HARDY'S INEQUALITY AND RELATED TOPICS

Alois Kufner

In this talk, we will deal mainly with the inequality

$$(\mathrm{H_k}) \qquad \Big(\int_a^b |u(t)|^q w_0(t)\ \mathrm{d}t\Big)^{1/q} \le c\Big(\int_a^b |u^{(k)}(t)|^p w_k(t)\ \mathrm{d}t\Big)^{1/p}$$

for a natural number $k > 1$ and parameters p, q, $0 < q < \infty$, $1 < p < \infty$. The inequality $(\mathrm{H_k})$ will be called a *k-th order Hardy inequality.* The *weight functions* w_0, w_k are measurable and positive a.e. in (a,b), and the inequality $(\mathrm{H_k})$ will be considered on the class K_k of functions which have absolutely continuous derivatives of order $k-1$ in (a,b) and for which the right hand side in $(\mathrm{H_k})$ is finite.

Problem No. 1. Under what (necessary and sufficient) conditions on p, q, w_0, w_k the inequality $(\mathrm{H_k})$ holds for all functions $u \in K_k$?

1° For $k = 1$, the problem is solved completely and the results can be found in [OK] for functions $u \in K_1$ satisfying some of the following *boundary conditions*:

$$(1) \qquad u(a) = 0 \quad \text{or} \quad u(b) = 0 \quad \text{or} \quad u(a) = u(b) = 0.$$

E.g., for $u \in K_1$ with $u(a) = 0$ and for p, q such that $1 < p \le q < \infty$, the necessary and sufficient condition for (H_1) to hold has the form

$$(2) \qquad B = \sup_{a<x<b} B(x) < \infty,$$

where

$$(3) \qquad B(x) = \Big(\int_x^b w_0(t)\ \mathrm{d}t\Big)^{1/q}\Big(\int_a^x w_1^{1-p'}(t)\ \mathrm{d}t\Big)^{1/p'}, \quad p' = \frac{p}{p-1}.$$

Remark. The necessary and sufficient condition changes slightly if we consider other boundary conditions from (1) and more substantially if we assume that $q < p$.

2° For $k > 1$, $(\mathrm{H_k})$ obviously becomes absurd if, e.g., u is a polynomial of order less than or equal $k-1$. Consequently, one has to add some conditions on $u \in K_k$. Inspired by (1), we assume that the functions $u \in K_k$ additionally satisfy

$$u^{(i)}(a) = 0 \quad \text{for } i \in M_0, \qquad u^{(j)}(b) = 0 \quad \text{for } j \in M_1, \tag{4}$$

where M_0, M_1 are subsets of the set $\{0, 1, \dots, k-1\}$. We denote this class of functions by

$$AC^{k-1}(a, b; M_0, M_1)$$

and assume in addition that

$$\#M_0 + \#M_1 = k. \tag{5}$$

3° For $(a, b) = (0, \infty)$, Problem No. 1 was completely solved

(i) for $M_0 = \{0, 1, \dots, k-1\}$, $M_1 = \emptyset$ by V. D. Stepanov in 1988 and

(ii) for $M_0 = \{0, 1, \dots, m-1\}$, $M_1 = \{m, m+1, \dots, k-1\}$ with $0 < m < k$ by H. P. Heinig and A. Kufner in 1989.

These choices of M_0, M_1 are the only reasonable ones for the case of an *unbounded* interval (a, b). The results as well as some counterexamples can be found in [OK, Chapter 10].

4° Therefore, we will assume in the sequel that

$$(a, b) = (0, 1).$$

Some partial results for $k = 2$ and $k = 3$ have been found in 1990 by A. Kufner and A. Wannebo (cf. [KW]), but the main problem now is the following question:

For what choice of M_0, M_1, the inequality $(\mathrm{H_k})$ *is meaningful?*

The answer was given by P. Drábek and A. Kufner in [DK]: Let us define the so-called *incidence matrix*

$$D = \begin{pmatrix} a_{00}, & a_{01}, & \dots, & a_{0,k-1} \\ a_{10}, & a_{11}, & \dots, & a_{1,k-1} \end{pmatrix}$$

with k zeros and k ones by

$$a_{js} = \begin{cases} 1 & \text{if } s \in M_j, \\ 0 & \text{if } s \notin M_j, \end{cases} \qquad j = 0, 1.$$

Definition. The sets M_0, M_1 are said to satisfy the *Pólya condition* if

$$\sum_{s=0}^{m}(a_{0s}+a_{1s}) \geq m+1 \quad \text{for } m=0,1,\dots,k-1. \tag{6}$$

(Let us note that due to (5), $\sum_{s=0}^{k-1}(a_{0s}+a_{1s})=k$.)

