EXTRAPOLATION OF REDUCED SOBOLEV IMBEDDINGS

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1. Background

The limiting imbeddings of Sobolev spaces attract a lot of attention nowadays. Various settings of the problem have been recently studied in many papers (cf. [FLS], [ET1], [EK], [KSchm], [KScho], …). The celebrated limiting imbedding theorem [Tru], [Po], [St] states in its basic form that the Sobolev space $W_p^m(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a sufficiently smooth boundary, $1 < p < \infty$, and $mp = N$, is imbedded into the Orlicz space $L_\Phi(\Omega)$ with $\Phi(t) = \exp t^{N/(N-m)} - 1$. The key to the proof is to expand Φ into its Taylor's series, taking into account the definition of the Luxemburg norm and showing that

$$\|u \mid L_q\| \le c q^{1-m/N} \|u \mid W_p^m\|, \quad q \text{ large.} \tag{1.1}$$

Equivalently, one can write

$$\sup_{1<q<\infty} q^{-(1-m/N)} \|u \mid L_q\| \le c\|u \mid W_p^m\|, \quad q \text{ large,} \tag{1.1'}$$

and it is easy to check that the expression on the left hand side of (1.1') is an equivalent norm in $L_\Phi(\Omega)$ (cf. [ET2]). Similarly, the condition

$$\lim_{p\to\infty} \frac{1}{p}\bigg(\int_\Omega |u(x)|^{Np/(N-m)}\,\mathrm{d}x\bigg)^{1/p} < \infty,$$

can be used as in the recent paper [FLS].

The conditions (1.1) and (1.1') express a control of the blow up of $\|u \mid L_q\|$ when q tends to ∞ (as is well known, $W_p^m(\Omega) \not\hookrightarrow L_\infty(\Omega)$ when $mp = N$). From the point of view of the scale of Lebesgue spaces, it is of course possible to transfer the situation to the case $q \nearrow q_0 < \infty$ and to balance the speed of blow up of $\|u \mid L_q\|$. Provided that the speed at which $\|u \mid L_q\|$ tends to infinity can be controlled in a suitable way by a power function, one gets the logarithmic Lebesgue spaces $L_{q_0} \log^\sigma L$ for negative σ's. The details can be found in [ET2], and from the point of view of this approach it is quite natural to use the term *logarithmic L_∞ spaces* for the exponential Orlicz spaces. More tedious considerations mimicking essentially the arguments from [ET2] make it possible to obtain a "finer logarithmic tuning" of the scale near L_{q_0} ($q_0 < \infty$) and near L_∞. Let us still observe that the above mentioned results can be expressed also in the language of suitable extrapolation methods (see [M] for the definitions of the Δ and Σ methods). The basic limiting situation repeats in a changed setting when considering W_p^{1+m}, $mp = N$. A function from W_p^{1+m} is generally not Lipschitz continuous, even though it is Hölder continuous with all $0 < \alpha < 1$. It has been shown by Brézis and Wainger [BW] that a better target space can be found, namely, the space $\mathrm{Lip} / \log^{1/p'}$, and this was generalized to Bessel potential logarithmic Sobolev spaces (with a non-integer smoothness parameter) in [EK]; at the same time a natural link between these two kinds of limiting imbeddings was found here.

There are far less results on the weighted limiting imbeddings; they are the subject of a recent paper [KScho].

In this contribution we shall follow another line of limiting considerations, which has been discussed in [KSchm]. The point here is a weakening of the assumptions on the smoothness: It is known that the Sobolev imbedding theorem holds with unchanged exponents if some of the highest order derivatives are neglected. Let $M(N,m)$ be a subset of all N-tuples $\alpha = (\alpha_1, \dots, \alpha_N)$, defined by

$$M(N,m) = \{\alpha \colon |\alpha| = m,\ \alpha_i = 0 \text{ or } \alpha_i = 1 \text{ for } 1 \le i \le N\}$$

and let $\mathcal{M} \subset M(N,m)$. We define the *reduced Sobolev space* $W_p^{\mathcal{M}}(\mathbb{R}^N)$, $1 < p < \infty$, as

$$W_p^{\mathcal{M}} = W_p^{\mathcal{M}}(\mathbb{R}^N) = \{f \in L_p \colon D^\beta f \in L_p \text{ for all } \beta \le \alpha \text{ and all } \alpha \in \mathcal{M}\}.$$

In the paper [Ad] there are considered sublimiting imbeddings for these spaces with $\mathcal{M} = M(N,m)$ and with subsets $\mathcal{M}$, satisfying the conditions

$$\sum_{\alpha \in \mathcal{M}} \alpha_i = k \ge 1 \quad \text{for all } i = 1, \dots, N.$$

It has been proved in [Ad] and [BIN] that then the sublimiting imbeddings hold with the same target space as in the case of W_p^m, that is,

$$W_p^{M(N,m)}(\mathbb{R}^N) \hookrightarrow L_q(\mathbb{R}^N) \quad \text{for } q = Np/(N-mp).$$

Related estimates for more general subsets $\mathcal{M}$ and mixed norm spaces can be found in [BIN, Theorem 15.7].

Here we are concerned with more general spaces $W_{\overline{p}}^{\mathcal{M}}(\mathbb{R}^N)$ corresponding to mixed norms and

$$\begin{aligned}\mathcal{M} &= \mathcal{S}(N,(n_k)) = \mathcal{S}(N,n_1,\dots,n_m)\\ &= \{\alpha = (\alpha^1,\dots,\alpha^m)\colon\ \alpha^j = (\alpha_1^j,\dots,\alpha_{n_j}^j),\ |\alpha^j| = 1,\ 1 \le j \le m\},\end{aligned}$$

where $n_1 + \cdots + n_m = N$ (see Definition 3.2 and Remark 3.3).

