ON COMPACT EMBEDDINGS OF SOBOLEV SPACES AND EXTENSION OPERATORS WHICH PRESERVE SOME SMOOTHNESS

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1. Introduction

Let $W_p^k(\Omega)$ denote the Sobolev space of functions $f \in L_p(\Omega)$ with distributional derivatives up to the kth order in $L_p(\Omega)$, having norm

$$\|f\|_{W_p^k(\Omega)} = \sum_{0 \le |\alpha| \le k} \|D^\alpha f\|_{L_p(\Omega)}$$

in the usual notation. It is well known that an open set Ω in $\mathbb{R}^n$ $(n \ge 2)$ with a sufficiently smooth boundary $\partial\Omega$ has the W_p^k-extension property, i.e., there exists a continuous linear map $E\colon W_p^k(\Omega) \to W_p^k(\mathbb{R}^n)$ such that $(Ef)(x) = f(x)$ for $x \in \Omega$. An example is a rotation of a set of the form

$$\{x = (x', x_n) \in \mathbb{R}^n \colon x_n < \Phi(x')\} \tag{1}$$

for some Lipschitz-continuous function Φ on $\mathbb{R}^{n-1}$. More generally, Ω has the W_p^k-extension property if $\partial\Omega$ satisfies the minimally smooth condition, i.e., it has a locally finite open covering $\{U_i\}$ which is such that $U_i \cap \Omega = U_i \cap \Omega_i$ for each i, where Ω_i is a rotation of a set of the form (1) for some Lipschitz function Φ_i, and the Lipschitz norms

$$\sup\{|\Phi_i(x) - \Phi_i(y)|\,|x-y|^{-1} \colon x, y \in \mathbb{R}^{n-1}, x \ne y\}$$

are bounded. In fact, Stein (see [7]), extending an earlier result of Calderón, proved the remarkable result that if

$$W(\Omega) = \bigcup_{\substack{k \in \mathbb{N}_0 \\ p \in [1,\infty]}} W_p^k(\Omega)$$

and Ω has a minimally smooth boundary, then there exists a linear extension $E\colon W(\Omega) \to W(\mathbb{R}^n)$ such that, for any $k \in \mathbb{N}_0, p \in [1,\infty], E\colon W_p^k(\Omega) \to W_p^k(\mathbb{R}^n)$ is continuous.

In [8] Vodop'yanov, Gol'dstein and Latfulin proved that a simply connected bounded domain Ω in $\mathbb{R}^2$ has the W_2^1-extension property if and only if it is a quasidisk, i.e., $\partial\Omega$ is the image of a circle under a quasi-conformal mapping of the plane into itself. This last condition was shown by Ahlfors [1] to be equivalent to the following property: there exists a constant c such that for all $x, y \in \partial\Omega$, $|x-z| \leq c|x-y|$, where z is any point on the subarc of $\partial\Omega$ of smaller diameter joining x and y. A quasidisk $\Omega \subset \mathbb{R}^2$ has the W_p^k-extension property for all $p \in [1,\infty), k \in \mathbb{N}$. The converse is however false for $p \neq 2$ (see [6, Theorem 1.5.2]). What is known is that if a simply connected $\Omega \subset \mathbb{R}^2$ and its complement have the W_p^k-extension property, then its boundary is a quasidisk, see [4]. For $n > 2$ the problem was investigated by Jones in [5] where a class of domains is introduced which includes those with minimally-smooth boundary and coincides with the class of quasidisks in the plane.

If Ω is a bounded open set with the W_p^1-extension property, then the embedding map $I\colon W_p^1(\Omega) \hookrightarrow L_p(\Omega)$ is compact. For if B is an open ball containing $\overline{\Omega}$ and $\phi \in C_0^\infty(B)$ is such that $\phi = 1$ on Ω, then $f \mapsto \phi E f$ maps $W_p^1(\Omega)$ continuously into $\mathring{W}_p^1(B)$ (the closure of $C_0^\infty(B)$ in $W_p^1(B)$) and since $\mathring{W}_p^1(B) \hookrightarrow L_p(B)$ is compact, so is $W_p^1(\Omega) \hookrightarrow L_p(\Omega)$. However the extension property is not necessary for the embedding I to be compact. For instance, if Ω is of the form (1) with Φ Hölder-continuous with exponent $\gamma \in (0,1)$ then, in general, it does not have the W_p^1-extension property, but it was proved in [2] that there exists a bounded linear extension of $W_p^1(\Omega)$ into $W_p^\gamma(\mathbb{R}^n)$, a Sobolev space of non-integer order. Since $W_p^\gamma(B) \hookrightarrow L_p(B)$ is still compact, it follows that $W_p^1(\Omega) \hookrightarrow L_p(\Omega)$ is compact for such domains.

