{\rho,\delta}$ or $\Psi^0_{1,\delta}$, respectively. In fact their proofs show, that the same is true for all generalized Sobolev spaces $H^s_2(\mathbb{R}^n)$ and $H^s_p(\mathbb{R}^n)$, s real. This result can be rephrased as

$$\Psi^0_{1,\delta} \cap \mathcal{L}(H^s_p)^{-1} = (\Psi^0_{1,\delta})^{-1}.$$

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In general a subalgebra $\mathcal{A}$ of an algebra $\mathcal{B}$ with the same unit is called a full subalgebra of $\mathcal{B}$ if the following relation holds:

$$\mathcal{A} \cap \mathcal{B}^{-1} = \mathcal{A}^{-1}.$$

Here $\mathcal{A}^{-1}$ and $\mathcal{B}^{-1}$ denote the groups of invertible elements in $\mathcal{A}$ and $\mathcal{B}$, respectively.

In the case when $\mathcal{A}$ is a continuously embedded Fréchet subalgebra of a Banach algebra $\mathcal{B}$, this property was called by Gramsch [Gr] spectral invariance with respect to $\mathcal{B}$. If $\mathcal{B}$ is additionally a C^*-algebra and $\mathcal{A} = \mathcal{A}^*$, then $\mathcal{A}$ is called a Ψ^*-algebra. By establishing the spectral invariance or even the Ψ^*-property one gains access to a variety of results—see for example [Gr], [GKa] and [GUW].

Spectral invariance is not an obvious property. For example, it is not true even for the natural Fréchet algebra of all operators of order 0 on $C^\infty(S^1)$ — see [Gr, 6.2]. In the case of pseudodifferential operators and function spaces, a simple consequence of spectral invariance of a class of pseudodifferential operators with respect to the operator algebras $\mathcal{L}(B^s_{p,q})$ and $\mathcal{L}(F^s_{p,q})$ is that the spectrum of each pseudodifferential operator belonging to this class is independent of the space parameters. Although this property is shared by many classes of pseudodifferential operators and various function spaces [Be], [Co2], [GUW], [LS1], [Sch1], [Sch2], [Ue] it however fails in slightly different situations and is far from being obvious. Examples illustrating this fact are the one-sided and the finite Hilbert transform [W]—see also [Jö, Chapter IV, §13]; some singular integral operators [GK] or the Laplace-Beltrami operator on the hyperbolic $\mathbb{H}^n$ space, considered in a L_p-setting [DST], and in particular [Dav, 5.7]. The p-dependence of the spectrum of uniformly elliptic operators on a Riemannian manifold M with bounded geometry was investigated in [Sh]. In particular, it was proved that this spectrum is p-independent if M is a manifold of subexponential growth—see [Sh, Proposition 2.6].

In [LS2] spectral invariance was shown for the class $\Psi^0_{1,\delta}$ with respect to Besov-Triebel-Lizorkin spaces with parameters $1 < p < \infty$, $0 < q < \infty$, s real. And for a subclass $\Psi_{\delta,sym}$ of $\Psi^0_{1,\delta}$, spectral invariance was proved with respect to Besov-Triebel-Lizorkin spaces with weights and all admissible parameters p, q and s, cf. [LT]. A remarkable consequence of this property is again that for each pseudodifferential operator belonging to $\Psi^0_{1,\delta}$ or $\Psi_{\delta,sym}$, respectively, its resolvent set and its spectrum are independent of the underlying space and coincide with its L_2 resolvent set and spectrum.

2. Notation and preliminary results

We are using the symbol class $S^m_{\rho,\delta}$ of standard pseudodifferential operators on $\mathbb{R}^n$, where $\mathbb{R}^n$ denotes the Euclidean n-space. The notation follows [Kg, Chap. 2].

Let $p(x,\xi)$ be a polynomially bounded complex-valued function defined on $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$. The pseudodifferential operator $P(x,D_x)$ with the symbol $p(x,\xi)$ is defined by

$$P(x,D_x)u(x) = (2\pi)^{-n} \int e^{ix\xi} p(x,\xi)\widehat{u}(\xi)\ \mathrm{d}\xi \quad \text{for } u \in S(\mathbb{R}^n),$$

where $S(\mathbb{R}^n)$ denotes the Schwartz class, and $\widehat{u}(\xi) = \int e^{-iy\xi}u(y)\ \mathrm{d}y$ is the Fourier transform of u.