For instance, in the non-mixed case $p = N/m = n_1 = \cdots = n_m$ we prove that

$$W_p^{\mathcal{M}}(\mathbb{R}^N) \hookrightarrow L_\Phi(\mathbb{R}^N) \quad \text{with } \Phi(t) \sim \exp t^{N/m(N-m)},$$

see Corollary 3.8 (ii).

The paper is divided into two major sections. In Section 2 we use an alternative classical analysis approach for some particular cases to get estimates of $\|u \mid L_q\|$ when $q \to \infty$. We get a sharp result, however, there are natural limits as to the generality in order that the proofs do not become technically unduly difficult and lengthy and, also, only particular subsets of integer derivatives up to a given order come into the play. The core of this contribution is Section 3, using the Fourier analysis approach to the fractional Sobolev spaces with dominating mixed derivatives as developed in [Sc1], [ScT, Chapter 2]. In particular, Theorem 3.5 is our most general result, covering especially the classical setting of the problem. Due to the limited extent of this paper we shall restrict ourselves only to the spaces of Lizorkin-Triebel type (the F-spaces); we refer to [KSchm] for the B-spaces. The crucial point is that

$$W_{\overline{p}}^{\mathcal{M}}(\mathbb{R}^N) = S_{\overline{p},\overline{2}}^{(1,\dots,1)}F(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m})$$

(see Definition 3.1) if $\mathcal{M} = \mathcal{S}(N,n_1,\dots,n_m)$, $N = n_1 + \cdots + n_m$, which enables us to prove imbeddings for the reduced Sobolev spaces via spaces with dominating mixed derivatives.

The spaces we consider are rather general, however, it should be pointed out that this is not so for the sake of the generality itself. Let us look more closely at the following illustrative example (see Definition 3.1 for the spaces in question). Put $N=3$, $m=2$, $\overline{p}=(3/2,3/2,3/2)$ and consider the space $S^{(1,1)}_{\overline{p},\overline{2}}F(\mathbb{R}^1\times\mathbb{R}^2)$. Clearly, if $f_1\in W^1_{3/2}(\mathbb{R}^1)$, $f_2\in W^1_{3/2}(\mathbb{R}^2)$ and $f=f_1\otimes f_2$, then, in general, $f\notin L_q$ for large q's. On the other hand, if $\overline{p}=(1,2,2)$, then $S^{(1,1)}_{\overline{p},\overline{2}}F(\mathbb{R}^1\times\mathbb{R}^2)\hookrightarrow L_\Phi$ with $\Phi(t)\sim\exp t^2$. We observe that

$$S^{(1,1)}_{\overline{p},\overline{2}}F=\{f\colon D^{(1,1,0)}f, D^{(1,0,1)}f, D^{(1,0,0)}f, D^{(0,1,0)}f, D^{(0,0,1)}f, f\in\mathcal{H}_{\overline{p}}\},$$

where $\mathcal{H}_{\overline{p}}=S^{(0,0)}_{\overline{p},\overline{2}}F$ is an appropriate substitute for $L_{\overline{p}}$. This shows that the seemingly too general mixed norm setting is in fact necessary and quite natural.

2. The classical technique approach

Here we shall deal with the spaces $W^{\mathcal{S}}_p(\mathbb{R}^N)$ with $N=2k$, $k\geq 2$, $p=N/2$, and $\mathcal{S}=\mathcal{S}(k)\subset M(N,2)$ satisfying

$$\sum_{\alpha\in\mathcal{S}}\alpha_j=k,\quad j=1,\dots,N. \tag{2.1}$$

The condition (2.1) means that if we write the elements of $\mathcal{S}$ as rows of a matrix, then the sum in each of the columns equals k.

A generalization of the considerations in this section to higher order spaces is possible, the proofs and the notation would become more complicated, the underlying idea being the same.

For $j=1,\dots,N$, let $S_j=\{l\neq j\colon \alpha_l=\alpha_j=1 \text{ for some } \alpha\in\mathcal{S}\}$.

For an N-tuple $\overline{p}=(p_1,\dots,p_N)$ with $0<p_i\leq\infty$ we shall denote by $\|u\mid L_{\overline{p}}(\mathbb{R}^N)\|$ the norm in the Lebesgue mixed norm space $L_{\overline{p}}(\mathbb{R}^N)$.

For $1<p\leq\infty$ the symbol $\|u\mid L_p(S_j)\|$ will stand for the L_p norm of u with respect to the variables $(x_l)_{l\in S_j}$. If S_j' is the complement of S_j with respect to $\{1,\dots,N\}$, then the meaning of $\|u\mid L_{q_1}(S_j)\mid L_{q_2}(S_j')\|$ is that we take first the L_{q_1} norm of u with respect to the variables in S_j and then the L_{q_2} norm of u with respect to the remaining variables.

We observe that by Minkowski's inequality, the value of $\|u\mid L_{\overline{p}}(\mathbb{R}^N)\|$ does not decrease after interchanging the order of integration with respect to x_i and x_{i+1} provided that $p_i\leq p_{i+1}$.

The symbol $L_\Phi=L_\Phi(\mathbb{R}^N)$ will stand for the (classical) Orlicz space in the sense of [KR], and $E_\Phi=E_\Phi(\mathbb{R}^N)$ will denote the closure of C_0^∞ in L_Φ. It is well known that $L_\Phi=E_\Phi$ if and only if the function Φ satisfies the Δ_2-condition, which is of course not the case of the exponential Orlicz spaces.

Now we are ready to state and to prove the following reduced limiting theorem.