5° In [K], the problem No. 1 was completely solved under the additional assumption

$$M_0 \cap M_1 = \emptyset \tag{7}$$

[i.e., $M_1=\{0,1,\dots,k-1\}\setminus M_0$] for $(a,b)=(0,1)$. For the particular case

$$1<p\leq q<\infty \quad \text{and} \quad M_0\neq\emptyset, \quad M_1\neq\emptyset,$$

the necessary and sufficient conditions for $(\mathrm{H_k})$ to hold for every $u\in AC^{k-1}(0,1;M_0,M_1)$ with M_0, M_1 satisfying the Pólya condition (6) and conditions (5) and (7) have the following form

$$\max(B_1,B_2)<\infty, \tag{8}$$

where $B_i=\sup\limits_{0<x<1}B_i(x)$, $i=1$, 2, and

$$\begin{aligned} B_1(x) &= \Big(\int_x^1 w_0(t)t^{\alpha_1 q}(1-t)^{\beta_1 q}\,\mathrm{d}t\Big)^{1/q} \\ &\quad\times\Big(\int_0^x w_k^{1-p'}(t)t^{\gamma_1 p'}(1-t)^{\delta_1 p'}\,\mathrm{d}t\Big)^{1/p'}, \\ B_2(x) &= \Big(\int_0^x w_0(t)t^{\alpha_2 q}(1-t)^{\beta_2 q}\,\mathrm{d}t\Big)^{1/q} \\ &\quad\times\Big(\int_x^1 w_k^{1-p'}(t)t^{\gamma_2 p'}(1-t)^{\delta_2 p'}\,\mathrm{d}t\Big)^{1/p'}, \end{aligned} \tag{9}$$

with certain non-negative integers α_1, β_1, γ_1, δ_1, α_2, β_2, γ_2, δ_2, which are defined by the following procedure:

Suppose that the set $\{0, 1, \dots, k-1\}$ is split into s successive groups G_1, G_2, …, G_s ($s \geq 2$) according to the following scheme:

$$\begin{aligned}
G_1 =& \{0, 1, \dots, k_1 - 1\}, \quad k_1 > 0,\\
G_2 =& \{k_1, k_1 + 1, \dots, k_1 + k_2 - 1\}, \quad k_2 > 0,\\
G_3 =& \{k_1 + k_2, k_1 + k_2 + 1, \dots, k_1 + k_2 + k_3 - 1\}, \quad k_3 > 0,\\
& \dots\\
G_s =& \{k_1 + k_2 + \dots + k_{s-1}, \dots, k-1\},
\end{aligned}$$

i.e., the group G_i contains k_i elements ($i = 1, 2, \dots, s$) with $k_i > 0$ and $k_1 + k_2 + \dots + k_s = k$.

(i) If $M_0 = \{G_1, G_3, \dots, G_{s-1}\}$ for s even, $M_0 = \{G_1, G_3, \dots, G_s\}$ for s odd, and—due to (5) and (7)— $M_1 = \{G_2, G_4, \dots\}$, then

$$\begin{aligned}
\alpha_1 = k_1 - 1, \quad \beta_1 = 0, \quad \gamma_1 = k_2, \quad \delta_1 = 0,\\
\alpha_2 = k_2, \quad \beta_2 = 0, \quad \gamma_2 = k_2 - 1, \quad \delta_2 = 0
\end{aligned}$$

for $s = 2$ (i.e., for $k = k_1 + k_2$),

$$\alpha_i = k_1, \quad \beta_i = \gamma_i = 0, \quad \delta_i = k_s, \quad i = 1, 2,$$

for s odd, and

$$\alpha_i = k_1, \quad \beta_i = 0, \quad \gamma_i = k_s, \quad \delta_i = 0, \quad i = 1, 2,$$

for s even, $s > 2$.

