HOMEOMORPHISMS WHICH ACT ON BESOV SPACES

Gérard Bourdaud and Winfried Sickel

1. Introduction

Every homeomorphism φ of the real line gives rise to a linear operator

$$T_\varphi \colon f \mapsto f \circ \varphi.$$

Our aim is to characterize the homeomorphisms for which T_φ is a bounded operator on the homogeneous Besov space $\dot{B}_p^{s,q}$ $(0 < s < 1)$ —or its inhomogeneous counterpart $B_p^{s,q}$.

A lot of work has been done on this problem by the Siberian school, including important results on the higher dimensional case. In some sense, the present paper is the continuation of Vodop'yanov's work [16].

Mostly, we shall have to reject pathological homeomorphisms. We consider the set $\mathcal{H}$ of all increasing homeomorphisms φ from $\mathbb{R}$ *onto* $\mathbb{R}$ such that

$$|E| = 0 \quad \Longrightarrow \quad |\varphi(E)| = 0,$$

for all Borel sets E. If $\varphi \in \mathcal{H}$, then one has

$$\int_{\mathbb{R}} f(\varphi(x))\varphi'(x)\,\mathrm{d}x = \int_{\mathbb{R}} f(x)\,\mathrm{d}x, \tag{1}$$

for all non-negative Lebesgue-measurable functions f. Similarly, we denote by $\mathcal{H}^{-1}$ the set of homeomorphisms φ such that $\varphi^{-1} \in \mathcal{H}$.

Let us recall that the homogeneous Besov space $\dot{B}_p^{s,q}$ $(0 < s < 1,\ 1 \leq p \leq \infty,\ 1 \leq q \leq \infty)$ is defined as the set of functions f such that

$$\|f\|_{\dot{B}_p^{s,q}} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x+h) - f(x)|^p\,\mathrm{d}x\right)^{q/p} \frac{\mathrm{d}h}{|h|^{1+sq}}\right)^{1/q} < \infty$$

(with the usual modifications for $p = \infty$ and/or $q = \infty$). The quotient space of $\dot{B}^{s,q}_p$ by the constant functions is a Banach space with respect to the above norm. The inhomogeneous Besov space is given by

$$B^{s,q}_p = \dot{B}^{s,q}_p \cap L^p.$$

This space turns into a Banach space if it is equipped with the norm

$$\|f\|_{B^{s,q}_p} = \|f\|_{\dot{B}^{s,q}_p} + \|f\|_p.$$

We shall also use the classical Sobolev spaces

$$\dot{W}^1_p = \{f : f' \in L^p\}$$

and $W^1_p = \dot{W}^1_p \cap L^p$. For more details about these function spaces like equivalent norms, embeddings etc., see e.g. [2], [4], [11], [14], [15].

Definition. A homeomorphism φ of the real line is *quasi-conformal* if there exists an $M \geq 1$ such that

$$\frac{1}{M} \leq \frac{\varphi(x+h) - \varphi(x)}{\varphi(x) - \varphi(x-h)} \leq M,$$

for all $x \in \mathbb{R}$ and $h > 0$.

Let us begin with the results of Vodop'yanov ([16, Theorems 2.1, 2.2]).

Theorem 1. *Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $0 < s < 1$, $s \neq 1/p$, and let φ be a homeomorphism of the real line. Then T_φ is an automorphism of the Besov space $\dot{B}^{s,q}_p$ (resp. $B^{s,q}_p$) if and only if φ and φ^{-1} are Lipschitz mappings.*

Theorem 2. *Let $2 \leq p < \infty$ and let φ be a homeomorphism of the real line. Then T_φ is an automorphism of the critical homogeneous Besov space $\dot{B}^{1/p,p}_p$ if and only if φ is quasi-conformal.*

Our results are the following:

Theorem 3. *Let $1 \leq p < \infty$, $1 \leq q \leq \infty$, $0 < s < 1/p$, and $\varphi \in \mathcal{H}$. Then the following three statements are equivalent:*

- *T_φ is a bounded operator on $\dot{B}^{s,q}_p$;*
- *T_φ is a bounded operator on $B^{s,q}_p$;*
- *φ^{-1} is a Lipschitz mapping.*

Theorem 4. *Let $1 < p \le \infty$, $1 \le q \le \infty$, $1/p < s < 1$, and $\varphi \in \mathcal{H}$. Then the following two statements are equivalent:*

- *T_φ is a bounded operator on $\dot{B}_p^{s,q}$;*
- *φ is a Lipschitz mapping.*