Theorem 2.1. *Let $k \geq 2$ be an integer, $N = 2k$, and let $\mathcal{S}$ be a subset of M satisfying the condition* (2.1). *If $u \in W_k^{\mathcal{S}}$ is supported in a unit ball, then $u \in E_\Phi(\mathbb{R}^N)$ with $\Phi(t) = \exp t^{N/2(N-1)} - 1$ and there exists $c > 0$ independent of u such that*

$$\|u \mid E_\Phi\| \leq c \sum_{\alpha \in \mathcal{S}} \|D^\alpha u \mid L_{N/2}\|.$$

Proof. Since there is no anisotropy as far as the integrability properties are concerned, the order of variables is not relevant. Hence after a suitable renumbering of variables we can switch to the following basic situation: For $l = 1, \ldots, N/2$ let S_l be the $N/2$-tuple whose j-th component equals $j + N/2$, $1 \leq j \leq N/2$. The $N/2$-tuples S_l for $l > N/2$ become all equal to $(1, \ldots, N/2)$ and it is obvious that among the couples $(l, j + N/2)$, $1 \leq l \leq N/2$, $1 \leq j \leq N/2$, there are already all couples of indices with respect to which the second order derivatives of $u \in W_{N/2}^{\mathcal{S}}$ are supposed to exist and to belong to $L_{N/2}$; altogether there are $N^2/4$ different couples when counted without respect to the order of their components.

It is also enough to assume that $u \in C_0^\infty$. First consider u as a function in variables $x_1, \ldots, x_{1+N/2}$ and invoke the classical limiting imbedding [Tru]; we have

$$\|u \mid L_r(S_{1+N/2})\| \leq c r^{1-2/N} \sum_{j=1}^{N/2} \|D_j u \mid L_{N/2}(S_{j+N/2})\|. \tag{2.2}$$

Taking the L_r norms with respect to the remaining variables, we obtain from (2.2) that

$$\|u \mid L_r(\mathbb{R}^N)\| \leq c r^{1-2/N} \sum_{j=1}^{N/2} \|D_j u \mid L_{N/2}(S_{j+N/2}) \mid L_r(S_1)\|. \tag{2.3}$$

Now the same argument applies to each derivative $D_j u$ from the right hand side of (2.3) since $D_j u$ $(1 \leq j \leq N/2)$ as functions in $x_{1+N/2}, \ldots, x_N$ have all first order derivatives. Hence

$$\|D_j u \mid L_r(S_1)\| \leq c r^{1-2/N} \sum_{j=1+N/2}^{N} \|D_i D_j u \mid L_{N/2}(S_1)\|$$

and

$$\|D_j u \mid L_r(S_1) \mid L_{N/2}(S_{1+N/2})\| \leq c r^{1-2/N} \sum_{j=1+N/2}^{N} \|D_i D_j u \mid L_{N/2}(\mathbb{R}^N)\|. \tag{2.4}$$

The repeated use of Minkowski's inequality gives the natural order of variables on the left hand side of (2.4). Combining this together with (2.3) we get

$$\|u \mid L_r(\mathbb{R}^N)\| \leq c r^{2(1-2/N)} \|u \mid W^{\mathcal{S}}_{N/2}(\mathbb{R}^N)\|. \tag{2.5}$$

The proof can be now concluded in a standard way, namely by expanding the function $\Phi(t) = \exp t^{N/2(N-1)} - 1$ into Taylor's series. □

Remark 2.2. Another natural idea to prove Theorem 2.1 seems to be a combinatorial refinement of Gagliardo's and Adams' technique from [Ad]. Nevertheless, a thorough analysis reveals that even though a certain exponential inequality can be proved, the target space for the resulting imbedding is worse. To be more specific, for spaces of second order such approach consists of going through estimates which give

$$\|u \mid L_q\| \leq c q \sum_{i=1}^{N} \|D_i u \mid L_{N/2}(S_{j+N/2}) \mid L_q(S_1)\|$$

and

$$\|D_i u \mid L_q(S_1) \mid L_{N/2}(S_{1+N/2})\| \leq c q \sum_{j \in S_i} \|D_j D_i u \mid L_{N/2}(\mathbb{R}^N)\|,$$

which pasted together yield only

$$\|u \mid L_q(\mathbb{R}^N)\| \leq c q^2 \|u \mid W^{\mathcal{S}}_{N/2}(\mathbb{R}^N)\|,$$

that is, an estimate worse for all dimensions N than that in (2.5).

3. The Fourier analysis approach

Let $N \geq 2$ and $\mathbb{R}^N = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$. A point $x \in \mathbb{R}^N$ will be written as $x = (x^1, \ldots, x^m)$ with $x^j \in \mathbb{R}^{n_j}$, $1 \leq j \leq m$. Let $\overline{p} = (p_1, \ldots, p_m)$, $0 < p_i \leq \infty$. We put

$$\|f \mid L_{\overline{p}}(\mathbb{R}^N)\| = \left(\int\limits_{\mathbb{R}^{n_m}} \left(\cdots \left(\int\limits_{\mathbb{R}^{n_1}} |f(x^1, \ldots, x^m)|^{p_1} \, \mathrm{d}x^1 \right)^{p_2/p_1} \cdots \right)^{p_m/p_{m-1}} \mathrm{d}x^m \right)^{1/p_m}$$

and ($\kappa = (\kappa_1, \ldots, \kappa_m)$)

$$\|a_\kappa \mid \ell_{\overline{p}}\| = \left(\sum_{\kappa_m=0}^{\infty} \left(\cdots \left(\sum_{\kappa_1=0}^{\infty} |a_{\kappa_1,\ldots,\kappa_m}|^{p_1} \right)^{p_2/p_1} \cdots \right)^{p_m/p_{m-1}} \right)^{1/p_m}$$

(with the usual modification if $p_j = \infty$).

Let $\{\phi_k^j(\xi^j)\}$ be the standard resolution of unity associated to dyadic balls in $\mathbb{R}^{n_j}$, $1 \leq j \leq m$, that is, a sequence of functions $\{\phi_k^j\}$ such that

$$\sum_{k=0}^{\infty} \phi_k^j(\xi^j) \equiv 1 \quad \text{in } \mathbb{R}^{n_j},$$
$$\operatorname{supp} \phi_0^j \subset \{\xi^j \in \mathbb{R}^{n_j} : |\xi^j| \leq 2\},$$
$$\operatorname{supp} \phi_k^j \subset \{\xi^j \in \mathbb{R}^{n_j} : 2^{k-1} \leq |\xi^j| \leq 2^{k+1}\},$$

with $\phi_k^j(\xi^j) = \phi^j(2^{-k}\xi^j)$, $k = 1, 2, \ldots$, where ϕ is a C^∞ function whose support is contained in $\{1/2 \leq \xi^j \leq 2\}$.