A function $p(x,\xi)$ belongs to the class $S^m_{\rho,\delta}$ $(0 \le \delta \le \rho \le 1,\ \delta < 1,$ m real), if for any multi-indices α, β, there exists a constant $c_{\alpha,\beta}$ such that

$$|p^{(\alpha)}_{(\beta)}(x,\xi)| \le c_{\alpha,\beta} < \xi >^{m-\rho|\alpha|+\delta|\beta|} \quad \text{for } (x,\xi) \in \mathbb{R}^n_x \times \mathbb{R}^n_\xi,$$

where $p^{(\alpha)}_{(\beta)}(x,\xi) = \partial^\alpha_\xi D^\beta_x p(x,\xi)$, and $< \xi > = (1+|\xi|^2)^{1/2}$.

Here, $\Psi^m_{\rho,\delta}$ is the class of all $P(x,D_x)$ with $p(x,\xi) \in S^m_{\rho,\delta}$. These operators map the Schwartz space $S(\mathbb{R}^n)$ continuously into itself. They can be extended to continuous operators on $S'(\mathbb{R}^n)$, the space of all complex-valued tempered distributions on $\mathbb{R}^n$.

The mapping $p(x,\xi) \mapsto P(x,D_x)$ is a bijection. For $p \in S^m_{\rho,\delta}$ define the semi-norms

$$|p|^{(m)}_{(l,k)} = \max_{|\alpha|\le k, |\beta| \le l} \sup_{x,\xi} \big\{ |p^{(\alpha)}_{(\beta)}(x,\xi)| < \xi >^{-m+\rho|\alpha|-\delta|\beta|} \big\}.$$

This countable family of semi-norms gives the usual Fréchet topology for $S^m_{\rho,\delta}$. It is well known that $\Psi^0_{\rho,\delta}$ and $\bigcup_m \Psi^m_{\rho,\delta}$, $0 \le \delta \le \rho \le 1$, $\delta < 1$, are operator algebras with respect to addition and composition of operators, cf. [Kg].

The space $L_p(\mathbb{R}^n)$, $1 < p < \infty$, is defined as usual. Moreover we denote by $H^s_p(\mathbb{R}^n)$ the generalized Sobolev- or Bessel-potential spaces, s real. We remark that $H^k_p(\mathbb{R}^n) = W^k_p(\mathbb{R}^n)$ if $k \in \mathbb{N}$ and $H^0_p(\mathbb{R}^n) = L_p(\mathbb{R}^n)$. Because all spaces in this paper are defined on $\mathbb{R}^n$, we omit $\mathbb{R}^n$ in the sequel. By $\mathcal{L}(H^s_p)$ we denote the set of all continuous linear operators $T\colon H^s_p \to H^s_p$.

Theorem 1. *Let $P(x,D_x) \in \Psi^0_{1,\delta}$ and $\delta < 1$. Then for all p with $1 < p < \infty$ there exist integers l, k and a constant c, independent of $P(x,D_x)$, such that*

$$\|P(x,D_x)u \mid L_p\| \le c\, |p|^{(0)}_{(l,k)}\, \|u \mid L_p\|$$

for all functions $u \in L_p(\mathbb{R}^n)$.

This was first proved by Illner [Il] in 1975. Later for example Bourdaud [Bo1] and Nagase [Na] considered non-regular symbols and obtained weaker conditions on $p(x, \xi)$. But the result is sharp with respect to the parameter p.

Theorem 2. *Let* $0 \le \delta < 1$ *and* $1 < p < \infty$. *Then* $\Psi^0_{1,\delta}$ *is a spectrally invariant subalgebra of* $\mathcal{L}(L_p)$, *that means*

$$\Psi^0_{1,\delta} \cap \mathcal{L}(L_p)^{-1} = (\Psi^0_{1,\delta})^{-1}.$$

This result was proved by Ueberberg [Ue] in 1988. Because of the lifting property in generalized Sobolev spaces and symbol classes of pseudodifferential operators, we have immediately the following property:

$$\Psi^0_{1,\delta} \cap \mathcal{L}(H^s_p)^{-1} = (\Psi^0_{1,\delta})^{-1}.$$

Proposition 1. *Let* $0 \le \delta < 1$, $1 < p < \infty$ *and* $P(x, D_x) \in \Psi^0_{1,\delta}$. *Then for the spectrum of* P *we have*

$$\sigma(P \mid \Psi^0_{1,\delta}) = \sigma(P \mid \mathcal{L}(H^s_p)) = \sigma(P \mid \mathcal{L}(L_2))$$

for all parameters $1 < p < \infty$ *and all real* s.

