ENTROPY NUMBERS IN WEIGHTED FUNCTION SPACES AND APPLICATIONS

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We intend to give a rather brief summary of recent investigations concerning entropy numbers in weighted function spaces. For the sake of brevity we will restrict ourselves to some special "model" cases. We will thus always refer to the original papers [3], [4] and [5] for details and generalizations. Most of the presented results have been obtained in a joint work with Prof. H. Triebel.

Introduction

We study function spaces of Lebesgue or (fractional) Sobolev type on $\mathbb{R}^n$, $L_p(\mathbb{R}^n)$ or $H_p^s(\mathbb{R}^n)$, resp., where $1 < p < \infty$, $s \in \mathbb{R}$. We assume the weight functions to be of polynomial type,

$$w(x) = (1 + |x|^2)^{\alpha/2}, \quad \alpha \in \mathbb{R}.$$

Then one can introduce weighted spaces in the following way:

$$L_p(\mathbb{R}^n, w(x)) : f \in S'(\mathbb{R}^n) \text{ with } \|f \mid L_p(\mathbb{R}^n, w(x))\| = \|w\, f \mid L_p(\mathbb{R}^n)\| < \infty,$$
$$H_p^s(\mathbb{R}^n, w(x)) : f \in S'(\mathbb{R}^n) \text{ with } \|f \mid H_p^s(\mathbb{R}^n, w(x))\| = \|\mathcal{F}^{-1}(1 + |x|^2)^{s/2} \mathcal{F} f \mid L_p(\mathbb{R}^n, w(x))\| < \infty,$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and its inverse, respectively, and $\| \cdot \mid L_p\|$ is the usual norm with respect to the Lebesgue measure. We adopt the following notations:

$$L_p(\alpha) := L_p(\mathbb{R}^n, (1 + |x|^2)^{\alpha/2}), 1 \le p \le \infty, \alpha \in \mathbb{R}, L_p = L_p(0) = L_p(\mathbb{R}^n),$$
$$H_p^s(\alpha) := H_p^s(\mathbb{R}^n, (1 + |x|^2)^{\alpha/2}), 1 < p < \infty, s \in \mathbb{R}, \alpha \in \mathbb{R}, \quad H_p^s = H_p^s(0) = H_p^s(\mathbb{R}^n).$$

One can show that $H_p^s(\alpha)$ is a Banach space for $1 < p < \infty$, $s \in \mathbb{R}$ and $\alpha \in \mathbb{R}$. Furthermore, the operator $f \longmapsto (1+|x|^2)^{\alpha/2} f$ is an isomorphic mapping from $H_p^s(\alpha)$ onto H_p^s, in particular, $\|(1+|x|^2)^{\alpha/2} f \mid H_p^s\|$ is an equivalent norm in $H_p^s(\alpha)$. In other words, one arrives at the same weighted Sobolev spaces by replacing the above definition by the one analogous to the L_p-case, i.e., for some $f \in S'$ and weight functions as above we get $f \in H_p^s(\mathbb{R}^n, w(x)) \Longleftrightarrow wf \in H_p^s$.

The main goal of this paper is a rather careful characterization of compact embedding operators by means of entropy numbers. In the first step we consider natural embeddings between weighted spaces of the above type.

Proposition. *Let $-\infty < s_2 < s_1 < \infty$, $1 < p_1 \leq p_2 < \infty$ and $\alpha_1, \alpha_2 \in \mathbb{R}$.*

(a) *$H_{p_1}^{s_1}(\alpha_1)$ is continuously embedded in $H_{p_2}^{s_2}(\alpha_2)$, $H_{p_1}^{s_1}(\alpha_1) \hookrightarrow H_{p_2}^{s_2}(\alpha_2)$, if, and only if, $s_1 - \frac{n}{p_1} \geq s_2 - \frac{n}{p_2}$ and $\alpha_2 \leq \alpha_1$.*