Let $\mathcal{S}'(\mathbb{R}^N)$ be the space of tempered distributions, and let $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse, respectively. For $f \in \mathcal{S}'(\mathbb{R}^N)$ we put

$$f_k(x) = f_{k_1,\ldots,k_m}(x^1, \ldots, x^m) := (\mathcal{F}^{-1}(\phi_{k_1}^1 \otimes \cdots \otimes \phi_{k_m}^m)\mathcal{F}f)(x^1, \ldots, x^m),$$

where $k = (k_1, \ldots, k_m)$, $k_j = 0, 1, 2, \ldots$, $j = 1, \ldots, m$, $x \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$.

Definition 3.1. Let $\overline{p} = (p_1, \ldots, p_m)$, $\overline{q} = (q_1, \ldots, q_m)$, $\overline{r} = (r_1, \ldots, r_m)$ with $0 < p_j < \infty$, $0 < q_j \leq \infty$, and $-\infty < r_j < \infty$, $j = 1, \ldots, m$. Then we define the *Lizorkin-Triebel space with dominating mixed smoothness*

$$S^{\overline{r}}_{\overline{p},\overline{q}}F(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}) = \{f \in \mathcal{S}'(\mathbb{R}^N) :$$
$$\|f \mid S^{\overline{r}}_{\overline{p},\overline{q}}F\| = \|2^{k_1 r_1 + \cdots + k_m r_m} f_{k_1,\ldots,k_m}(x) \mid \ell_{\overline{p}} \mid L_{\overline{q}}\| < \infty\}.$$

We shall use the notation $\overline{k} = (k, \ldots, k)$ (an m-tuple).

Definition 3.2. Let $\overline{p} = (p_1, \dots, p_m)$, $1 < p_j < \infty$, and $-\infty < r_j < \infty$, $j = 1, \dots, m$. The *fractional Sobolev space with dominating mixed derivatives* is defined as

$$\begin{aligned} S^{\overline{r}}_{\overline{p}} H(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}) = \{ & f \in \mathcal{S}'(\mathbb{R}^N) : \\ & \|f \mid S^{\overline{r}}_{\overline{p}} H\| = \|\mathcal{F}^{-1}(1 + |\xi^1|^2)^{r_1/2} \dots (1 + |\xi^m|^2)^{r_m/2} \mathcal{F} f \mid L_{\overline{p}}\| < \infty \}. \end{aligned}$$

Remark 3.3. The F-spaces of the above type have been intensively studied in [Sc1]. The model case $N = 2$, $n_1 = n_2 = 1$ can be found in [ScT, Chapter 2]. All the spaces are quasi-Banach spaces (Banach spaces if $\min(p_j, q_j) \geq 1$) and independent of the choice of the systems $\{\phi^j\}$ (up to equivalence of the resulting quasi-norms). If $0 < p_1 = \cdots = p_m < \infty$, $0 < q_1 = \cdots = q_m \leq \infty$, $n_1 = \cdots = n_m = 1$, then $S^{\overline{r}}_{\overline{p},\overline{q}} F(\mathbb{R}^N)$ is a special case of non-homogeneous general function spaces in the sense of [Tri3–5].

The corresponding spaces of Besov type can be considered, too. Limiting imbeddings for them have been proved in [KSchm]. For the spaces $S^{\overline{r}}_{\overline{p},\overline{q}} B(\mathbb{R}^N)$ with $n_1 = \cdots = n_m = 1$, $r_j \geq 0$, $p_j \geq 1$, $q_j \geq 1$ we refer to the book by Amanov [Am] and the literature quoted there. The remark on non-homogeneous general function spaces concerning [Tri3–5] applies as well.

Let r_j be non-negative integers and $1 < p_j < \infty$, $j = 1, \dots, m$. We define the *Sobolev space with dominating mixed derivatives*

$$\begin{aligned} S^{\overline{r}}_{\overline{p}} W(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}) = \{ & f \in L_{\overline{p}}(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}) : \\ & \|f \mid S^{\overline{r}}_{\overline{p}} W\| = \sum_{\substack{\theta_j = 0,1;\ |\alpha^j| = r_j \\ j = 1, \dots, m}} \|D^{(\theta_1 \alpha^1, \dots, \theta_m \alpha^m)} f \mid L_{\overline{p}}\| < \infty \}. \end{aligned}$$

Using the arguments from [Sc1, Thm. A.2] (see also [Sc2, Thm. 3] and [ScT, Thm. 2.3.1] for the case $n_1 = \cdots = n_m$) one can prove that $S^{\overline{r}}_{\overline{p},\overline{2}} F(\mathbb{R}^N) = S^{\overline{r}}_{\overline{p}} H(\mathbb{R}^N)$ for $1 < p_j < \infty$ and $-\infty < r_j < \infty$ $(1 \leq j \leq m)$ with equivalent norms. Moreover, if r_j are non-negative integers, then

$$S^{\overline{r}}_{\overline{p},\overline{2}} F(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}) = S^{\overline{r}}_{\overline{p}} W(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}).$$

We observe that by standard arguments (Fourier multipliers of Mikhlin-Hörmander type) the norm in $S^{\overline{r}}_{\overline{p}} W(\mathbb{R}^N)$ is equivalent to

$$\sum_{\substack{0 \leq |\alpha^j| \leq r_j \\ j = 1, \dots, m}} \|D^{(\alpha^1, \dots, \alpha^m)} f \mid L_{\overline{p}}\|.$$

The space $S^{\overline{1}}_{\overline{p}}W(\mathbb{R}^N)$ is of special interest: It consists of those $f \in L_{\overline{p}}(\mathbb{R}^N)$ for which $D^{(\alpha^1,\dots,\alpha^m)}f \in L_{\overline{p}}(\mathbb{R}^N)$ and $|\alpha^j| = 0$ or $|\alpha^j| = 1$ for all $j = 1,\dots,m$. Its special non-mixed case is the reduced Sobolev space. This also establishes connections with the spaces $W^{\mathcal{M}}_p(\mathbb{R}^N)$ with $\mathcal{M} = \mathcal{S}(N, n_1,\dots,n_m)$ introduced earlier.