Remark 1. In the L_2-case, similar results had been proved before by Calderón and Vaillancourt [CV1], [CV2] in 1971/72 (boundedness of pseudodifferential operators in L_2) and Beals [Be] in 1977/79 (spectral invariance) for the subalgebra $\Psi^0_{\rho,\delta}$, $0 \le \delta \le \rho \le 1$, $\delta < 1$.

In the case $\Psi^0_{1/2,1/2}$ the spectral invariance has been proved independently by Coifmann and Meyer [CM] in 1978.

Concerning $\delta = \rho = 1$, it was shown first by Ching [Ch] in 1972, that $\Psi^0_{1,1}$ contains pseudodifferential operators which are not continuous in L_2. Furthermore, $\Psi^0_{1,1}$ does not form an algebra and is not self-adjoint. But pseudodifferential operators of this class are of special interest, for their important role in the paradifferential calculus of Bony—see [Bon]. Therefore by Bourdaud [Bo2] it was proved that the operators of the subclass $\widetilde{\Psi}^0_{1,1} = \Psi^0_{1,1} \cap (\Psi^0_{1,1})^*$ are L_2-continuous and form an algebra—see also [Hö2]. Some references concerning mapping properties of pseudodifferential operators belonging to the exotic class $\Psi^0_{1,1}$ in general function spaces may be found in [TrII, 6.3.4, Remark 4].

Remark 2. In the case of weighted function spaces of Bessel-potential type, first results are contained in [Sch1] and [Sch2]. With respect to some Besov spaces of variable order, this problem was investigated in [LS1]. Also for boundary value problems spectral invariance was established by Schrohe in [Sch3]. Further results, concerning pseudodifferential operators of class $\Psi^m_{\rho,\delta}$ with $0<\rho<1$ and negative real m, are contained in [AH].

Recently, spectral invariance of pseudodifferential operators in the sense of the Weyl-Hörmander calculus was considered with respect to generalized Sobolev spaces in [BC].

3. Function spaces

Now we introduce the required function spaces and collect some basic properties. Here, we follow closely [TI] and [TII] in the unweighted case, and [HT1], [HT2] in the weighted case. More details, explanations and references to the literature may be found there.

Let L_p with $0<p\le\infty$, be the usual quasi-Banach space with respect to the Lebesgue measure, quasi-normed by $\|\cdot \mid L_p\|$. Let $\varphi\in S$ be such that

$$\operatorname{supp}\varphi\subset\{\xi\in\mathbb{R}^n : |\xi|<2\}\quad\text{ and }\quad \varphi(\xi)=1 \text{ if } |\xi|\le 1,$$

put $\varphi_o=\varphi$ and for each $j\in\mathbb{N}$ let $\varphi_j(\xi)=\varphi(2^{-j}\xi)-\varphi(2^{-j+1}\xi)$. Then, since $1=\sum_{j=0}^{\infty}\varphi_j(\xi)$ for all $\xi\in\mathbb{R}^n$, the φ_j form a dyadic resolution of unity. Given any $f\in S'$, we denote by $\widehat{f}$ and $f^\vee$ its Fourier transform and its inverse Fourier transform, respectively. Then $(\varphi_j\widehat{f})^\vee$ is an analytic function on $\mathbb{R}^n$.

Definition 1. Let $s\in\mathbb{R}$, $0<q\le\infty$ and let $\{\varphi_j\}_{j=0}^{\infty}$ be the above dyadic resolution of unity.

(i) Let $0<p\le\infty$. The space $B^s_{p,q}$ is the collection of all $f\in S'$ such that

$$\|f \mid B^s_{p,q}\| = \Big(\sum_{j=0}^{\infty} 2^{jsq}\|(\varphi_j\widehat{f})^\vee \mid L_p\|^q\Big)^{1/q}$$

is finite (with the usual modification if $q=\infty$).

(ii) Let $0<p<\infty$. The space $F^s_{p,q}$ is the collection of all $f\in S'$ such that

$$\|f \mid F^s_{p,q}\| = \Big\|\Big(\sum_{j=0}^{\infty} 2^{jsq}|(\varphi_j\widehat{f})^\vee(\cdot)|^q\Big)^{1/q} \,\Big|\, L_p\Big\|$$

is finite (with the usual modification if $q=\infty$).