(ii) If $M_0 = \{G_2, G_4, \dots\}$, $M_1 = \{G_1, G_3, \dots\}$ then—using the substitution $t \mapsto 1 - t$ —we obtain that ($i = 1$, 2)

$$\alpha_i = \gamma_i = 0, \quad \beta_1 = k_1, \quad \gamma_1 = k_2 - 1, \quad \beta_2 = k_1 - 1, \quad \gamma_2 = k_2$$

for $s = 2$,

$$\alpha_i = 0, \quad \beta_i = k_1, \quad \gamma_i = k_s, \quad \delta_i = 0$$

for s odd, and

$$\alpha_i = 0, \quad \beta_i = k_1, \quad \gamma_i = 0, \quad \delta_i = k_s$$

for s even, $s > 2$.

The functions $B_i(x)$ from (9) are closely related to the function $B(x)$ from (3). Using the necessary and sufficient condition for $k = 1$ in the case

$$p > q,$$

we can again formulate necessary and sufficient conditions even for $k > 1$.

For the case

$$M_0 = \{0, 1, \dots, k-1\}, \quad M_1 = \emptyset,$$

the necessary and sufficient condition for $(\mathrm{H_k})$ has again the form (8), but with functions $B_1(x)$, $B_2(x)$ in (9) replaced by

$$\widetilde{B}_1(x) = \Big(\int_x^1 (t-x)^{(k-1)q} w_0(t)\ \mathrm{d}t\Big)^{1/q} \Big(\int_0^x w_k^{1-p'}(t)\ \mathrm{d}t\Big)^{1/p'},$$

$$\widetilde{B}_2(x) = \Big(\int_x^1 w_0(t)\ \mathrm{d}t\Big)^{1/q} \Big(\int_0^x w_k^{1-p'}(t)(x-t)^{(k-1)p'}\ \mathrm{d}t\Big)^{1/p'}.$$

The proof follows the same lines as the one in [St] for $(a, b) = (0, \infty)$. The case $M_0 = \emptyset$, $M_1 = \{0, 1, \dots, k-1\}$ can be again solved by using the substitution $t \mapsto 1 - t$.

6° The proof of the result mentioned in 5° for M_0, M_1 satisfying (5), (7) and, moreover, $M_0 \neq \emptyset$, $M_1 \neq \emptyset$, can be found in [K]. If (7) is *not fulfilled*, i.e., if

$$M_0 \cap M_1 \neq \emptyset, \tag{10}$$

necessary and sufficient conditions can be found in [KW] and [K] for $k = 2, 3, 4$. For an *arbitrary* integer $k > 1$, the following *conjecture* has been formulated in [K]:

Suppose that the sets M_0, M_1 satisfy (5) and (10) [and, of course, the Pólya condition (6)]. Then, for $1 < p \le q < \infty$, the necessary and sufficient conditions for $(\mathrm{H_k})$ to hold for every $u \in AC^{k-1}(0, 1; M_0, M_1)$ have again the form (8) with $B_i(x)$ given by (9) $(i = 1, 2)$, but this time the integers $\alpha_1, \dots, \delta_2$ are defined by the following procedure:

(i) Define $\widetilde{M}_1$ by

$$\widetilde{M}_1 = \{0, 1, \dots, k-1\} \setminus M_0.$$

Then the pair M_0, $\widetilde{M}_1$ satisfies the conditions of 5° (i) (if $G_1 \subset M$) or of 5° (ii) (if $G_1 \subset \widetilde{M}_1$) and there exist uniquely determined non-negative integers

$$\alpha_{i,1}, \quad \beta_{i,1}, \quad \gamma_{i,1}, \quad \delta_{i,1} \qquad (i = 1, 2)$$

according to 5°.

(ii) Define $\widetilde{M}_0$ by

$$\widetilde{M}_0 = \{0, 1, \ldots, k-1\} \setminus M_1.$$

Then the pair $\widetilde{M}_0$, M_1 again satisfies the conditions of 5° and there exist uniquely determined non-negative integers

$$\alpha_{i,2}, \quad \beta_{i,2}, \quad \gamma_{i,2}, \quad \delta_{i,2} \qquad (i = 1, 2).$$

(iii) The constants $\alpha_1, \ldots, \delta_2$ corresponding to the *initial* pair M_0, M_1 are now defined by

$$\alpha_i = \alpha_{i,1}, \quad \beta_i = \beta_{i,2}, \quad \gamma_i = \gamma_{i,1}, \quad \delta_i = \delta_{i,2} \qquad (i = 1, 2).$$

7° The foregoing conjecture was confirmed by the results of [KW] for $k = 2, 3$ as well as by a direct computation of all admissible cases for $k = 4$, but a general proof was given recently by G. Sinnamon [Si].