In what follows we shall pursue a finer analysis of the asymptotic behaviour of the constant $c = c(q)$ in the imbedding inequality

$$\|f \mid L_{\overline{q}}(\mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m})\| \le c\,\|f \mid S^{\overline{r}}_{\overline{p},\overline{2}}F(\mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m})\|$$

when $r_j = n_j/p_j$ $(j = 1,\dots,m)$ and $q \to \infty$. Let us recall that, in general,

$$S^{\overline{r}}_{\overline{p},\overline{q}}F(\mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m}) \subset S^{\overline{t}}_{\overline{u},\overline{v}}F(\mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m}) \tag{3.1}$$

provided that $0 < p_j < u_j < \infty$, $0 < q_j, v_j \le \infty$, and

$$r_j - \frac{n_j}{p_j} = t_j - \frac{n_j}{u_j} \quad (j = 1,\dots,m).$$

In particular, we have

$$S^{\overline{r}}_{\overline{p},\overline{q}}F(\mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m}) \subset L_{\overline{u}}(\mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m}) \tag{3.2}$$

if $r_j \ge n_j(1/p_j - 1/u_j)$ and $1 < p_j < u_j < \infty$ $(j = 1,\dots,m)$.

Proofs of the imbeddings (3.1) and (3.2) can be found in [ScT, Thm. 2.4.1] (the model case $n_1 = n_2 = 1$, $m = 2$); a detailed proof in the general case can be found in [Sc1, Chapter 6] or in [Sc3]. These imbeddings are sharp and cannot be improved with respect to the scale of the spaces involved. This follows from [SiT], taking into account that

$$\|f \mid S^{\overline{r}}_{\overline{p},\overline{q}}F\| \sim \prod_{j=1}^{m} \|f^j \mid F^{r_j}_{p_j,q_j}(\mathbb{R}^{n_j})\|$$

for all $f = f^1 \otimes \dots \otimes f^m$, where $f^j \in F^{r_j}_{p_j,q_j}(\mathbb{R}^{n_j})$.

Of crucial importance for establishing the limiting imbeddings for the F-spaces are Lemma 3.4 and Theorem 3.5. We shall not give the full proofs; an interested reader can look into [KSchm].

Lemma 3.4. *Let $1 < p_j < q_j < \infty$, $r_j = n_j/p_j$ $(j = 1, \dots, m)$. If $2 \le p_1 \le p_2 \le \dots \le p_m$, then there exists a constant c independent of $\overline{q}$ and f such that*

$$\|f \mid L_{\overline{q}}(\mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m})\| \le c \prod_{j=1}^{m} q_j^{1-1/p_j} \|f \mid S^{\overline{r}}_{\overline{p},\overline{2}}F(\mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m})\|$$

for all $f \in S^{\overline{r}}_{\overline{p},\overline{2}}F(\mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m})$.

Proof. Step 1. Let $f_k(x) = f_{k_1,\dots,k_m}(x^1, \dots, x^m)$ be the function from Definition 3.1. Then

$$(3.3) \quad \|f \mid L_{\overline{q}}\| = \Big\| \sum_{k_m=0}^{\infty} \dots \sum_{k_1=0}^{\infty} f_{k_1,\dots,k_m}(x^1, \dots, x^m) \Bigm| L_{\overline{q}} \Big\|$$
$$\le \|f_{k_1,\dots,k_m}(x^1, \dots, x^m) \mid \ell_{1|k_1}(L_{q_1}) \mid \dots \mid \ell_{1|k_m}(L_{q_m})\|,$$

where $\ell_{\kappa_j|k_j}(L_{q_j})$ denotes the L_{q_j} norm with respect to $x^j \in \mathbb{R}^{n_j}$ and the ℓ_{κ_j} norm with respect to k_j (the subindex k_j will be sometimes omitted if a misunderstanding cannot occur).

Step 2. An inequality of Nikol'skii type (see [ScT, Proposition 2.4.1 (i)]) applied to the right hand side of (3.3) gives

$$\|f \mid L_{\overline{q}}\|$$
$$\le c \Big\| \prod_{j=1}^{m} 2^{k_j n_j (1/p_j - 1/q_j)} f_{k_1,\dots,k_m}(x^1, \dots, x^m) \Bigm| \ell_{1|k_1}(L_{p_1}) \Bigm| \dots \Bigm| \ell_{1|k_m}(L_{p_m}) \Big\|$$

with c independent of $\overline{q}$. Using Hölder's inequality m-times we arrive at

$$\|f \mid L_{\overline{q}}\| \le c \prod_{j=1}^{m} \Big(\sum_{k_j=0}^{\infty} 2^{-k_j n_j p'_j / q_j} \Big)^{1/p'_j} \times$$
$$\times \Big\| \prod_{j=1}^{m} 2^{k_j n_j / p_j} f_{k_1,\dots,k_m} \Bigm| \ell_{p_1}(L_{p_1}) \Bigm| \dots \Bigm| \ell_{p_m}(L_{p_m}) \Big\|,$$

where $1/p_j + 1/p'_j = 1$ and c is independent of f and $\overline{q}$.