The theory of these spaces has been developed systematically by Triebel in [TrI] and [TrII]. In particular, both $B^s_{p,q}$ and $F^s_{p,q}$ are quasi-Banach spaces, independent of the system $\{\varphi_j(\xi)\}_{j=0}^{\infty}$ in the sense of equivalent quasi-norms. If $p \geq 1$ and $q \geq 1$, then both $B^s_{p,q}$ and $F^s_{p,q}$ are Banach spaces. In this case, their properties may also be found in [Tr].

The two scales cover many well-known classical spaces:

(i) $B^s_{\infty,\infty}$ with $s > 0$ are the Hölder-Zygmund spaces;

(ii) $B^s_{p,q}$ with $s > 0$, $1 < p < \infty$, $1 \leq q \leq \infty$ are the classical Besov spaces;

(iii) $F^s_{p,2}$ with $-\infty < s < \infty$, $1 < p < \infty$ are the Bessel-potential spaces H^s_p including the Sobolev spaces $F^s_{p,2} = W^s_p$ for $1 < p < \infty$ and $s = 0, 1, 2, \dots$ as a sub-case;

(iv) $F^0_{p,2}$ with $0 < p < \infty$ are the (non-homogeneous) Hardy spaces.

We will use the following results on boundedness of pseudodifferential operators in function spaces of Besov-Triebel-Lizorkin type.

Theorem 3. *Let s, m be real, $0 < q \leq \infty$, and $\delta < 1$.*

(i) *Let $0 < p \leq \infty$ and $P(x, D_x) \in \Psi^m_{1,\delta}$. Then there exist integers l, k and a constant c such that*

$$\|P(x, D_x)u \mid B^s_{p,q}\| \leq c\,|p|^{(m)}_{(l,k)}\,\|u \mid B^{s+m}_{p,q}\|$$

for all $u \in B^{s+m}_{p,q}$.

(ii) *Let $0 < p < \infty$ and $P(x, D_x) \in \Psi^m_{1,\delta}$. Then there exist integers l, k and a constant c such that*

$$\|P(x, D_x)u \mid F^s_{p,q}\| \leq c\,|p|^{(m)}_{(l,k)}\,\|u \mid F^{s+m}_{p,q}\|$$

for all $u \in F^{s+m}_{p,q}$.

It was Päivärinta who first showed (i) and (ii) for $0 < p, q < \infty$ [Pä]. In view of the lifting property of the Bessel-potential operator on the scales of spaces $B^s_{p,q}$ and $F^s_{p,q}$, this result can also be deduced from the results in [Ma] and [Ya], which also partly cover more general classes of pseudodifferential operators and function spaces, see also [TrII, 6.2.2].

In what follows a weighted counterpart of the spaces $B^s_{p,q}$ and $F^s_{p,q}$ will be considered. Let $L_p(w(x))$ be the weighted generalization, quasi-normed by

$$\|f \mid L_p(w(\cdot))\| = \|wf \mid L_p\|$$

where $w(x) > 0$ is a weight function on $\mathbb{R}^n$ and $0 < p \leq \infty$.

Definition 2. The class W of admissible weight functions is the collection of all positive C^∞-functions $w(x)$ on $\mathbb{R}^n$ with the following properties:

(i) for any multi-index β there exists a positive constant c_β with

$$|D^\beta w(x)| \leq c_\beta w(x) \quad \text{for all } x \in \mathbb{R}^n,$$

(ii) there exist two constants $c > 0$ and $\alpha > 0$ such that

$$0 < w(x) \leq cw(y) < x - y >^{\alpha} \quad \text{for all } x \in \mathbb{R}^n \text{ and all } y \in \mathbb{R}^n.$$

Definition 3. Let $w(x) \in W$. Let $s \in \mathbb{R}$, $0 < q \leq \infty$ and let $\{\varphi_j\}_{j=0}^{\infty}$ be the above dyadic resolution of unity.

(i) Let $0 < p \leq \infty$. The space $B^s_{p,q}(w(x))$ is the collection of all $f \in S'$ such that

$$\|f \mid B^s_{p,q}(w(\cdot))\| = \Big(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \widehat{f})^{\vee} \mid L_p(w(\cdot))\|^q\Big)^{1/q}$$

is finite (with the usual modification if $q = \infty$).

(ii) Let $0 < p < \infty$. The space $F^s_{p,q}(w(x))$ is the collection of all $f \in S'$ such that

$$\|f \mid F^s_{p,q}(w(\cdot))\| = \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \widehat{f})^{\vee}(\cdot)|^q\Big)^{1/q} \Bigm| L_p(w(\cdot)) \Big\|$$

is finite (with the usual modification if $q = \infty$).