The proof is based on the fact the inequality ($\mathrm{H_k}$) on $AC^{k-1}(0, 1; M_0, M_1)$ is equivalent to the inequality

$$\Big(\int_0^1 |(Tf)(x)|^q w_0(x)\,\mathrm{d}x\Big)^{1/q} \le C\Big(\int_0^1 f^p(x) w_k(x)\,\mathrm{d}x\Big)^{1/p}$$

for measurable non-negative functions f, where

$$u(x) = (Tf)(x) = \int_0^x K_1(x,t) f(t)\,\mathrm{d}t + \int_x^1 K_2(x,t) f(t)\,\mathrm{d}t,$$

and u is the solution of the boundary value problem

$$\begin{gathered} u^{(k)} = f \quad \text{in } (0,1), \\ u^{(i)}(0) = 0 \quad \text{for } i \in M_0, \qquad u^{(j)}(1) = 0 \quad \text{for } j \in M_1. \end{gathered}$$

Since it can be shown that the kernels $K_i(x,t)$ are equivalent to the product $x^{\alpha_i}(1-x)^{\beta_i}t^{\gamma_i}(1-t)^{\delta_i}$, the result follows by using the (first order) Hardy inequality. For details see [K] and [Si].

Remark. The above-mentioned necessary and sufficient conditions concern only the case $1 < p \le q < \infty$, but obviously, they can be modified to the case

$$p > q, \quad 0 < q < \infty, \quad 1 < p < \infty,$$

and also to the "limiting" cases $p = 1$ and $p, q = \infty$, using the corresponding modified conditions for the (first order) Hardy inequality.

8° Open problems. There are some problems which still remain open.

(i) A counterpart of the third condition in (1) is the choice of M_0, M_1 in such a way that condition (5) is violated, i.e., the number of boundary conditions exceeds k:

$$\#M_0 + \#M_1 > k.$$

In this case we talk about an *overdetermined Hardy inequality.*

This problem is open even in the simplest case $k = 2$, where we have the following possibilities:

(i-1) $u(0) = u(1) = 0$, $u'(0) = 0$ [i.e., $M_0 = \{0,1\}$, $M_1 = \{0\}$, $\#M_0 + \#M_1 = 3$] or, equivalently, $u(0) = u(1) = 0$, $u'(1) = 0$.

(i-2) $u(0) = 0$, $u'(0) = u'(1) = 0$ [i.e., $M_0 = \{0,1\}$, $M_1 = \{1\}$, $\#M_0 + \#M_1 = 3$] or, equivalently, $u(1) = 0$, $u'(1) = u'(1) = 0$.

(i-3) $u(0) = u(1) = u'(0) = u'(1) = 0$ [i.e., $M_0 = M_1 = \{0,1\}$, $\#M_0 + \#M_1 = 4$].

Obviously *sufficient* conditions can be derived easily by putting together the conditions for certain subsets of M_0, M_1. For example, in the case (i-1) we can use the conditions for $u(0) = u(1) = 0$ together with the conditions for $u(1) = u'(0) = 0$, or the conditions for $u(0) = u'(0) = 0$ together with the conditions for $u(0) = u(1) = 0$, etc. Thus, the problem consists in establishing *necessary and sufficient* conditions for $(\mathrm{H_k})$ to hold in the case of overdetermined boundary conditions.

(ii) Another possibility how to make inequality $(\mathrm{H_k})$ meaningful is to replace the boundary conditions (4) where only *single derivatives* appear, by conditions expressed by a *combination of derivatives.*

E.g., for $k = 2$, we can consider functions $u \in K_2$ that satisfy mixed boundary conditions of the form

$$\begin{aligned} \alpha u(0) + \beta u'(0) &= 0, \quad |\alpha| + |\beta| > 0, \\ \gamma u(1) + \delta u'(1) &= 0, \quad |\gamma| + |\delta| > 0. \end{aligned} \tag{11}$$

Of course, for bigger integers k, the situation becomes more complicated. A *general* approach is still an open problem.