The question we wish to discuss is the following: if Ω is such that $W_p^1(\Omega) \hookrightarrow L_p(\Omega)$ is compact, does there exist a function space $\Lambda(\mathbb{R}^n)$ and a continuous extension $W_p^1(\Omega) \to \Lambda(\mathbb{R}^n)$ such that the map $f \mapsto \phi f\colon \Lambda(\mathbb{R}^n) \to L_p(B)$ is compact, where $\overline{\Omega} \subset B$ and $\phi \in C_0^\infty(B)$. Our approach is based on spaces of functions with an amount of smoothness which is sufficiently small to be retained on setting a function on Ω to be zero outside Ω, but is nevertheless adequate for the compactness of $\Lambda(\mathbb{R}^n) \to L_p(\Omega)$. These are the spaces of generalized smoothness defined in Section 2.

2. Spaces of generalized smoothness

Definition 2.1. Let $0 < p \le \infty$. A function λ will be called an L_p-smoothness if it is positive and non-decreasing on $(0,\infty)$ and is such that

1. $\lim_{h\to 0+} \lambda(h) = 0$,
2. $h^{1/p'}/\lambda(h) \in L_p(0,1)$, where $1/p' = 1 - 1/p$,
3. $[h^{1/p}\lambda(h)]^{-1} \in L_p(1,\infty)$ if $0 < p < \infty$.

Example 2.2. (i) $\lambda(h) = h^l$, $l \in \mathbb{R}$, is an L_p-smoothness function if and only if $0 < l < 1$ for $0 < p < \infty$ and $0 < l \le 1$ for $p = \infty$.

(ii) If λ is a positive non-decreasing function satisfying 3. in Definition 2.1 and $\lambda(h) = h^l |\ln h|^\gamma$, for small enough h, then λ is an L_p-smoothness function if and only if one of the following conditions holds:

(a) $l = 0$, $\gamma < 0$,
(b) $0 < l < 1$, $\gamma \in (-\infty,\infty)$,
(c) $l = 1$, $\gamma > 1/p$ for $0 < p < \infty$, or $\gamma \ge 0$ for $p = \infty$.

Definition 2.3. Let λ be an L_p-smoothness function, $0 < p \le \infty$, and $(a,b) \subset (-\infty,\infty)$. We shall say that $f \in W_p^{\lambda(\cdot)}(a,b)$ if f is measurable on (a,b) and

$$\|f\|_{W_p^{\lambda(\cdot)}(a,b)} := \|f\|_{L_p(a,b)} + \|f\|_{w_p^{\lambda(\cdot)}(a,b)} < \infty, \tag{2}$$

where

$$\|f\|_{w_p^{\lambda(\cdot)}(a,b)} = \begin{cases} \left(\int\limits_a^b \int\limits_a^b \frac{|f(x)-f(y)|^p}{\lambda(|x-y|)^p} \frac{\mathrm{d}x\ \mathrm{d}y}{|x-y|} \right)^{1/p} & \text{for } 0 < p < \infty \\ \sup\limits_{\substack{x,y\in(a,b)\\ x\neq y}} \frac{|f(x)-f(y)|}{\lambda(|x-y|)} & \text{for } p = \infty. \end{cases} \tag{3}$$

We shall denote by $w_p^{\lambda(\cdot)}(a,b)$ the set of all those measurable functions f which are such that $\|f\|_{w_p^{\lambda(\cdot)}(a,b)} < \infty$.

We shall concentrate on domains Ω in $\mathbb{R}^2$ of the form

$$\Omega = \{(x_1,x_2) \in \mathbb{R}^2 : a < x_1 < b, \phi_1(x_1) < x_2 < \phi_2(x_1)\}, \tag{4}$$

$$\Omega = \{(x_1,x_2) \in \mathbb{R}^2 : c < x_2 < d, \psi_1(x_2) < x_1 < \psi_2(x_2)\}, \tag{5}$$

but our analysis applies to more general domains in $\mathbb{R}^n$, $n \ge 2$.