(b) *$H_{p_1}^{s_1}(\alpha_1)$ is compactly embedded in $H_{p_2}^{s_2}(\alpha_2)$, $H_{p_1}^{s_1}(\alpha_1) \hookrightarrow\hookrightarrow H_{p_2}^{s_2}(\alpha_2)$, if, and only if, $s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2}$ and $\alpha_2 < \alpha_1$.*

Recall that for $\alpha_1 = \alpha_2 = 0$ we have the unweighted case and hence the part (a) is nothing else than the well-known embedding theorem in $\mathbb{R}^n$. In contrast to that there is no counterpart of (b) in the unweighted situation in $\mathbb{R}^n$ as there is no compact embedding. In the sequel we will merely deal with this more interesting case and ask "how compact" these embeddings are. In this framework it turns out to be more convenient to restrict the regarded situation further. In view of the above proposition it is sufficient to concentrate on the case of only one weighted space (taking the other one unweighted):

$$\mathrm{id}\colon H_{p_1}^{s_1}(\alpha) \longrightarrow H_{p_2}^{s_2},$$
$$s_2 < s_1,\ 1 < p_1 \leq p_2 < \infty,\ \delta := \left(s_1 - \frac{n}{p_1}\right) - \left(s_2 - \frac{n}{p_2}\right) > 0,\ \alpha > 0,$$

(in the above notation this refers to $\alpha = \alpha_1 - \alpha_2$).

An application of Hölder's inequality shows that one can even admit certain parameters $p_2 < p_1$.

(∗) Let $\alpha > 0$, $1 < p_1 < \infty$, $\dfrac{1}{p_0} = \dfrac{1}{p_1} + \dfrac{\alpha}{n}$, $p_0 < p_2 < \infty$, $s_2 < s_1$ and $\delta > 0$. Then $\mathrm{id}\colon H_{p_1}^{s_1}(\alpha) \longrightarrow H_{p_2}^{s_2}$ is compact.

Entropy numbers

Let $T \in \mathcal{L}(A,B)$, i.e., $T\colon A \longrightarrow B$ be a continuous linear operator between complex Banach spaces A and B. Let U_A be the unit ball in A, $U_A = \{a \in A\colon \|a \mid A\| \le 1\}$. If T is compact, then for any $\varepsilon > 0$ there are finitely many balls of radius ε covering the image of the unit ball $T(U_A)$. This leads to the concept of entropy numbers.

Definition. Let $k \in \mathbb{N}$ and $T \in \mathcal{L}(A,B)$. Then the k-th entropy number $e_k(T)$ is the infimum of all numbers $\varepsilon > 0$ such that there exist 2^{k-1} balls in B of radius ε covering $T(U_A)$.

We want to mention two elementary properties of entropy numbers. First, it is known that

$$\lim_{n\to\infty} e_n(T) = 0 \quad \Longleftrightarrow \quad T \text{ compact}.$$

Therefore we may regard these quantities as a measure of compactness. In our case we are only interested in the rate of their decay. Moreover, the multiplicativity of entropy numbers turns out very useful for our later purposes: Let $R \in \mathcal{L}(A,B)$, $S \in \mathcal{L}(B,C)$ and $T \in \mathcal{L}(C,D)$. Then

$$e_k(T \circ S \circ R) \le \|T\|\, e_k(S)\, \|R\|, \quad k \in \mathbb{N}.$$

Before presenting our main result we would like to mention an earlier one by Edmunds and Triebel, which showed itself both as a forerunner and an essential tool for proving our proposition. This result is related to function spaces on domains.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞ boundary $\partial\Omega$. For $s \in \mathbb{R}$, $1 < p < \infty$, one introduces