Step 3. Since $\sum_{\ell=0}^{\infty} \left(2^{-np'}\right)^{\ell/q} \sim q$ for $q \to \infty$ (with n and p' fixed), we get

(3.4)

$$\|f \mid L_{\overline{q}}\| \le c \prod_{j=1}^{m} q_j^{1-1/p_j} \times$$
$$\times \Big\| \prod_{j=1}^{m} 2^{k_j n_j / p_j} f_{k_1,\dots,k_m} \Bigm| \ell_{p_1}(L_{p_1}) \Bigm| \dots \Bigm| \ell_{p_m}(L_{p_m}) \Big\|.$$

If $p_j \le p_k$, then the $\ell_{p_k}(L_{p_j})$-norm is majorized by the $L_{p_j}(\ell_{p_k})$-norm and the same relation holds between the ℓ_p and ℓ_2 norms provided that $p \ge 2$. Therefore, it follows from (3.4) that

$$\|f \mid L_{\overline{q}}\| \le c \prod_{j=1}^{m} q_j^{1-1/p_j} \|f \mid S^{\overline{r}}_{\overline{p},\overline{2}}F(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m})\|.$$

□

Theorem 3.5. *Let $1 \le p_j < q_j < \infty$, $r_j = n_j/p_j$ $(j = 1, \dots, m)$. Then there exists a constant c independent of $\overline{q}$ and f such that*

$$\begin{aligned} \|f \mid L_{\overline{q}}(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m})\| \\ \le c \prod_{j=1}^{m} q_j^{1-1/p_j} \|f \mid S^{\overline{r}}_{\overline{p},\overline{2}}F(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m})\| \end{aligned} \tag{3.5}$$

for all $f \in S^{\overline{r}}_{\overline{p},\overline{2}}F(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m})$.

Sketch of the proof. It is enough to consider only $m = 2$. The proof goes along the same lines if $m > 2$, but it is more complicated from the technical point of view. Moreover, a complete proof even in this simplified form would be rather long because it uses a lot of non-folklore arguments from the interpolation theory. Therefore we restrict ourselves to explaining the main points of the proof.

Step 1. Four special cases can be distinguished: (1) $p_1 = p_2 = 1$, (2) $p_1 = 1$, $p_2 \ge 2$, (3) $p_1 \ge 2$, $p_2 = 1$, and (4) $2 \le p_1 \le p_2$. The remaining combinations of p_1 and p_2 can be handled by complex interpolation. Indeed,

$$[S^{\overline{r}}_{\overline{p},\overline{q}}F, S^{\overline{t}}_{\overline{u},\overline{v}}F]_\theta = S^{\overline{c}}_{\overline{a},\overline{b}}F \tag{3.6}$$

for $1 \le p_j, q_j, u_j, v_j < \infty$, $-\infty < r_j, t_j < \infty$ $(j = 1, \dots, m)$, $0 < \theta < 1$, and $\dfrac{1}{a_j} = \dfrac{1-\theta}{p_j} + \dfrac{\theta}{u_j}$, $\dfrac{1}{b_j} = \dfrac{1-\theta}{q_j} + \dfrac{\theta}{v_j}$, $c_j = (1-\theta)r_j + \theta t_j$ $(j = 1, \dots, m)$. The proof of (3.6) goes via iteration and retraction to the complex interpolation of vector-valued L_p-spaces and vector-valued sequence spaces, which is dealt with in [Tri1, Thms. 1.18.1, 1.18.4, and 2.4.1 (d)].

The case (4) was considered in Lemma 3.4. The core of the proof for the remaining three cases consists in showing that

$$\|f \mid L_{\overline{\infty}}\| \le c\|f \mid S^{(n_1,n_2)}_{\overline{1},\overline{2}}F\|, \tag{3.7}$$

$$\|f \mid L_{\overline{1}}\| \le c\|f \mid S^{(n_1,n_2)}_{\overline{1},\overline{2}}F\|, \tag{3.8}$$

$$\|f \mid L_{(1,\infty)}\| \le c\|f \mid S^{(n_1,n_2)}_{\overline{1},\overline{2}}F\|, \tag{3.9}$$

$$\|f \mid L_{(\infty,1)}\| \le c\|f \mid S^{(n_1,n_2)}_{\overline{1},\overline{2}}F\|. \tag{3.10}$$

Step 2. We shall give a sketch of the proof of (3.7). Combining an idea from [Ja] with our technique to prove Lemma 3.4, we get, applying the inequality of Nikol'skii type ([ScT, Prop. 2.4.1 (i)], with $\overline{q} = \overline{\infty}$ and $\overline{p} = \overline{2}$), that

$$\begin{aligned} \|f \mid L_{\overline{q}}\| &\le \|f_{k_1,k_2}(x^1,x^2) \mid \ell_1(L_\infty) \mid \ell_2(L_\infty)\| \\ &\le \|2^{k_1 n_1/2} 2^{k_2 n_2/2} f_{k_1,k_2}(x^1,x^2) \mid \ell_1(L_2) \mid \ell_1(L_2)\| \\ &= \|f_{k_1,k_2}(x^1,x^2) \mid \ell_1^{n_1/2}(L_2) \mid \ell_1^{n_2/2}(L_2)\|. \end{aligned} \tag{3.11}$$

If $0 < u < 1 < v < 2$ and $0 < \theta < 1$ are such that

$$\frac{1-\theta}{u} + \frac{\theta}{v} = 1$$

and if we define r and t by

$$n_2 - \frac{n_2}{u} = r - \frac{n_2}{2} \quad \text{and} \quad n_2 - \frac{n_2}{v} = t - \frac{n_2}{2},$$

then

$$(1-\theta)r + \theta t = \frac{n_2}{2}.$$

By real interpolation (see [Pe, Chapter 5, Thms. 2,4] and for special cases also [Tri1, Sections 1.18.2, 1.18.4, 1.18.6] and [BL, Sections 5.2 and 5.6]), we get that

$$\Big(\ell^r_\infty(L_2(A)), \ell^t_\infty(L_2(A))\Big)_{\theta,1} = \ell_1^{n_2/2}(L_2(A)), \tag{3.12}$$

where $A = \ell_1^{n_1/2}(L_2)$.