Theorem 5. *Let $1 < p < \infty$, $1 \le q \le \infty$, $1/p < s < 1$, and $\varphi \in \mathcal{H}$. Then the following two statements are equivalent:*

- *T_φ is a bounded operator on $B_p^{s,q}$;*
- *φ is a Lipschitz mapping and $\sup_{x\in\mathbb{R}}\big(\varphi^{-1}(x+1) - \varphi^{-1}(x)\big) < \infty$.*

Theorem 6. *Let $1 < p < \infty$ and $\varphi \in \mathcal{H} \cap \mathcal{H}^{-1}$. Then T_φ is a bounded operator on the critical homogeneous Besov space $\dot{B}_p^{1/p,p}$ if and only if φ is quasi-conformal.*

Our paper is organized as follows. In the Section 2 we prove the *necessity* of the conditions given above and in the Section 3 the *sufficiency*. Then we examine the critical inhomogeneous Besov space, for which we have only partial results. As usual, C denotes a constant whose exact value is not important. We shall often say that "φ acts on a space E" if T_φ is a bounded operator on E. If $1 \le p \le +\infty$, p' is the conjugate exponent, given by $p' = p/(p-1)$.

2. Proofs: necessary conditions

2.1. The case $0 < s < 1/p$.

Let us consider $a < b$ and

$$f = \chi_{[a,b]}.$$

It is well known that f belongs to $\dot{B}_p^{s,q}$ and that a homogeneity argument gives

$$\|f\|_{\dot{B}_p^{s,q}} \le C(b-a)^{1/p-s} \tag{2}$$

and

$$\|f\|_p \le C(b-a)^{1/p}.$$

For $0 < b-a \le 1$, one arrives at

$$\|f\|_{\dot{B}_p^{s,q}} \le \|f\|_{B_p^{s,q}} \le C(b-a)^{1/p-s}.$$

The boundedness of T_φ implies

$$(3)\quad \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(\varphi(x+h)) - f(\varphi(x))|^p \, \mathrm{d}x \right)^{q/p} \frac{\mathrm{d}h}{|h|^{1+sq}} \leq C(b-a)^{q(1/p-s)}.$$

For $x \in [\varphi^{-1}(a), \varphi^{-1}(b)]$ and $h > \varphi^{-1}(b) - \varphi^{-1}(a)$, we have $|f(\varphi(x+h)) - f(\varphi(x))| = 1$ and the estimates (2) and (3) give

$$\left(\varphi^{-1}(b) - \varphi^{-1}(a)\right)^{q(1/p-s)} \leq C(b-a)^{q(1/p-s)}.$$

Thus

$$\frac{\varphi^{-1}(b) - \varphi^{-1}(a)}{b-a} \leq C.$$

We leave to the reader the easy modification of the last proof for the cases $p = \infty$ or $q = \infty$.

2.2. The case $1/p < s < 1$; necessity of $\varphi' \in L^\infty$.

We consider the family $(\psi_\varepsilon)_{\varepsilon>0}$ of continuous compactly-supported functions such that

$$\psi'_\varepsilon = \frac{1}{\varepsilon}(\chi_{[-1-\varepsilon,-1]} - \chi_{[1,1+\varepsilon]}).$$

These functions have the following properties:

- $\operatorname{supp} \psi_\varepsilon \subset [-\varepsilon - 1, \varepsilon + 1]$,
- $\psi_\varepsilon(x) = 1$ for $x \in [-1, +1]$,
- $\|\psi_\varepsilon\|_{\dot{B}^{s,q}_p} \leq C\varepsilon^{1/p-s}$.

In order to prove the third property, one may use the embedding

$$\dot{B}^{s-1/p+1,q}_1 \subset \dot{B}^{s,q}_p$$

(cf. [2, (6.3.1)]).

We obtain

$$\|\psi_\varepsilon\|_{\dot{B}^{s,q}_p} \leq C\|\psi'_\varepsilon\|_{\dot{B}^{s-1/p,q}_1},$$

and the inequality (2) leads to

$$\|\psi'_\varepsilon\|_{\dot{B}^{s-1/p,q}_1} \leq C\varepsilon^{1/p-s}.$$

On the other hand, one has

$$\|\psi_\varepsilon\|_p \leq (2+2\varepsilon)^{1/p};$$

thus, for $\varepsilon \leq 1$, the inhomogeneous norm satisfies the same estimate as does the homogeneous one.