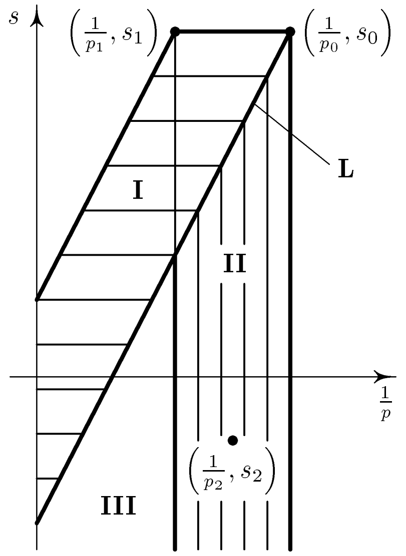
$$H^s_p(\Omega) = \{f \in D'(\Omega)\colon \exists g \in H^s_p(\mathbb{R}^n),\ g_{|\Omega} = f\}.$$

Let id_Ω be the compact embedding operator $\mathrm{id}_\Omega\colon H^{s_1}_{p_1}(\Omega) \longrightarrow H^{s_2}_{p_2}(\Omega)$, where $1 < p_1,\ p_2 < \infty$ and $s_1 - s_2 > n \max(\frac{1}{p_1} - \frac{1}{p_2}, 0)$. Then

$$e_k(\mathrm{id}_\Omega\colon H^{s_1}_{p_1}(\Omega) \longrightarrow H^{s_2}_{p_2}(\Omega)) \sim k^{-(s_1-s_2)/n},\ k \in \mathbb{N}$$

(see [1] and [2]).

THE MAIN THEOREM



I: $0 < \delta < \alpha$

II: $\delta > \alpha,\ \frac{1}{p_1} \le \frac{1}{p_2} < \frac{1}{p_0}$

III: $\delta > \alpha,\ \frac{1}{p_2} < \frac{1}{p_1}$

L: $\delta = \alpha,\ \frac{1}{p_2} < \frac{1}{p_0}$

Theorem. *Let the assumptions* $(*)$ *be satisfied. Using the above notation we have*

(a) *in* **I**: $\quad e_k(\mathrm{id}) \sim k^{-(s_1-s_2)/n}$,

(b) *in* **II**: $\quad e_k(\mathrm{id}) \sim k^{-\alpha/n+1/p_2-1/p_1}$,

(c) *in* **III**:
$$ck^{-\alpha/n+1/p_2-1/p_1} \le e_k(\mathrm{id}) \le c_\varepsilon k^{-\alpha/n+1/p_2-1/p_1}(1+\log k)^{\varepsilon-1/p_2+1/p_1},$$

(d) *on* **L** *(with additional assumptions on* p_1 *and* p_2*)*:
$$e_k(\mathrm{id}) \sim k^{-(s_1-s_2)/n}(1+\log k)^{\alpha/n}.$$

Let us add a few remarks. In view of the above proposition one realizes that the situation of weighted function spaces on $\mathbb{R}^n$ is similar to the one of function spaces on domains if the power of the weight is “large enough” $(\alpha > \delta)$. In **II** only the p-parameters (and the weight) determine the behaviour of entropy numbers, whereas the situation in **III** is not so clear at the moment. In the above diagram we indicated those “level lines” at which the exponent of k (in (a) and (b)) is constant. The most interesting, but also difficult, case to handle is the critical one where $\delta = \alpha$ (on $\mathbf{L}$). Especially in the more general case of B-spaces or F-spaces (for simplicity excluded here) the results are rather surprising, we refer to [5] for more details. Finally we want to repeat once more that all the above considerations have been done in the wider context of Besov and Triebel-Lizorkin spaces (including also quasi-Banach cases), see [3] and [4].

Applications

We intend to give an example of an application of the above result. A wider range of such examples is presented in [4].

Let B be the operator $Bf = b_2(x)(\mathrm{id} - \Delta)^{-m} b_1(x) f$, where b_1 and b_2 are assumed to belong to spaces of type $L_r(\alpha)$, Δ is the Laplacian, and $m \in \mathbb{N}$. Then some typical questions are, for example, in which spaces L_p, $1 < p < \infty$, the operator $B \colon L_p \longrightarrow L_p$ acts compactly and whether we can say anything about its eigenvalues.