Remark 3. We took over the above definitions from [HT1] and [HT2]. Weighted spaces of the above type even in the more general context of ultra-distributions and for larger classes of weight functions have been treated in [ST, sect. 5.1] (see also the references given there).

Again the spaces $B^s_{p,q}(w(x))$ and $F^s_{p,q}(w(x))$ are quasi-Banach spaces. They are independent of the chosen dyadic resolution of unity $\{\varphi_j\}_{j=0}^{\infty}$ in the sense of equivalent quasi-norms. The operator $f \mapsto wf$ is an isomorphic map from $B^s_{p,q}(w(x))$ onto $B^s_{p,q}$ and from $F^s_{p,q}(w(x))$ onto $F^s_{p,q}$. A proof of these properties may be found in [HT1]. In our context the following assertion is of interest.

Theorem 4. *Let $s \in \mathbb{R}$, $0 < q \leq \infty$, $0 < p \leq \infty$ (with $p < \infty$ in the F-case), $w(x) \in W$, $0 \leq \delta < 1$ and $P(x, D_x) \in \Psi^0_{1,\delta}$. Then $P(x, D_x)$ induces a bounded map from $B^s_{p,q}(w(x))$ into itself and from $F^s_{p,q}(w(x))$ into itself. There exist integers l, k and a constant c such that*

$$\|P(x, D_x)u \mid B^s_{p,q}(w(x))\| \leq c\,|p|^{(0)}_{(l,k)}\, \|u \mid B^s_{p,q}(w(x))\|$$

for all $u \in B^s_{p,q}(w(x))$, and

$$\|P(x, D_x)u \mid F^s_{p,q}(w(x))\| \leq c\,|p|^{(0)}_{(l,k)}\, \|u \mid F^s_{p,q}(w(x))\|$$

for all $u \in F^s_{p,q}(w(x))$.

This has been proved in [HT2]—Theorem and Remark 1 in Subsection 3.1.

Remark 4. The estimates in both theorems imply that the topology of $\Psi^0_{1,\delta}$ is stronger than the corresponding operator topology.

4. Results

Theorem 5. *Let $1 < p < \infty$, $0 < q < \infty$, s real and $\delta < 1$. Then $\Psi^0_{1,\delta}$ is a spectrally invariant subalgebra of $\mathcal{L}(B^s_{p,q})$ and $\mathcal{L}(F^s_{p,q})$. That means*

$$\Psi^0_{1,\delta} \cap \mathcal{L}(B^s_{p,q})^{-1} = \Psi^0_{1,\delta} \cap \mathcal{L}(F^s_{p,q})^{-1} = (\Psi^0_{1,\delta})^{-1}.$$

The theorem generalizes the result of Ueberberg—Proposition 1—and was proved in [LS2]. To establish Theorem 5, we applied the so-called commutator method, already used by Beals and Ueberberg. More precisely, we first proved a criterion for an operator in Besov-Triebel-Lizorkin spaces to be a pseudodifferential operator with symbol in the class $S^0_{1,\delta}$. This criterion is expressed in terms of the mapping properties of commutators of the operator in question with operators of multiplication by coordinate functions in $\mathbb{R}^n$ and differentiation with respect to the coordinates. Afterwards, applying a lifting property and a certain representation formula, the theorem follows.

Again the following consequences are immediate.

Corollary 1. *Let $1 < p^* < \infty$, $0 < q^* < \infty$, s^* real, and $0 \le \delta < 1$. If $P \in \Psi^0_{1,\delta}$ is invertible in $\mathcal{L}(B^{s*}_{p*,q*})$ or $\mathcal{L}(F^{s*}_{p*,q*})$, then P is also invertible in $\mathcal{L}(B^s_{p,q})$ and $\mathcal{L}(F^s_{p,q})$ for all $1 < p < \infty$, $0 < q \le \infty$ and s real.*

Corollary 2. *The spectrum of a pseudodifferential operator $P \in \Psi^0_{1,\delta}$ is the same on all the spaces $B^s_{p,q}$ and $F^s_{p,q}$, $1 < p < \infty$, $0 < q < \infty$ and s real. Since $B^0_{2,2} = L_2$, it coincides with its L_2 spectrum:*

$$\sigma(P \mid \mathcal{L}(B^s_{p,q})) = \sigma(P \mid \mathcal{L}(F^s_{p,q})) = \sigma(P \mid \mathcal{L}(L_2)).$$

So far we have not succeeded in using the commutator method in case $0 < p \le 1$, too. So an alternative idea consists in application of further properties of the function spaces—e.g., duality and interpolation. But for this reason, we have to restrict the class of pseudodifferential operators under consideration.