Step 1 of the proof of [ScT, Prop. 2.4.1] shows that (for fixed k_2)

$$\begin{aligned} &\|f_{k_1,k_2}(x^1,x^2) \mid \ell_1^{n_1/2}(L_2) \mid L_2\| \\ &\qquad \le c\|2^{k_2 n_2(1/u-1/2)} f_{k_1,k_2}(x^1,x^2) \mid \ell_1^{n_1/2}(L_2) \mid L_u\|. \end{aligned}$$

Multiplying by $2^{k_2 r}$ and taking sup-norms with respect to k_2 at first on the right hand side and then on the left hand side, we get

$$\begin{aligned} &\| f_{k_1,k_2}(x^1,x^2) \mid \ell_1^{n_1/2}(L_2) \mid \ell_\infty^r(L_2) \| \\ &\qquad \le c \| f_{k_1,k_2}(x^1,x^2) \mid \ell_1^{n_1/2}(L_2) \mid L_u(\ell_\infty^{n_2}) \|, \end{aligned} \tag{3.13}$$

and, moreover, the same inequality holds with v and t instead of u and r, respectively. One more invoking of real interpolation shows that

$$(L_u(B), L_v(B))_{\theta,1} = L_1(B), \tag{3.14}$$

where $B = \ell_\infty^{n_2}(\ell_1^{n_1/2}(L_2))$. The interpolation formulas (3.12) and (3.14) can be applied to systems $\{f_{k^1,k_2}(x^1,x^2)\}$ of entire analytic functions (cf. [Tri 2, Subsection 2.4.9] or [Pä] for the complex method). Consequently, we get from (3.13) that

$$\begin{aligned} &\| f_{k_1,k_2}(x^1,x^2) \mid \ell_1^{n_1/2}(L_2) \mid \ell_1^{n_2/2}(L_2) \| \\ &\qquad \le c \| f_{k_1,k_2}(x^1,x^2) \mid \ell_1^{n_1/2}(L_2) \mid L_1(\ell_\infty^{n_2}) \|. \end{aligned} \tag{3.15}$$

If u, v, θ are as above and r and t are given by

$$n_1 - \frac{n_1}{u} = r - \frac{n_1}{2} \quad \text{and} \quad n_1 - \frac{n_1}{v} = t - \frac{n_1}{2},$$

respectively, then $(1-\theta)r + \theta t = n_1/2$.

For fixed x^2, k_1, k_2 we shall use Nikol'skii's inequality with respect to x^1 (cf. [ScT, Thm. 1.6.2]) to show that

$$\begin{aligned} &\| 2^{k_2 n_2} f_{k_1,k_2}(x^1,x^2) \mid L_{2|x^1} \| \\ &\qquad \le c \| 2^{k_1 n_1 (1/u - 1/2)} 2^{k_2 n_2} f_{k_1,k_2}(x^1,x^2) \mid L_{u|x^1} \|, \end{aligned} \tag{3.16}$$

where $L_{u|x^1}$ indicates that the L_u-(quasi-)norm is to be taken with respect to x^1. Multiplying (3.16) by $2^{k_1 r}$ and taking sup-norms with respect to k_1, k_2, we get

$$\begin{aligned} &\| f_{k_1,k_2}(x^1,x^2) \mid L_{2|x^1} \mid \ell^{n_2}_{\infty|k_2} \mid \ell^r_{\infty|k_1} \| \\ &\qquad \le c \| f_{k_1,k_2}(x^1,x^2) \mid \ell_{\overline{\infty}}^{(n_1,n_2)} \mid L_{u|x^1} \|. \end{aligned} \tag{3.17}$$

Integration of (3.17) with respect to x^2 yields

$$\|f_{k_1,k_2}(x^1,x^2) \mid L_{2|x^1} \mid \ell^{n_2}_{\infty|k_2} \mid \ell^{r}_{\infty|k_1} \mid L_{1,x^2}\| \leq c\|f_{k_1,k_2}(x^1,x^2) \mid \ell^{(n_1,n_2)}_{\overline{\infty}} \mid L_{u|x^1} \mid L_{1|x^2}\|. \tag{3.18}$$

Real interpolation with the parameters θ and 1 on both sides of (3.18) gives

$$\|f_{k_1,k_2}(x^1,x^2) \mid L_{2|x^1} \mid \ell^{n_2}_{\infty|k_2} \mid \ell^{n_1/2}_{1|k_1} \mid L_{1,x^2}\| \leq c\|f_{k_1,k_2}(x^1,x^2) \mid \ell^{(n_1,n_2)}_{\overline{\infty}} \mid L_{\overline{1}}\| \tag{3.19}$$

and we see that the inequalities (3.11), (3.15), (3.17) can be combined to obtain

$$\|f \mid L_{\overline{\infty}}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})\| \leq c\|f_{k_1,k_2}(x^1,x^2) \mid \ell^{(n_1,n_2)}_{\overline{\infty}} \mid L_{\overline{1}}\| \leq c\|f \mid S^{(n_1,n_2)}_{\overline{1},\overline{2}}F\|. \tag{3.20}$$

Thus (3.7) is proved.

The inequality (3.8) follows simply by Hölder's inequality.

The proof (3.9) uses similar arguments as that of (3.7).

Finally, to prove (3.10) we start from

$$\begin{aligned}
&\|f(x^1,x^2) \mid L_{(\infty,1)}\| \\
&\qquad \leq c\sum_{k_2=0}^{\infty}\sum_{k_1=0}^{\infty} 2^{-k_2n_2}\|2^{k_1n_1/2+k_2n_2} f_{k_1,k_2}(x^1,x^2) \mid L_{2|x^1} \mid L_{1|x^2}\| \\
&\qquad \leq c\|f_{k_1,k_2}(x^1,x^2) \mid \ell^{n_1/2}_{1}(L_2) \mid L_1(\ell^{n_2}_{\infty})\|,
\end{aligned}$$

and then it is enough to use (3.19).