Definition 2.4. Let λ be an L_p-smoothness function, $0 < p \leq \infty$, and suppose that Ω has the form (4), (5). We shall write $f \in W_p^{\lambda(\cdot)}(\Omega)$ if f is measurable on Ω and

$$\|f\|_{W_p^{\lambda(\cdot)}(\Omega)} := \|f\|_{L_p(\Omega)} + \|f\|_{w_{p,1}^{\lambda(\cdot)}(\Omega)} + \|f\|_{w_{p,2}^{\lambda(\cdot)}(\Omega)} < \infty, \tag{6}$$

where

$$\|f\|_{w_{p,1}^{\lambda(\cdot)}(\Omega)} = \big\| \, \|f(\cdot, x_2)\|_{w_p^{\lambda(\cdot)}(\psi_1(x_2),\psi_2(x_2))} \big\|_{L_p(c,d)} \tag{7}$$

and

$$\|f\|_{w_{p,2}^{\lambda(\cdot)}(\Omega)} = \big\| \, \|f(x_1, \cdot)\|_{w_p^{\lambda(\cdot)}(\phi_1(x_1),\phi_2(x_1))} \big\|_{L_p(a,b)}. \tag{8}$$

The $W_p^{\lambda(\cdot)}(\cdot)$ are our spaces of generalized smoothness. Many other examples occur in the literature of spaces of L_p functions satisfying some generalized smoothness condition; see [3, Section 3] for references. Our main concern is with domains of the type (4), (5) which are strongly degenerate, for instance

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| < a, \phi(|x_1|) < x_2 < \phi(a)\} \tag{9}$$

where $\phi(x)$ tends arbitrarily slowly to zero as x tends to zero, and hence the cusp is of arbitrary sharpness. When we refer to (9) below we shall assume that ϕ is non-negative, continuous and increasing on $[0, a)$ with $\phi(0) = 0$, $\phi'(a) > 0$.

One of our main theorems in [3], namely Theorem 5.2, describes circumstances under which the extension by zero operator, defined by

$$T_0 f(x) = \begin{cases} f(x), & x \in \Omega \\ 0, & x \notin \Omega, \end{cases}$$

maps $W_p^{\lambda(\cdot)}(\Omega)$ continuously into $W_p^{\nu(\cdot)}(\mathbb{R}^2)$. A corollary of this result is

Lemma 2.5. *Let Ω be of the form (9), $0 < p < \infty$, and $\lambda(\cdot)$ an L_p-smoothness function which is such that*

$$\lim_{h \to 0+} \frac{h\lambda'(h)}{\lambda(h)} = c_1, \quad \limsup_{h \to 0+} \frac{\lambda(2h)}{\lambda(h)} < 2^{1/p}. \tag{10}$$

Suppose also that

$$\lim_{h\to 0+} \frac{h\phi'(h)}{\phi(h)} = c_2, \tag{11}$$

and for some $h_0 > 0$ and all $h \in (0, h_0)$

$$c_3 h \le \phi(h) \le 2\phi(h/2). \tag{12}$$

Then $\lambda \circ \phi$ is an L_p-smoothness function and

$$T_0 \colon W_p^{\lambda(\cdot)}(\Omega) \to W_p^{\lambda\circ\phi(\cdot)}(\mathbb{R}^2). \tag{13}$$

If $1 \le p < \infty$ and $\lambda(\cdot)$ is an L_p-smoothness function, it is easy to show that there is a continuous embedding

$$W_p^1(\Omega) \hookrightarrow W_p^{\lambda(\cdot)}(\Omega) \tag{14}$$

(cf. [3, Lemma 3.7]), and from this and [3, Theorem 5.2] it follows that there exists an L_p-smoothness function $\lambda(\cdot)$ such that

$$T_0 \colon W_p^1(\Omega) \to W_p^{\lambda(\cdot)}(\mathbb{R}^2) \tag{15}$$

is continuous. Finally, the embedding

$$W_p^{\lambda(\cdot)}(\mathbb{R}^2) \to L_p(\Omega) \tag{16}$$

is compact and hence so is the embedding

$$W_p^1(\Omega) \to L_p(\Omega). \tag{17}$$

Detailed proofs of these results may be found in [3, Section 5]. Thus for a domain Ω of the form (9) which can have a cusp of arbitrary severity, there is a continuous extension (15) and the spaces $W_p^{\lambda(\cdot)}(\mathbb{R}^2)$ retain sufficient smoothness for (16), and consequently (17), to be compact.