If $A \in \Psi^0_{1,\delta}$, then A' and A^* stand for the dual (transpose) and adjoint operator, respectively. We have $A' \in \Psi^0_{1,\delta}$, $A^* \in \Psi^0_{1,\delta}$ and in the sense of the dual pairing $S - S'$,

$$\langle A\varphi, \psi\rangle = \langle \varphi, A'\psi\rangle = \langle \varphi, \overline{A^*\overline{\psi}}\rangle.$$

Definition 4. Let $0 \leq \delta < 1$. Then

$$\Psi_{\delta,sym} = \{A \in \Psi^0_{1,\delta} \ : A = A' = A^*\}.$$

The elements of $\Psi_{\delta,sym}$ are self-dual in the $S - S'$ pairing and self-adjoint in L_2.

Remark 5. If in the context of oscillatory integrals, $A \in \Psi^0_{1,\delta}$ is written as

$$A\varphi = (2\pi)^{-n} \int\int e^{i(x-y)\xi} \ a(x,y,\xi)\varphi(y) \ \mathrm{d}y \ \mathrm{d}\xi$$

with the amplitude $a(x,y,\xi)$, then the dual operator and the adjoint operator have the amplitudes

$$a'(x,y,\xi) = a(y,x,-\xi) \quad \text{and} \quad a^*(x,y,\xi) = \overline{a(y,x,\xi)}.$$

Hence, an operator $A \in \Psi^0_{1,\delta}$ belongs to $A \in \Psi_{\delta,sym}$ if and only if

$$a(x,y,\xi) = a(y,x,-\xi) = \overline{a(y,x,\xi)},$$

see [Tre, I.3 and IV.2] for details.

Remark 6. Let $a(x,\xi)$ belong to $S^0_{1,\delta}$ and let further $a(x,\xi)$ be real-valued and rotation invariant in ξ. In the spirit of the Weyl calculus denote by A^w the operator

$$A^w\varphi = (2\pi)^{-n} \int\int e^{i(x-y)\xi} a\Big(\frac{x+y}{2},\xi\Big)\varphi(y) \ \mathrm{d}y \ \mathrm{d}\xi, \quad \varphi \in S.$$

Notice that A^w differs from the pseudodifferential operator A defined in a usual way by the symbol $a(x,\xi)$. But it can be shown that A^w belongs also to $\Psi^0_{1,\delta}$. Then, with respect to the remark given above, it is an element of $\Psi_{\delta,sym}$, too.

Remark 7. We were interested in the subclass $\Psi_{\delta,sym}$ mostly for technical reasons. On the other hand, in [HT2] the authors dealt with operators of this type in connection with estimates of the "negative spectrum" of symmetrically perturbed positive self-adjoint pseudodifferential operators in L_2.

Theorem 6. *Let s be real, $0 < q \le \infty$, $0 < p \le \infty$ ($0 < p < \infty$ in the case of F-spaces), $w(x) \in W$ and $0 \le \delta < 1$.*

Then

$$\Psi_{\delta,sym} \cap \mathcal{L}(B^s_{p,q}(w(x)))^{-1} = \Psi_{\delta,sym} \cap \mathcal{L}(F^s_{p,q}(w(x)))^{-1} = \Psi^{-1}_{\delta,sym}.$$

This theorem was proved in [LT]. The main tools of the proof were duality and interpolation properties of $B^s_{p,q}(w(x))$ and $F^s_{p,q}(w(x))$. Again, we get immediately:

Corollary 3. *For a pseudodifferential operator $A \in \Psi_{\delta,sym}$, its spectrum is the same in all spaces $B^s_{p,q}(w(x))$ and $F^s_{p,q}(w(x))$ for $0 < q \le \infty$, $0 < p \le \infty$ ($p < \infty$ in the F-case), $w(x) \in W$ and s real,*

$$\sigma(A \mid \mathcal{L}(B^s_{p,q}(w(x)))) = \sigma(A \mid \mathcal{L}(F^s_{p,q}(w(x)))) = \sigma(A \mid \mathcal{L}(L_2)),$$

and it coincides with its L_2-spectrum.