Let us consider three numbers $a < b < c$ such that

$$\varphi(c) - \varphi(b) \leq \frac{\varphi(b) - \varphi(a)}{2} \leq 1. \tag{4}$$

Define

$$r = \frac{\varphi(b) - \varphi(a)}{2}, \quad x_0 = \frac{\varphi(b) + \varphi(a)}{2}, \quad \varepsilon = \frac{\varphi(c) - \varphi(b)}{r},$$

and

$$f(x) = \psi_\varepsilon\Big(\frac{x - x_0}{r}\Big).$$

The condition (4) implies

$$\|f\|_{B_p^{s,q}} \leq C\big(\varphi(c) - \varphi(b)\big)^{1/p-s}.$$

The boundedness of T_φ leads to

$$\int_{c-b}^{c-a} \bigg(\int_{c-h}^{b} \mathrm{d}x\bigg)^{q/p} \frac{\mathrm{d}h}{h^{1+sq}} \leq C\big(\varphi(c) - \varphi(b)\big)^{q(1/p-s)}.$$

By the change of the variables $z = (c-b)^{-1}h$ we derive from this

$$\int_{1}^{\frac{c-a}{c-b}} (z-1)^{q/p} \frac{\mathrm{d}z}{z^{1+sq}} \leq C\left(\frac{\varphi(c) - \varphi(b)}{c-b}\right)^{q(1/p-s)}. \tag{5}$$

Let us choose a, b, c in the following manner: b is a Lebesgue point of φ', a and c are sufficiently close to b in order to guarantee (4) and

$$c - b \leq \frac{c-a}{2}.$$

Then the term on the left-hand side of (5) is greater than the positive constant

$$\int_1^2 (z-1)^{q/p} \frac{\mathrm{d}z}{z^{1+sq}};$$

by taking $c \to b+$, one gets $\varphi'(b) \leq C$.

2.3. The inhomogeneous space $\dot{B}^{s,q}_p$.
It remains to demonstrate the necessity of

$$\sup_{x\in\mathbb{R}}\big(\varphi^{-1}(x+1)-\varphi^{-1}(x)\big)<\infty.$$

We test T_φ on the functions

$$f_a(x)=\psi_1(x-a),\quad a\in\mathbb{R}.$$

Obviously, the family $\{f_a\}_{a\in\mathbb{R}}$ is uniformly bounded in all spaces under consideration here. The assumed boundedness of T_φ results in

$$\|f_a\circ\varphi\|_p\le C$$

for all $a\in\mathbb{R}$. Integrating $|f_a\circ\varphi|^p$ over the interval $[\varphi^{-1}(a),\varphi^{-1}(a+1)]$ yields

$$\varphi^{-1}(a+1)-\varphi^{-1}(a)\le C.$$

2.4. The critical homogeneous space.
Following an idea of L. Ahlfors [1], we reduce the problem to the *normalized* homeomorphisms, i.e., the homeomorphisms φ such that $\varphi(0)=0$ and $\varphi(1)=1$.

Let $\mathcal{G}$ be the set of the affine transforms

$$x\mapsto rx+a$$

$(r\neq 0,\ a\in\mathbb{R})$. Let us observe that, for all $g\in\mathcal{G}$, T_g is an *isometry* of $\dot{B}^{1/p,q}_p$ onto itself; this implies:

Lemma 1. *Let φ be a homeomorphism such that T_φ is a bounded operator on $\dot{B}^{1/p,q}_p$. Then, for every $g\in\mathcal{G}$, $T_{\varphi\circ g}$ and $T_{g\circ\varphi}$ are also bounded operators on $\dot{B}^{1/p,q}_p$; moreover, one has*

$$\|T_\varphi\|=\|T_{\varphi\circ g}\|=\|T_{g\circ\varphi}\|,$$

where $\|\cdot\|$ denotes the operator norm in $\mathcal{L}(\dot{B}^{1/p,q}_p)$.