3. An example

Let

$$\Omega = \{(x,y) \colon 0 < |x| < a \text{ if } 0 < y \le b,\ 0 \le |x| < a \text{ if } b < y < a\}, \tag{18}$$

$$\widetilde{\Omega} = (\overline{\Omega})^\circ = (-a,a) \times (0,a). \tag{19}$$

It is clear that Ω is not a quasidisk and there is no continuous extension of $W_p^1(\Omega)$ into $W_p^1(\widetilde{\Omega})$; it is easy to see this directly since the function f defined by $f(x,y) = \theta(x)\phi(y)$, where θ is the Heaviside function and $\phi \in C_0^\infty[0,b)$ is equal to 1 in a right neighbourhood of $y=0$, is in $W_p^1(\Omega)$ but not in $W_p^1(\widetilde{\Omega})$. However

$$W_p^1(\Omega) \hookrightarrow L_p(\Omega) \tag{20}$$

is compact. For we can write $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$, where the Ω_i are rectangles, and the embeddings $W_p^1(\Omega_i) \hookrightarrow L_p(\Omega_i)$ are compact.

We claim that there exists an L_p-smoothness function $\lambda(\cdot)$ such that

$$W_p^{\lambda(\cdot)}(\Omega) = W_p^{\lambda(\cdot)}(\widetilde{\Omega}) \tag{21}$$

in the sense of being isomorphic, and consequently, in view of [3, Theorem 5.2], there is a continuous extension of $W_p^{\lambda(\cdot)}(\Omega)$ into $W_p^{\nu(\cdot)}(\mathbb{R}^2)$ for some L_p-smoothness function ν, which in turn confirms the compactness of (20) on using the compactness of (16). The domain (18) is not of the form (9) but the methods developed for proving the results quoted in Section 2 can still be used. The subsequent analysis serves a dual purpose: in proving (21) it illustrates some of the methods in [3]; also it motivates the Hardy-type inequality for differences discussed in Section 4 below.

By analogy with Definition 2.4, we define the following norm on $W_p^{\lambda(\cdot)}(\Omega)$:

$$\|f\|_{W_p^{\lambda(\cdot)}(\Omega)} := \|f\|_{L_p(\Omega)} + \|f\|_{w_{p,1}^{\lambda(\cdot)}(\Omega)} + \|f\|_{w_{p,2}^{\lambda(\cdot)}(\Omega)},$$

where

$$\begin{aligned} \|f\|^p_{w_{p,1}^{\lambda(\cdot)}(\Omega)} &= \big\| \, \|f(\cdot,x_2)\|_{w_p^{\lambda(\cdot)}(0,a)} \big\|^p_{L_p(0,b)} + \big\| \, \|f(\cdot,x_2)\|_{w_p^{\lambda(\cdot)}(-a,0)} \big\|^p_{L_p(0,b)} \\ &\quad + \big\| \, \|f(\cdot,x_2)\|_{w_p^{\lambda(\cdot)}(-a,a)} \big\|^p_{L_p(b,a)}, \\ \|f\|^p_{w_{p,2}^{\lambda(\cdot)}(\Omega)} &= \big\| \, \|f(x_1,\cdot)\|_{w_p^{\lambda(\cdot)}(0,a)} \big\|_{L_p(-a,a)}. \end{aligned}$$