Remark 8. The technique used in the proof of Theorem 6 turns out to be applicable in more general situations, provided that the mapping properties needed, interpolation and duality results are at hand.

For example, this is true in the case of Besov spaces of variable order of differentiation, introduced in [Le1]. The mapping properties of pseudo-differential operatorsbelonging to $\Psi^0_{1,\delta}$ [Le1], the interpolation properties of these spaces [Le2] and the duality results [Le3] guarantee that the class $\Psi_{\delta,sym}$ is spectral invariant with respect to $\mathcal{L}(B^{s,a}_{p,q})$.

References

[AH] J. Alvarez and J. Hounie, *Spectral invariance and tameness of pseudo-differential operators on weighted Sobolev spaces*, J. Operator Theory **30** (1993), 41–67.

[Be] R. Beals, *Characterisation of pseudodifferential operators and applications*, Duke Math. J. **44** (1977), 45–57; **46** (1979), 215.

[Bon] J.-M. Bony, *Calcul symbolique et propagation des singularités pour les équation aux dérivées partielles non linéaires*, Ann. Scuola Norm. Sup. Pisa **14** (1981), 209–246.

[BC] J.-M. Bony and J.-Y. Chemin, *Espaces functionnels associé au calcul de Weyl-Hörmander*, Bull. Soc. Math. France **122** (1994), 77–118.

[Bo1] G. Bourdaud, *L^p-estimates for certain non-regular pseudo-differential operators*, Comm. Partial Differential Equations **7** (1982), 1023–1033.

[Bo2] G. Bourdaud, *Une algebre maximale d'opérateurs pseudodifférentiels*, Comm. Partial Differential Equations **13** (1988), 1059–1083.

[CV1] A. P. Calderón and R. Vaillancourt, *On the boundedness of pseudo-differential operators*, J. Math. Soc. Japan **23** (1971), 374–378.

[CV2] A. P. Calderón and R. Vaillancourt, *A class of bounded pseudo-differential operators*, Proc. Nat. Acad. Sci. U.S.A. **69** (1972), 1185–1187.

[Ch] C.-H. Ching, *Pseudodifferential operators with non-regular symbols*, J. Differential Equations **11** (1972), 436–447.

[CM] R. Coifman and Y. Meyer, *Au delà des opérateurs pseudo-différentiels*, Astérisque **57** (1978).

[Co1] H. O. Cordes, *On a class of C^*-algebras*, Math. Ann. **170** (1967), 283–313.

[Co2] H. O. Cordes, *On some C^*-algebras and Fréchet-*-algebras of pseudodifferential operators*, Proc. Sympos. Pure Math. **43** (1985), 79–104.

[Da] E. B. Davies, *Heat kernels and spectral theory*, Cambridge University Press, Cambridge, 1989.

[DST] E. B. Davies, B. Simon and M. Taylor, *L^p spectral theory of Kleinian groups*, J. Funct. Anal. **78** (1988), 116–136.

[GK] I. Gohberg and N. Krupnik, *Einführung in die Theorie der eindimensionalen Integraloperatoren*, Birkhäuser, Basel, 1979.

[Gr] B. Gramsch, *Relative Inversion in der Störungstheorie von Operatoren und Ψ-Algebren*, Math. Ann. **269** (1984), 27–71.

[GK] B. Gramsch and W. Kaballo, *Decompositions of meromorphic Fredholm resolvents and Ψ^*-algebras*, Integral Equations Operator Theory **12** (1989), 23–41.

[GUW] B. Gramsch, J. Ueberberg and K. Wagner, *Spectral invariance and submultiplicativity for Fréchet algebras with applications to pseudo-differential operators and Ψ^*-quantization*, Operator calculus and spectral theory (Lambrecht, 1991), Oper. Theory Adv. Appl., 57, Birkhäuser, Basel, 1992, pp. 71–98.

[HT1] D. Haroske and H. Triebel, *Entropy numbers in weighted function spaces and eigenvalue distributions of some degenerate pseudodifferential operators I*, Math. Nachr. **167** (1994), 131–156.

[HT2] D. Haroske and H. Triebel, *Entropy numbers in weighted function spaces and eigenvalue distributions of some degenerate pseudodifferential operators II*, Math. Nachr. **168** (1994), 109–137.

[Hö1] L. Hörmander, *The analysis of linear differential operators III*, Springer, Berlin, 1985.

[Hö2] L. Hörmander, *Pseudo-differential operators of type 1,1*, Comm. Partial Differential Equations **13** (1988), 1085–1111.