Lemma 2. *There exists a constant $C(p) > 0$ such that for every normalized homeomorphism φ acting on $\dot{B}_p^{1/p,p}$ one has*

$$\varphi^{-1}\left(\tfrac{1}{2}\right) \leq 1 - \exp\left(-1 - C(p)\|T_\varphi\|^p\right).$$

Proof. Let us consider

$$f(x) = \psi_2(4x-1).$$

We have $f(x) = 1$ for $0 < x < 1/2$ and $f(x) = 0$ for $x > 1$; if $a = \varphi^{-1}(1/2)$, we obtain

$$\int_{1-a}^{1} \left(\int_{1-h}^{a} \mathrm{d}x \right) \frac{\mathrm{d}h}{h^2} \leq \|T_\varphi\|^p \|\psi_2\|^p_{\dot{B}_p^{1/p,p}}.$$

It is then easy to arrive at the desired inequality. □

Lemma 3. *Let $1 < p < \infty$ and $\varphi \in \mathcal{H} \cap \mathcal{H}^{-1}$. If T_φ is a bounded operator on $\dot{B}_p^{1/p,p}$, then $T_{\varphi^{-1}}$ is a bounded operator on $\dot{B}_{p'}^{1/p',p'}$, with the same operator norm.*

Proof. Let f be a C^∞ function in $\dot{B}_{p'}^{1/p',p'}$. One has

$$\|T_{\varphi^{-1}}(f)\|_{\dot{B}_{p'}^{1/p',p'}} = \|T_{\varphi^{-1}}(f)'\|_{\dot{B}_{p'}^{-1/p,p'}} = \|(f' \circ \varphi^{-1})(\varphi^{-1})'\|_{\dot{B}_{p'}^{-1/p,p'}}.$$

Then we notice that $\dot{B}_{p'}^{-1/p,p'}$ is the dual space of $\dot{B}_p^{1/p,p}$; this gives

$$\begin{aligned}
\|T_{\varphi^{-1}}(f)\|_{\dot{B}_{p'}^{1/p',p'}} &= \sup \left\{ \left| \int (f' \circ \varphi^{-1})(\varphi^{-1})' g \,\mathrm{d}x \right| : \|g\|_{\dot{B}_p^{1/p,p}} \leq 1 \right\} \\
&= \sup \left\{ \left| \int f'(g \circ \varphi) \,\mathrm{d}x \right| : \|g\|_{\dot{B}_p^{1/p,p}} \leq 1 \right\} \\
&\leq \|T_\varphi\| \, \|f'\|_{\dot{B}_{p'}^{-1/p,p'}} \leq \|T_\varphi\| \, \|f\|_{\dot{B}_{p'}^{1/p',p'}}.
\end{aligned}$$

Hence we deduce

$$\|T_{\varphi^{-1}}\|_{\mathcal{L}(\dot{B}_{p'}^{1/p',p'})} \leq \|T_\varphi\|_{\mathcal{L}(\dot{B}_p^{1/p,p})}.$$

We obtain the *equality* of the two norms by replacing φ by φ^{-1} in the above proof. □

Lemma 4. *There exists a function $\delta_p\colon (0,+\infty) \to (0,1)$ such that for every normalized homeomorphism φ acting on $\dot{B}_p^{1/p,p}$ one has*

$$\delta_p(\|T_\varphi\|) \le \varphi\big(\tfrac{1}{2}\big) \le 1 - \delta_p(\|T_\varphi\|).$$

Proof. By Lemma 3, we can apply Lemma 2 to φ^{-1}; we obtain

$$\varphi\big(\tfrac{1}{2}\big) \le 1 - \delta_p(\|T_\varphi\|), \tag{6}$$

where

$$\delta_p(t) = \exp\big(-1 - C(p')t^{p'}\big).$$

Then we consider $\sigma \in \mathcal{G}$ such that $\sigma(x) = 1 - x$; by Lemma 1, we can apply (6) to the normalized homeomorphism $\sigma \circ \varphi \circ \sigma$; this gives

$$\varphi\big(\tfrac{1}{2}\big) \ge \delta_p(\|T_\varphi\|).$$

□

Now we consider $\varphi \in \mathcal{H} \cap \mathcal{H}^{-1}$ such that T_φ is a bounded operator on $\dot{B}_p^{1/p,p}$. We are going to prove that φ is quasi-conformal. For any interval $[a, b]$, we define $g_1, g_2 \in \mathcal{G}$ by the following formulae:

$$g_1(t) = (b-a)t + a, \quad g_2(t) = \frac{t - \varphi(a)}{\varphi(b) - \varphi(a)} .$$

By Lemma 1, $\varphi_0 = g_2 \circ \varphi \circ g_1$ is a normalized homeomorphism such that $\|T_{\varphi_0}\| = \|T_\varphi\|$. If we apply Lemma 4 to φ_0, we obtain

$$\delta_p(\|T_\varphi\|)\big(\varphi(b) - \varphi(a)\big) \le \varphi\Big(\frac{a+b}{2}\Big) - \varphi(a) \le \big(1 - \delta_p(\|T_\varphi\|)\big)\big(\varphi(b) - \varphi(a)\big),$$

from which we deduce

$$\begin{aligned} \frac{\delta}{1-\delta}\left(\varphi\Big(\frac{a+b}{2}\Big) - \varphi(a)\right) &\le \varphi(b) - \varphi\Big(\frac{a+b}{2}\Big) \\ &\le \frac{1-\delta}{\delta}\left(\varphi\Big(\frac{a+b}{2}\Big) - \varphi(a)\right) . \end{aligned}$$

3. Proofs: sufficient conditions

Our main tool will be the interpolation theory. Therefore, we begin with a collection of obvious results.