We shall prove that for $1 \le p < \infty$ and a suitable L_p-smoothness function $\lambda(\cdot)$,

$$\|f\|_{W_p^{\lambda(\cdot)}(\widetilde{\Omega})} \asymp \|f\|_{W_p^{\lambda(\cdot)}(\Omega)}, \tag{22}$$

where $\asymp$ indicates that the quotient of the two sides is bounded between two positive constants. We have

$$\begin{aligned}
\|f\|^p_{w^{\lambda(\cdot)}_{p,1}(\widetilde{\Omega})} &= \int_0^a \Big\{ \int_{-a}^a \int_{-a}^a \frac{|f(x_1,x_2)-f(y_1,x_2)|^p}{\lambda(|x_1-y_1|)^p|x_1-y_1|} \, \mathrm{d}x_1 \, \mathrm{d}y_1 \Big\} \, \mathrm{d}x_2 \\
&= \int_0^b \Big\{ \|f(\cdot,x_2)\|^p_{w^{\lambda(\cdot)}_p(-a,0)} + \|f(\cdot,x_2)\|^p_{w^{\lambda(\cdot)}_p(0,a)} + 2K^p_p(f,x_2) \Big\} \, \mathrm{d}x_2 \\
&\quad + \int_b^a \Big\{ \int_{-a}^a \int_{-a}^a \frac{|f(x_1,x_2)-f(y_1,x_2)|^p}{\lambda(|x_1-y_1|)^p|x_1-y_1|} \, \mathrm{d}x_1 \, \mathrm{d}y_1 \Big\} \, \mathrm{d}x_2,
\end{aligned}$$

where

$$K^p_p(f,x_2) = \int_{-a}^0 \int_0^a \frac{|f(x_1,x_2)-f(y_1,x_2)|^p}{\lambda(|x_1-y_1|)^p|x_1-y_1|} \, \mathrm{d}x_1 \, \mathrm{d}y_1.$$

Thus

$$\begin{aligned}
&K_p(f,x_2) \\
&\quad \le \Big(\int_{-a}^0 \int_0^a \frac{|f(x_1,x_2)-f(-y_1,x_2)|^p}{\lambda(|x_1-y_1|)^p|x_1-y_1|} \, \mathrm{d}x_1 \, \mathrm{d}y_1 \Big)^{1/p} \\
&\qquad + \Big(\int_{-a}^0 \int_0^a \frac{|f(y_1,x_2)-f(-y_1,x_2)|^p}{\lambda(|x_1-y_1|)^p|x_1-y_1|} \, \mathrm{d}x_1 \, \mathrm{d}y_1 \Big)^{1/p} \\
&\quad = \Big(\int_0^a \int_0^a \frac{|f(x_1,x_2)-f(y_1,x_2)|^p}{\lambda(|x_1+y_1|)^p|x_1+y_1|} \, \mathrm{d}x_1 \, \mathrm{d}y_1 \Big)^{1/p} \\
&\qquad + \Big(\int_0^a |f(y_1,x_2)-f(-y_1,x_2)|^p \Big(\int_{-a}^0 \frac{\mathrm{d}x_1}{\lambda(|x_1-y_1|)^p|x_1-y_1|} \Big) \, \mathrm{d}y_1 \Big)^{1/p} \\
&\quad \le \|f(\cdot,x_2)\|_{w^{\lambda(\cdot)}_p(0,a)} + \mathcal{T}_p(f,x_2)
\end{aligned}$$

where $\mathcal{T}_p(f,x_2)$ is the last expression in the penultimate line. Also we have

$$\mathcal{T}_p(f,x_2) \le \mathcal{K}_p(f,x_2) + \|f(\cdot,x_2)\|_{w^{\lambda(\cdot)}_p(0,a)}$$

and so

$$\mathcal{T}_p(f, x_2) + \|f(\cdot, x_2)\|_{w_p^{\lambda(\cdot)}(0,a)} \asymp \mathcal{K}_p(f, x_2) + \|f(\cdot, x_2)\|_{w_p^{\lambda(\cdot)}(0,a)}. \tag{23}$$

Let

$$\lambda^*(y)^{-p} := \int_y^\infty \frac{du}{\lambda(u)^p u}, \quad 0 < y < \infty, \tag{24}$$

and $\lambda^*(-y) = \lambda^*(y)$. Then

$$\mathcal{T}_p(f, x_2) \leq \|f(\cdot, x_2)/\lambda^*(\cdot)\|_{L_p(-a,a)}. \tag{25}$$

If we can prove that

$$\|f(\cdot, x_2)/\lambda^*(\cdot)\|_{L_p(0,a)} \leq K(a)\{\|f(\cdot, x_2)\|_{L_p(0,a)} + \|f(\cdot, x_2)\|_{w_p^{\lambda(\cdot)}(0,a)}\}, \tag{26}$$

it will follow from (23) and (24) that

$$\|f\|_{w_{p,1}^{\lambda(\cdot)}(\widetilde{\Omega})} \leq K(a)\|f\|_{w_{p,1}^{\lambda(\cdot)}(\Omega)},$$

and hence (22). For the more complicated case connected with an investigation of the domain Ω defined in (9), we shall need a more precise inequality than (26), and this is the inequality discussed in Section 4 below. It is the heart of the method in [3]. The function $\lambda^*(\cdot)$ defined in (24) is, for $1 \leq p < \infty$, a modified L_p-smoothness function as defined in [3, Definition 3.8].