To give a satisfactory answer using our above result we need another important tool, namely Carl's celebrated inequality:

Let X be a complex Banach space and let $B \colon X \longrightarrow X$ be linear and compact. Let $e_k(B)$ be the k-th entropy number of B and $\{\mu_k\}$ the sequence of its eigenvalues counted with respect to their algebraic multiplicity and in a decreasing order, $|\mu_1| \geq |\mu_2| \geq \cdots \geq 0$. Then

$$\Big(\prod_{m=1}^{k} |\mu_m| \Big)^{1/k} \leq \inf_{n \in \mathbb{N}} 2^{n/2k} e_n(B), \;\; k \in \mathbb{N},$$

in particular,

$$|\mu_k| \leq \sqrt{2}\, e_k(B), \quad k \in \mathbb{N}.$$

Now we are able to show all this is consistent.

Proposition 1. *Let $m \in \mathbb{N}$, α_1 and α_2 be real numbers with $\alpha = \alpha_1 + \alpha_2 > 0$, $1 \leq r_1 \leq \infty$, $1 \leq r_2 \leq \infty$, $\frac{1}{r_1} + \frac{1}{r_2} < \min\big(1, \frac{2m}{n}\big)$, and $b_1 \in L_{r_1}(\alpha_1)$, $b_2 \in L_{r_2}(\alpha_2)$.*

For any p, $1 < p < \infty$, which satisfies $\frac{1}{r_2} < \frac{1}{p} < 1 - \frac{1}{r_1}$, the operator

$$B = b_2 \circ (\mathrm{id} - \Delta)^{-m} \circ b_1$$

is compact in L_p.

Furthermore,
(i) *if* $\frac{1}{r_1}+\frac{1}{r_2}>\frac{2m-\alpha}{n}$, *then*

$$|\mu_k| \le c\|b_1 \mid L_{r_1}(\alpha_1)\|\,\|b_2 \mid L_{r_2}(\alpha_2)\| k^{-2m/n},$$

(ii) *if* $\frac{1}{r_1}+\frac{1}{r_2}=\frac{2m-\alpha}{n}$, *then*

$$|\mu_k| \le c\|b_1 \mid L_{r_1}(\alpha_1)\|\,\|b_2 \mid L_{r_2}(\alpha_2)\| \times k^{-2m/n}(1+\log k)^{2m/n+\min(1/p-1/r_2,1-1/p-1/r_1)},$$

(iii) *if* $\frac{1}{r_1}+\frac{1}{r_2}<\frac{2m-\alpha}{n}$, *then*

$$|\mu_k| \le c_\varepsilon\|b_1 \mid L_{r_1}(\alpha_1)\|\,\|b_2 \mid L_{r_2}(\alpha_2)\| \times k^{-\alpha/n-1/r_1-1/r_2}(1+\log k)^{\varepsilon+1/r_1+1/r_2}.$$

(In the case $r_1=r_2=\infty$ *one can choose* $\varepsilon=0$.*)*

Let us briefly point out some ideas of the proof to explain where the assumptions are used and in what way the above theorem and Carl's inequality are involved.

We decompose B as $B=b_2\circ \mathrm{id}\circ(\mathrm{id}-\Delta)^{-m}\circ b_1$ where

$$\begin{cases} b_1\colon L_p \longrightarrow L_q(\alpha_1) \quad \text{with } \frac{1}{q}=\frac{1}{p}+\frac{1}{r_1}, \\ (\mathrm{id}-\Delta)^{-m}\colon L_q(\alpha_1) \longrightarrow H_q^{2m}(\alpha_1), \\ \mathrm{id}\colon H_q^{2m}(\alpha_1) \longrightarrow L_t(-\alpha_2), \\ b_2\colon L_t(-\alpha_2) \longrightarrow L_p \quad \text{with } \frac{1}{p}=\frac{1}{t}+\frac{1}{r_2}. \end{cases}$$