- By means of the formula
$$\int_{\mathbb{R}} |f'(\varphi(x))|^p \varphi'(x)^p \, \mathrm{d}x = \int_{\mathbb{R}} |f'(z)|^p \varphi'(\varphi^{-1}(z))^{p-1} \, \mathrm{d}z$$
we see that $\varphi' \in L^\infty$ is sufficient for T_φ to be a bounded operator on $\dot{W}^1_p$ $(1 < p < \infty)$.
- Again by the above formula it follows that there is no additional condition in case of $\dot{W}^1_1$.
- The formula (1) yields the sufficiency of $(\varphi^{-1})' \in L^\infty$ in case of L^p-spaces $(1 \le p < \infty)$.
- Using the two preceding statements we get that $(\varphi^{-1})' \in L^\infty$ is a sufficient condition in case of W^1_1.
- $\varphi' \in L^\infty$ is a sufficient condition in case of the Hölder spaces $\dot{B}^{s,\infty}_\infty$ $(0 < s < 1)$.

3.1. The case $0 < s < 1/p$.

We first deal with homogeneous spaces. We employ the real interpolation formula
$$(\dot{W}^1_1, L^1)_{\Theta,q} = \dot{B}^{s,q}_1, \quad s = 1 - \Theta, \quad 0 < \Theta < 1$$
(cf. [4, (6.2.4)]). Then the statements 2 and 3 yield the boundedness of T_φ on $\dot{B}^{s,q}_1$ $(0 < s < 1,\ 1 \le q \le \infty)$ provided that $(\varphi^{-1})' \in L^\infty$. We continue with the help of the complex interpolation formula
$$[\dot{B}^{s_0,1}_1, L^{p_1}]_\Theta = \dot{F}^{s,q}_p, \quad s = (1-\Theta)s_0,$$
$$\frac{1}{p} = (1-\Theta) + \frac{\Theta}{p_1}, \quad \frac{1}{q} = (1-\Theta) + \frac{\Theta}{2}, \quad 0 < \Theta < 1$$
(cf. [6]). (The Triebel-Lizorkin spaces $\dot{F}^{s,q}_p$ will not play an important role here, cf. [14], [15], [6] for details.) If $s_0 \uparrow 1$ and $p_1 \uparrow \infty$, then each $s < 1/p$ is admissible: we obtain the boundedness of T_φ on $\dot{F}^{s,q}_p$ for $1 < p < \infty$, $0 < s < 1/p$ and some q, which depends on s and p. To switch to Besov spaces and to remove this dependence on q we employ
$$(\dot{F}^{s_0,q_0}_p, L^p)_{\Theta,q} = \dot{B}^{s,q}_p, \quad s = (1-\Theta)s_0, \quad s_0 > 0, \quad 0 < \Theta < 1$$
(cf. [5]).

Now, consider the inhomogeneous spaces. But there is no principal difference. Instead of the statement 2 one starts with the statement 4. For the complex interpolation formula we refer the reader to [14, (2.4.2)] and [2, (6.4)].

3.2. The homogeneous space for $1/p < s < 1$.
We proceed as above. We first employ

$$[\dot{B}^{s_0,\infty}_\infty, \dot{W}^1_{p_1}]_\Theta = \dot{F}^{s,q}_p, \quad s = (1-\Theta)s_0 + \Theta, \qquad (7)$$
$$\frac{1}{p} = \frac{\Theta}{p_1}, \quad \frac{1}{q} = \frac{\Theta}{2}, \quad 0 < \Theta < 1$$

(cf. [6]), if $1 < p_1 < \infty$. Further, we need the interpolation formula

$$(\dot{F}^{s_0,q_0}_p, \dot{W}^1_p)_{\Theta,q} = \dot{B}^{s,q}_p, \quad s = (1-\Theta)s_0 + \Theta, \quad 0 < \Theta < 1 \qquad (8)$$

for $1 < p < \infty$, $0 < s_0 < 1$ (cf. [6]). Here our strategy looks as follows. We first apply (7) with $s_0 \downarrow 0$ and $p_1 \downarrow 1$. Afterwards we use (8).