For illustration, let us consider two special cases of the conditions (11):

(ii-1) If we consider functions $u \in K_2$ which satisfy

$$u(0) = 0, \quad u(1) - u'(1) = 0$$

[i.e., we take $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = -1$ in (11)], then it can be shown that for such functions the Hardy inequality (H_2) *cannot hold.*

(ii-2) If we consider functions $u \in K_2$ which satisfy

$$u'(0) = 0, \quad u(1) - u'(1) = 0$$

[i.e., we take $\beta = 0$, $\alpha = \gamma = -\delta = 1$ in (11)], then the Hardy inequality (H_2) with $1 < p \leq q < \infty$ holds for such functions if and only if (8) is satisfied, where now

$$B_1(x) = \Big(\int_x^1 w_0(t) t^q \,\mathrm{d}t\Big)^{1/q} \Big(\int_0^x w_2^{1-p'}(t) \,\mathrm{d}t\Big)^{1/p'},$$

$$B_2(x) = \Big(\int_0^x w_0(t) \,\mathrm{d}t\Big)^{1/q} \Big(\int_x^1 w_2^{1-p'}(t) t^{p'} \,\mathrm{d}t\Big)^{1/p'}$$

[i.e., we have again the functions from (9) with the particular choice $\alpha_1 = \gamma_2 = 1$, $\alpha_2 = \beta_i = \gamma_2 = \delta_i = 0$, $i = 1, 2$].

(iii) If the sets M_0, M_1 satisfy the Pólya condition, then the right hand side of $(\mathrm{H_k})$ is a *norm* on the linear space $AC^{k-1}(0,1;M_0,M_1)$. Let us denote its closure by

$$W^{k,p}(M_0, M_1; w_k);$$

if we suppose in addition that

$$w_k^{1-p'} \in L^1_{\mathrm{loc}}(0,1),$$

then $W^{k,p}(M_0, M_1; w_k)$ is a Banach space and the inequality $(\mathrm{H_k})$ can be considered as a *continuous imbedding* of the weighted Sobolev space $W^{k,p}(M_0, M_1; w_k)$ into the weighted Lebesgue space $L^q(w_0)$:

$$W^{k,p}(M_0, M_1; w_k) \hookrightarrow L^q(w_0). \tag{12}$$

Using known results for the first order Hardy inequality, it is easy to show that the imbedding (12) is *compact*, if

$$\lim_{x\to 0} B_i(x) = \lim_{x\to 1} B_i(x) = 0, \quad i = 1, 2. \tag{13}$$

Of course, the conditions (13) are only *sufficient* for the compactness. The question of finding *necessary and sufficient* conditions is still open.

9° Finally, let us shortly deal with the so-called *fractional order Hardy inequalities*, i.e., the inequalities of the form

$$\Big(\int_a^b |u(t)|^q w_0(t)\ \mathrm{d}t\Big)^{1/q} \le C\Big(\int_a^b |u^{(\lambda)}(t)|^p w_\lambda(t)\ \mathrm{d}t\Big)^{1/p} \tag{H_λ}$$

and

$$\Big(\int_a^b |u^{(\lambda)}(t)|^q w_\lambda(t)\ \mathrm{d}t\Big)^{1/q} \le C\Big(\int_a^b |u'(t)|^p w_1(t)\ \mathrm{d}t\Big)^{1/p} \tag{$(\mathrm{H}_\lambda)^*$}$$

with

$$0 < \lambda < 1.$$

Problem No. 2. Under what conditions on q, p, w_0, w_1 and w_λ the inequalities (H_λ) and $(\mathrm{H}_\lambda)^*$ hold, where $u^{(\lambda)}$ denotes the fractional order derivative of u?

For simplicity, we will deal with the particular case

$$(a, b) = (0, \infty).$$

The first problem which occurs is *how to understand* the right hand side of (H_λ) and the left hand side of $(\mathrm{H}_\lambda)^*$. Since, for the non-weighted case, the "fractional order λ" expression

$$\int_0^\infty |u^{(\lambda)}(t)|^r\ \mathrm{d}t$$

is understood as

$$J_r = \int_0^\infty \int_0^\infty \frac{|u(x) - u(y)|^r}{|x-y|^{1+\lambda r}} \, \mathrm{d}x \, \mathrm{d}y,$$

we can define the integral on the right hand side of (H_λ) for $(a,b) = (0,\infty)$ as

$$J_p(w) = \int_0^\infty \int_0^\infty \frac{|u(x) - u(y)|^p}{|x-y|^{1+\lambda p}} w_\lambda(x,y) \, \mathrm{d}x \, \mathrm{d}y. \tag{14}$$