Step 3. The following mixed norm estimates hold:

$$\|f \mid L_{(q_1,\infty)}\| \leq c q_1^{1-1/p_1}\|f \mid S^{(n_1/p_1,n_2)}_{(p_1,1),\overline{2}}F\| \quad \text{if } 1\leq p_1<q_1<\infty, \tag{3.21}$$

$$\|f \mid L_{(\infty,q_2)}\| \leq c q_2^{1-1/p_2}\|f \mid S^{(n_1,n_2/p_2)}_{(1,p_2),\overline{2}}F\| \quad \text{if } 2\leq p_2<q_2<\infty, \tag{3.22}$$

$$\|f \mid L_{(q_1,1)}\| \leq c q_1^{1-1/p_1}\|f \mid S^{(n_1/p_1,n_2)}_{(p_1,1),\overline{2}}F\| \quad \text{if } p_1\geq 2, \tag{3.23}$$

$$\|f \mid L_{(1,q_2)}\| \leq c q_2^{1-1/p_2}\|f \mid S^{(n_1,n_2/p_2)}_{(1,p_2),\overline{2}}F\| \quad \text{if } p_2\geq 2. \tag{3.24}$$

We shall briefly go through the proof of (3.21). Similarly as in the proof of Lemma 3.4 (cf. (3.4) and the preceding estimates) we obtain

$$\begin{aligned} (3.25) \qquad & \|f(x^1,x^2) \mid L_{(q_1,\infty)}\| \\ & \leq c q_1^{1-1/p_1} \|f_{k_1,k_2}(x^1,x^2) \mid \ell_{p_1}^{n_1/p_1}(L_{p_1}) \mid \ell_1^{n_2/2}(L_2)\| \end{aligned}$$

and, analogously as in the proof of (3.15), it is possible to show that (3.25) implies

$$\begin{aligned} (3.26) \qquad & \|f(x^1,x^2) \mid L_{(q_1,\infty)}\| \\ & \leq c q_1^{1-1/p_1} \|f_{k_1,k_2}(x^1,x^2) \mid \ell_{p_1}^{n_1/p_1}(L_{p_1}) \mid L_1(\ell_\infty^{n_2})\|. \end{aligned}$$

Since $p_1 \geq 2$, (3.26) yields (3.21).

The proof of the remaining estimates (3.22)–(3.24) makes use of a similar technique.

Step 4. Now we are in a position to complete the proof of (3.5). By (3.7) and (3.8)–(3.10) we get (3.5) for $p_1 = p_1 = 1$. To get (3.5) for $p_1 = 1$ and $p_2 \geq 2$ we use (3.21), (3.23) and the mixed norm interpolation inequality

$$\|f \mid L_{(q_1,q_2)}\| \leq \|f \mid L_{(q_1,\infty)}\|^{1-1/q_2} \|f \mid L_{(q_1,1)}\|^{1/q_2}.$$

If $p_1 \geq 2$ and $p_2 \geq 2$, we proceed analogously, using (3.22), (3.24), and the inequality

$$\|f \mid L_{(q_1,q_2)}\| \leq \|f \mid L_{(\infty,q_2)}\|^{1-1/q_1} \|f \mid L_{(1,q_2)}\|^{1/q_1}$$

to prove (3.5) again. By virtue of our consideration at the beginning of the proof we are done. □

Remark 3.6. The estimate (3.5) cannot be improved with respect to the exponents $1 - 1/p_j$. It is enough to take $\widetilde{f}(x) = f(x^1) \otimes \cdots \otimes f(x^m)$, where f^j are the "extremal functions" [ET2, Subsection 2.7.1].

Corollary 3.7. *Under the assumptions of Theorem 3.5 the estimate*

$$\|f \mid L_q\| \leq c q^{m - \sum_{j=1}^{m} 1/p_j} \|f \mid S_{\overline{p},\overline{2}}^{\overline{r}} F(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m})\|$$

holds for every $q > \max\limits_{j=1,\ldots,m} p_j$.

Of particular interest are the reduced Sobolev spaces described in Sections 1 and studied in Section 2. In this case we have

Corollary 3.8. *Let* $\mathcal{M} = \mathcal{S}(N, n_1, \dots, n_m)$.

(i) *If* $q_j > n_j$ $(j = 1, \dots, m)$, $q > \max_j n_j$, *then*

$$\|f \mid L_{\overline{q}}\| \le c \prod_{j=1}^{m} q_j^{1-1/n_j} \|f \mid W_{\overline{n}}^{\mathcal{M}}(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m})\|,$$

where $\overline{n} = (n_1, \dots, n_m)$ *and*

$$\|f \mid L_q\| \le cq^{m(1-\sum_{j=1}^{m} 1/n_j)} \|f \mid W_{\overline{n}}^{\mathcal{M}}(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m})\|.$$

(ii) *If* $n_1 = \cdots = n_m = n > m$, *then*

$$\|f \mid L_q\| \le cq^{m(1-1/n)} \|f \mid W_n^{\mathcal{M}}(\mathbb{R}^N)\|.$$

This is a consequence of Theorem 3.6, Corollary 3.7 and the identity

$$W_p^{\mathcal{M}}(\mathbb{R}^N) = S_{\overline{p},\overline{2}}^{(1,\dots,1)} F(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}),$$

which follows from the observations following Definition 3.2.

Remark 3.9. Let us illustrate the previous corollary by a typical example. Let

$$\mathcal{S} = \{(1,0,1,0), (1,0,0,1), (0,1,1,0), (0,1,0,1)\}$$

and let $W_2^{\mathcal{S}}$ be the corresponding reduced Sobolev space. Then $m = n = 2$, thus for $q > 2$ we have

$$\|f \mid L_q\| \le cq \|f \mid W_2^{\mathcal{S}}\|$$

in contrast to

$$\|f \mid L_q\| \le cq^{1/2} \|f \mid W_2^2\|.$$

Both the last imbeddings are sharp.

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