3.3. The inhomogeneous space for $s > 1/p$.
What we need is the following inequality

$$\sum_{j\in\mathbb{Z}} \Big(\sup_{j\le x\le j+1} |f(x)| \Big)^p \le C \|f\|^p_{B^{s,q}_p}. \qquad (9)$$

Let us assume (9) for a while: we get

$$\begin{aligned}\|f\circ\varphi\|^p_p &= \sum_{j\in\mathbb{Z}} \int_j^{j+1} |f(x)|^p (\varphi^{-1})'(x)\,\mathrm{d}x \\ &\le \sum_{j\in\mathbb{Z}} \Big(\sup_{j\le x\le j+1} |f(x)|^p \Big) \big(\varphi^{-1}(j+1) - \varphi^{-1}(j)\big) \le C\|f\|^p_{B^{s,q}_p}.\end{aligned}$$

Together with Subsection 3.2, this proves sufficiency for inhomogeneous Besov spaces. It remains to prove (17). Using the embedding

$$B^{s,q}_p \subset B^{s-\varepsilon,p}_p,$$

it suffices to prove it with $p = q$. We start from the so-called ℓ^p-localization property of $B^{s,p}_p$:

$$\sum_{j\in\mathbb{Z}} \|\theta(\cdot - j)f\|^p_{B^{s,p}_p} \le C(\theta)\|f\|^p_{B^{s,p}_p}, \qquad (10)$$

for all $\theta \in \mathcal{D}(\mathbb{R})$ (cf. for instance [3]). The hypothesis $s > 1/p$ leads to

$$\sum_{j\in\mathbb{Z}} \|\theta(\cdot - j)f\|^p_\infty \le C(\theta)\|f\|^p_{B^{s,p}_p}.$$

A suitable choice of θ gives (9).

3.4. The critical homogeneous space.
Step 1: $2 \le p < +\infty$. For the sake of completeness, we give here Vodop'yanov's proof. Let φ be a quasi-conformal homeomorphism of the real line. Following Ahlfors [1, Chapter IV, Theorem 2], there exists a quasi-conformal homeomorphism $\widetilde{\varphi}$ of the whole plane $\mathbb{R}^2$ such that $\widetilde{\varphi}(x,0) = (\varphi(x),0)$ $(x \in \mathbb{R})$. Next we consider the bounded *trace* operators:

$$Tr\colon \dot{B}_p^{2/p,p}(\mathbb{R}^2) \to \dot{B}_p^{1/p,p}(\mathbb{R}) \quad (p>2), \quad Tr\colon \dot{W}_2^1(\mathbb{R}^2) \to \dot{B}_2^{1/2,2}(\mathbb{R}).$$

These operators satisfy

$$Tr(f)(x) = f(x,0) \quad (x \in \mathbb{R}).$$

The existence of the bounded *extension* operators E, such that $Tr \circ E = Id$, is a classical property of the Besov and Sobolev spaces. Thus we have reduced the problem to proving that the quasi-conformal mapping $\widetilde{\varphi}$ defines a bounded operator $T_{\widetilde{\varphi}}$ on both $\dot{B}_p^{2/p,p}(\mathbb{R}^2)$ and $\dot{W}_2^1(\mathbb{R}^2)$.

For the Sobolev space $\dot{W}_2^1(\mathbb{R}^2)$ this follows from the estimate

$$\left|\frac{\partial\widetilde{\varphi}}{\partial x}(x,y)\right|^2 + \left|\frac{\partial\widetilde{\varphi}}{\partial y}(x,y)\right|^2 \le C \det J\widetilde{\varphi}(x,y)) \text{ (a.e.)},$$

where $J\widetilde{\varphi}(x,y)$ is the Jacobian matrix of $\widetilde{\varphi}$ (see [9], [7]). For the Besov space, we use the interpolation formula

$$\dot{B}_p^{2/p,p}(\mathbb{R}^2) = \left(BMO(\mathbb{R}^2), \dot{W}_2^1(\mathbb{R}^2)\right)_{2/p,p}$$

(see [12]) and Reimann's theorem [13], which asserts that $T_{\widetilde{\varphi}}$ is a bounded operator on $BMO(\mathbb{R}^2)$.
Step 2: $1 < p < 2$. We use the following two classical properties: as $\widetilde{\varphi}$ is a quasi-conformal mapping of the plane, the same is true for $\widetilde{\varphi}^{-1}$ ([1, Chapter II]); the trace of $\widetilde{\varphi}^{-1}$ on the real line, i.e. φ^{-1}, is a quasi-conformal mapping ([1, Chapter IV, Theorem 1]). Thus we can apply the previous step to φ^{-1}, then we conclude with the help of Lemma 3.