The inequality

$$\int_0^x |u(t)|^p t^{-\lambda p} \, \mathrm{d}t \le C J_p \tag{15}$$

was derived by G. N. Jakovlev in 1961 and rediscovered by P. Grisvard in 1969. It holds for $u \in C_0^\infty(0,\infty)$, $0 < \lambda < 1$, $\lambda \ne 1/p$, and it is in fact the inequality (H_λ) with $p = q$, $w_\lambda(t) \equiv 1$ and $w_0(t) = t^{-\lambda p}$.

Analogously, it can be shown that

$$J_p \le C \int_0^\infty |u'(t)|^p t^{(1-\lambda)p} \, \mathrm{d}t,$$

which is $(\mathrm{H}_\lambda)^*$ for $p = q$, $w_\lambda(t) \equiv 1$, $w_1(t) = t^{(1-\lambda)p}$.

A. Kufner and L. E. Persson derived in [KP] the inequality

$$\int_0^\infty |u(x)|^p w_0(x) \, \mathrm{d}x \le C \int_0^\infty \int_0^\infty \frac{|u(x) - u(y)|^p}{|x-y|^{1+\lambda p}} w_\lambda(x) \, \mathrm{d}x \, \mathrm{d}y \tag{16}$$

for $p > 1$, $\lambda \ge -1/p$ and

$$w_\lambda(x) = w_0(x) x^{\lambda p} + w_1(x) x^{(\lambda-1)p},$$

provided that

$$\sup_{0<x<\infty} \Big(\int_0^x w_0(t) \, \mathrm{d}t \Big)^{1/p} \Big(\int_x^\infty w_1^{1-p'}(t) \, \mathrm{d}t \Big)^{1/p'} < \infty.$$

As a particular case, we obtain (16) with $w_0(x) = x^{\alpha-\lambda p}$, $w_\lambda(x) = x^\alpha$ provided that $\alpha - \lambda p > -1$.

In [KP], inequalities of the type (H_λ) and $(\mathrm{H}_\lambda)^*$ have been treated also via interpolation theory. Recently, H. P. Heinig, A. Kufner and L. E. Persson have derived some slightly more general results including, e.g., more complicated weight functions. These results will be published separately, therefore, let us only mention one result:

The inequality (H_λ) *holds for* $p = q$ *with the right hand side according to* (14) *with*

$$w_\lambda(x,y) = v(x)v(y)$$

provided that

$$w_0(x) = x^{-\lambda p}\frac{v(x)V(x)}{\lambda}, \quad V(x) = \int_0^x v(t)\ \mathrm{d}t.$$

The following problems connected with fractional order Hardy inequalities are still open:

(i) All conditions guaranteeing the validity of (H_λ) are *only sufficient.*

(ii) Mostly, only the case $p = q$ is solved, whereas for $p < q$, only partial results involving mixed norms on the right hand side are available.

References

[DK] P. Drábek and A. Kufner, *The Hardy inequality and Birkhoff interpolation*, Bayreuth. Math. Schriften **47** (1994), 99–105.

[K] A. Kufner, *Higher order Hardy inequalities*, Bayreuth. Math. Schr. **44** (1993), 105–146.

[KP] A. Kufner and L. E. Persson, *Hardy inequality of fractional order via interpolation*, Inequalities and applications (R. P. Agarwal, ed.), World Scientific, Singapore, 1994, pp. 417–430.

[KW] A. Kufner and A. Wannebo, *Some remarks on the Hardy inequality for higher order derivatives*, General Inequalities, Vol. IV, Birkhäuser, 1992, pp. 33–48.

[OK] B. Opic and A. Kufner, Pitman Research Notes in Math. Series, vol. 219, Longman Scientific & Technical, Harlow, 1990.

[Si] G. Sinnamon, *Kufner's conjecture for higher order Hardy inequalities*, to appear in Real Anal. Exchange.

[St] V. D. Stepanov, *Two-weighted estimates of Riemann-Liouville integrals*, Math. SSSR Izv. **36** (1991), no. 3, 669–681.