[Il] R. Illner, *A class of L^p-bounded pseudodifferential operators*, Proc. Amer. Mat. Soc. **51** (1975), 347–355.

[Jö] K. Jörgens, *Lineare Integraloperatoren*, B.G. Teubner, Stuttgart, 1970.

[Kg] H. Kumano-go, *Pseudo-differential operators*, Massachusetts Institute of Technology Press, Cambridge (Massachusetts)-London, 1981.

[Le1] H.-G. Leopold, *On Besov spaces of variable order of differentiation*, Z. Anal. Anwendungen **8** (1989), 69–82.

[Le2] H.-G. Leopold, *Interpolation of Besov spaces of variable order of differentiation*, Arch. Math. **53** (1989), 178–187.

[Le3] H.-G. Leopold, *On function spaces of variable order of differentiation*, Forum Math. **3** (1991), 1–21.

[LS1] H.-G. Leopold and E. Schrohe, *Spectral invariance for algebras of pseudodifferential operators on Besov spaces of variable order of differentiation*, Math. Nachr. **156** (1992), 7–23.

[LS2] H.-G. Leopold and E. Schrohe, *Spectral invariance for algebras of pseudodifferential operators on Besov-Triebel-Lizorkin spaces*, Manuscripta Math. **78** (1993), 99-110.

[LT] H.-G. Leopold and H. Triebel, *Spectral invariance for pseudo- differential operators on weighted function spaces*, Manuscripta Math. **83** (1994), 315–325.

[Ma] J. Marshall, *Weighted parabolic Triebel-spaces of product type. Fourier multipliers and pseudodifferential operators*, Forum Math. **3** (1991), 479–511.

[Na] M. Nagase, *On the boundedness of pseudo-differential operators in L^p-spaces*, Sci. Rep. College Ed. Osaka Univ. **32** (1983), 9–13.

[Pä] L. Päivärinta, *Pseudodifferential operators in Hardy-Triebel-spaces*, Z. Anal. Anwendungen **2** (1983), 235–242.

[Sch1] E. Schrohe, *Boundedness and spectral invariance for standard pseudodifferential operators on anisotropically weighted L^p-Sobolev spaces*, Integral Equations Operator Theory **13** (1990), 271–284.

[Sch2] E. Schrohe, *Spectral invariance, ellipticity and the Fredholm property for pseudodifferential operators on weighted Sobolev spaces*, Ann. Global Anal. Geom. **10** (1992), 237–254.

[Sch3] E. Schrohe, *Functional calculus and Fredholm criteria for boundary value problems on noncompact manifolds*, Operator calculus and spectral theory (Lambrecht, 1991), Oper. Theory Adv. Appl., 57, Birkhäuser, Basel, 1992, pp. 255–269.

[ST] H.-J. Schmeißer and H. Triebel, *Topics in Fourier analysis and function spaces*, Wiley, Chichester, 1987.

[Sh] M. A. Shubin, *Spectral theory of elliptic operators on non-compact manifolds*, Méthodes semi-classiques, Vol. 1 (Nantes, 1991), Astérisque **207** (1992), 37–108.

[Ta] M. E. Taylor, *Pseudodifferential operators*, Princeton University Press, Princeton (New Jersey), 1981.

[Tre] F. Treves, *Introduction to pseudodifferential and Fourier integral operators I, II*, Plenum Press, New York-London, 1980.

[Tr] H. Triebel, *Interpolation theory, function spaces, differential operators*, North-Holland and Deutscher Verlag der Wissenschaften, Amsterdam-New York-Oxford and Berlin, 1978; sec. ed., Barth, Heidelberg, Leipzig, 1995.

[TrI] H. Triebel, *Theory of function spaces*, Birkhäuser, Basel, 1983.

[TrII] H. Triebel, *Theory of function spaces II*, Birkhäuser, Basel, 1992.

[Ue] J. Ueberberg, *Zur Spektralinvarianz von Algebren von Pseudodifferentialoperatoren in der L_p-Theorie*, Manuscripta Math. **61** (1988), 459–475.

[Ya] M. Yamazaki, *A quasi-homogeneous version of paradifferential operators I: Boundedness on spaces of Besov type*, J. Fac. Sci. Univ. Tokyo Sect. I A Math. **33** (1986), 131–174.

[W] H. Widom, *Singular integral equations in L^p*, Trans. Amer. Math. Soc. **97** (1960), 131–161.