Remark. It is a nice idea of Vodop'yanov to transfer the problem from the real line to the plane. Interpolation with $BMO(\mathbb{R})$ is an alternative method, but it does not give the optimal result. Let us explain why. We start with the interpolation formula

$$\left(BMO(\mathbb{R}), \dot{F}_1^{1,2}(\mathbb{R})\right)_{1/p,p} = \dot{B}_p^{1/p,p}(\mathbb{R}).$$

Recall that

$$\dot F_1^{1,2}(\mathbb{R}) = \{f\colon f' \in \dot F_1^{0,2}(\mathbb{R})\}$$

and that $\dot F_1^{0,2}(\mathbb{R})$ is the Hardy space whose dual space is $BMO(\mathbb{R})$. By duality—as in Lemma 3—and interpolation, we obtain the following result: *if φ acts on $BMO(\mathbb{R})$, then φ acts on $\dot B_p^{1/p,p}(\mathbb{R})$.* P. Jones [8] studied the homeomorphisms which act on $BMO(\mathbb{R})$: they are characterized by $\varphi' \in A_\infty$ (the weight class of Muckenhoupt). But $\varphi' \in A_\infty$ is a stronger property than quasi-conformality.

4. Remarks on the critical inhomogeneous space

As far as we know, the full characterization of admissible changes of variables in $B_p^{1/p,p}$ is an open problem. First we recall that the condition

$$\sup_{x\in\mathbb{R}} \big(\varphi^{-1}(x+1) - \varphi^{-1}(x)\big) < \infty \tag{11}$$

is *necessary* for T_φ to act on $B_p^{1/p,p}$ (the proof given in (2.3) is true for all $s > 0$).

Proposition 1 ([16]). *Let $s > 0$. If φ is a homeomorphism acting on $\dot B_p^{s,p}$ and satisfying the condition (11), then φ acts also on $B_p^{s,p}$.*

Proof. We shall follow Vodop'yanov once again. Let θ be a positive C^∞ function such that

$$\sum_{j\in\mathbb{Z}} \theta(x-j) = 1$$

and—say— $\operatorname{supp}\theta \subset [-1/4, 5/4]$. Then we have

$$\sum_{j\in\mathbb{Z}} \theta(\varphi(x)-j)^p \ge 2^{-p},$$

which implies

$$\|f\circ\varphi\|_p^p \le 2^p \sum_j \int_{\mathbb{R}} |f_j\circ\varphi|^p \,\mathrm{d}x,$$

where $f_j(x) = f(x)\theta(x-j)$. Now we consider

$$\alpha = \sup_{x\in\mathbb{R}} (\varphi^{-1}(x+1) - \varphi^{-1}(x)).$$

Let us recall the Friedrichs inequality

$$\|f\|_p \leq C(I)\|f\|_{\dot{B}_p^{s,q}},$$

which is satisfied by all functions f with support in an open interval I (see e.g. [10, p. 350]). We apply it to the function

$$(f_j \circ \varphi)(x + \varphi^{-1}(j))$$

and the interval $(-\alpha, 2\alpha)$; this leads to

$$\|f \circ \varphi\|_p^p \leq C \sum_j \|f_j \circ \varphi\|_{\dot{B}_p^{s,p}}^p \leq C \sum_j \|f_j\|_{B_p^{s,p}}^p \leq C\|f\|_{B_p^{s,p}}^p$$

(the last inequality follows from (10)). □

Proposition 1 and Theorem 6 imply the following:

Proposition 2. *Let $1 < p < +\infty$ and $\varphi \in \mathcal{H} \cap \mathcal{H}^{-1}$. If φ is quasi-conformal and satisfies* (11)*, then φ acts on $B_p^{1/p,p}$.*

Vodop'yanov conjectured that the above condition is also *necessary* for φ to act on $B_p^{1/p,p}$ (see [16, Theorem 2.2]). We shall show that it is false:

Proposition 3. *Let $1 < p < +\infty$ and $\varphi \in \mathcal{H} \cap \mathcal{H}^{-1}$. If φ has the two following properties:*

- *there exists $q > 1$ such that*

$$\sup_{a \in \mathbb{R}} \int_a^{a+1} (\varphi^{-1})'(x)^q \,\mathrm{d}x < +\infty, \tag{12}$$

- *there exist some constants $C > 0$ and $\delta > 0$ such that, for almost all (x, y) with $|x - y| < \delta$, one has*

$$\left(\frac{\varphi(x) - \varphi(y)}{x - y}\right)^2 \frac{1}{\varphi'(x)} \frac{1}{\varphi'(y)} \leq C, \tag{13}$$

then T_φ is a bounded operator on $B_p^{1/p,p}$.