4. A Hardy-type inequality for differences

The following results are proved in [3].

Theorem 4.1. *Let $0 < p < \infty$, $0 < A \leq \infty$, $0 \leq B < \infty$, and let ω be a non-negative measurable function on $(0, A)$ which is such that*

$$v(x) = B + \int_x^A \omega(t)\ \mathrm{d}t < \infty \tag{27}$$

for all $x \in (0, A)$. Suppose that there exists $\alpha \in (1, 2)$ such that

$$v(x) \leq \alpha v(2x), \quad x \in (0, A/2). \tag{28}$$

Then, for all $a \in (0, A]$ and all $f \in L_p(0, a)$,

$$(29) \qquad \Big(\int_0^a |f(x)|^p v(x) \, \mathrm{d}x \Big)^{1/p} \le c_1 \Big\{ \Big(v(a) \int_0^a |f(x)|^p \, \mathrm{d}x \Big)^{1/p} + \Big(\int_0^a \int_0^a |f(x) - f(y)|^p \omega(|x-y|) \, \mathrm{d}x \, \mathrm{d}y \Big)^{1/p} \Big\},$$

where c_1 is independent of f and a.

When $a = \infty$, we have for all $f \in L_p(0, \infty)$,

$$(30) \quad \Big(\int_0^\infty |f(x)|^p v(x) \, \mathrm{d}x \Big)^{1/p} \le c_1 \Big(\int_0^\infty \int_0^\infty |f(x) - f(y)|^p \omega(|x-y|) \, \mathrm{d}x \, \mathrm{d}y \Big)^{1/p}.$$

Corollary 4.2. *Let $0 < p < \infty$, $0 < A \le \infty$, and let ω be a non-negative measurable function on $(0, A)$ which satisfies (27). Suppose also that*

$$(31) \qquad \limsup_{x \to 0+} \{ v(x)/v(2x) \} < 2.$$

Then for all $a \in (0, A]$ and $f \in L_p(0, A)$,

$$(32) \quad \Big(\int_0^a |f(x)|^p v(x) \, \mathrm{d}x \Big)^{1/p} \le c_2 \Big\{ \Big([1 + v(a)] \int_0^a |f(x)|^p \, \mathrm{d}x \Big)^{1/p} + \Big(\int_0^a \int_0^a |f(x) - f(y)|^p \omega(|x-y|) \, \mathrm{d}x \, \mathrm{d}y \Big)^{1/p} \Big\},$$

where c_2 does not depend on f and a.

For (26) we need $\omega(h) = \lambda(h)^{-p} h^{-1}$, $A = \infty$, $B = 0$, and hence $v(t) = \lambda^*(t)^{-p}$ in (27). An example of an L_p-smoothness function $\lambda(\cdot)$ which satisfies (31) is $\lambda(h) = h^l$, $0 < l < 1/p$. This case was considered by Yakovlev in [9].

It is particularly relevant to our objective to note that the hypothesis of Theorem 4.1 allows $v(x)$ to tend arbitrarily slowly to ∞ as $x \to 0$. Indeed, for any non-increasing, non-negative function ψ defined on $(0, A)$ which is such that $\lim_{x \to 0+} \psi(x) = \infty$ there exists a function v satisfying the hypothesis and which is such that $\lim_{x\to 0+} v(x) = \infty$ and $v(x) \le \psi(x)$. The piecewise linear function defined by

$$v(A) = \psi(A), \quad v(2^{-k}A) = \min\{\psi(2^{-k+1}A), \alpha v(2^{-k+1}A)\}, \quad k = 1, 2, \dots,$$

has these properties.

For a discussion of the sharpness of the inequalities (29), (30) and (32) we refer the reader to [3, Section 2].

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