We know that $\mathrm{id}\colon H_q^{2m}(\alpha_1) \longrightarrow L_t(-\alpha_2)$ is compact if $2m-\frac{n}{q}+\frac{n}{t}>0$ ("$\delta>0$") and $\alpha=\alpha_1-(-\alpha_2)>0$ (and then, by Hölder's inequality for b_1 and b_2, also B is compact). Note that

$$2m-\frac{n}{q}+\frac{n}{t}>0 \iff 2m-\frac{n}{p}-\frac{n}{r_1}+\frac{n}{p}-\frac{n}{r_2}>0 \iff \frac{2m}{n}>\frac{1}{r_1}+\frac{1}{r_2}.$$

The multiplicativity of entropy numbers then gives

$$e_k(B) \le c\|b_1 \mid L_{r_1}(\alpha_1)\|\,\|b_2 \mid L_{r_2}(\alpha_2)\|\, e_k(\mathrm{id}).$$

By the above theorem we have to distinguish the cases "$\delta < \alpha$", "$\delta = \alpha$" and "$\delta > \alpha$". This corresponds to the cases (i)–(iii) in the above proposition. Finally Carl's inequality yields the desired result.

Similarly we can handle more complicated operators B, e.g. degenerate pseudodifferential operators, because only the mapping property of the part $(\mathrm{id} - \Delta)^{-m}$ is used (and not its particular structure). So we will finish this short report with another example which seems much more complicated at the first glance. In fact, using known results about mapping properties of ΨDO's, see e.g. [4], our methods apply to these cases in the same way.

Proposition 2. *Suppose $\varkappa > 0$, $0 \le \gamma \le 1$, and $b(x,D) \in \Psi^{-\varkappa}_{1,\gamma}$, $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha = \alpha_1 + \alpha_2 > 0$, $1 \le r_1 \le \infty$, $1 \le r_2 \le \infty$ with $\frac{1}{r_1} + \frac{1}{r_2} < \min\left(1, \frac{\varkappa}{n}\right)$, and $b_1 \in L_{r_1}(\alpha_1)$, $b_2 \in L_{r_2}(\alpha_2)$.*

For any p, $1 < p < \infty$, satisfying additionally $\frac{1}{r_2} < \frac{1}{p} < 1 - \frac{1}{r_1}$, and any $\beta \in \mathbb{R}$ the operator

$$B = b_2 \circ b(x,D) \circ b_1$$

is compact in $L_p(\beta)$.

Furthermore,

(i) *if $\frac{1}{r_1} + \frac{1}{r_2} > \frac{\varkappa - \alpha}{n}$, then*

$$|\mu_k| \le c \|b_1 \mid L_{r_1}(\alpha_1)\| \, \|b_2 \mid L_{r_2}(\alpha_2)\| \, k^{-\varkappa/n},$$

(ii) *if $\frac{1}{r_1} + \frac{1}{r_2} = \frac{\varkappa - \alpha}{n}$, then*

$$|\mu_k| \le c \|b_1 \mid L_{r_1}(\alpha_1)\| \, \|b_2 \mid L_{r_2}(\alpha_2)\| \times k^{-\varkappa/n} (1 + \log k)^{\varkappa/n + \min(1/p - 1/r_2, 1 - 1/p - 1/r_1)},$$

(iii) *if $\frac{1}{r_1} + \frac{1}{r_2} < \frac{\varkappa - \alpha}{n}$, then*

$$|\mu_k| \le c_\varepsilon \|b_1 \mid L_{r_1}(\alpha_1)\| \, \|b_2 \mid L_{r_2}(\alpha_2)\| \times k^{-\alpha/n - 1/r_1 - 1/r_2} (1 + \log k)^{\varepsilon + 1/r_1 + 1/r_2}.$$

(In the case of $r_1 = r_2 = \infty$ one can choose $\varepsilon = 0$.)

References

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