Proof. Recall that

$$\|f\|_p + \left(\iint_{|x-y|<\delta} (x - y)^{-2}|f(x) - f(y)|^p \,\mathrm{d}x \,\mathrm{d}y\right)^{1/p}$$

is an equivalent norm in $B_p^{1/p,p}$ (cf. [15, (2.6.1)]). In order to estimate $\|f\circ\varphi\|_p$, we take

$$r=\frac{qp}{q-1};$$

the inequality $p\le r<+\infty$ yields the embeddings

$$B_p^{1/p,p}\subset B_r^{1/r,p}\subset L^r;$$

then (10) leads to

$$\sum_{j\in\mathbb{Z}}\left(\int_j^{j+1}|f(x)|^r\,\mathrm{d}x\right)^{p/r}\le C\|f\|^p_{B_p^{1/p,p}};$$

this implies

$$\begin{aligned}\|f\circ\varphi\|_p^p&=\sum_{j\in\mathbb{Z}}\int_j^{j+1}|f(x)|^p(\varphi^{-1})'(x)\,\mathrm{d}x\\&\le\sum_{j\in\mathbb{Z}}\left(\int_j^{j+1}|f(x)|^r\,\mathrm{d}x\right)^{p/r}\left(\int_j^{j+1}(\varphi^{-1})'(x)^q\,\mathrm{d}x\right)^{1-p/r}\\&\le C\|f\|^p_{B_p^{1/p,p}}.\end{aligned}$$

On the other hand, (13) implies

$$\begin{aligned}&\iint_{|x-y|<\delta}(x-y)^{-2}|(f\circ\varphi)(x)-(f\circ\varphi)(y)|^p\,\mathrm{d}x\,\mathrm{d}y\\&\qquad\le C\iint_{|x-y|<\delta}\frac{\varphi'(x)\varphi'(y)}{(\varphi(x)-\varphi(y))^2}|f(\varphi(x))-f(\varphi(y))|^p\,\mathrm{d}x\,\mathrm{d}y\\&\qquad\le C\iint(u-v)^{-2}|f(u)-f(v)|^p\,\mathrm{d}u\,\mathrm{d}v.\end{aligned}$$

□

It may be convenient to replace (13) by a more manageable condition. That is the purpose of the next proposition.

Proposition 4. *For every $\varphi \in \mathcal{H}$, the following statement implies* (13)*: there exist some $\delta > 0$ and $C > 0$ such that*

$$\varphi'(y) \leq C\varphi'(x) \tag{14}$$

for almost all x, y satisfying $|x - y| < \delta$.

Proof. A classical theorem on product measures is the following: if N is a negligible subset of $\mathbb{R}^2$, then, for almost every $x \in \mathbb{R}$, one has $(x, y) \notin N$ for almost every $y \in \mathbb{R}$. Thus we get a negligible set $M \subset \mathbb{R}$ and, for each $x \notin M$, a negligible $A_x \subset \mathbb{R}$ such that

$$\varphi'(y) \leq C\varphi'(x),$$

as soon as $y \notin A_x$ and $|y - x| < \delta$. Then for all $x \notin M$ and $y \in (x, x + \delta)$ we have

$$\frac{\varphi(y) - \varphi(x)}{y - x} = \frac{1}{y - x} \int_x^y \varphi'(z)\, \mathrm{d}z \leq C\varphi'(x),$$

and the same is true for $x - \delta < y < x$. We have obtained

$$\left(\frac{\varphi(y) - \varphi(x)}{y - x}\right)^2 \leq C^2 \varphi'(x)\varphi'(y),$$

for all $(x, y) \in (\mathbb{R} \setminus M)^2$ such that $|x - y| < \delta$. □

Proposition 5. *Let $1 < p < \infty$.*

- *There exists a C^1-diffeomorphism φ which acts on $\dot{B}_p^{1/p,p}$ but not on $B_p^{1/p,p}$.*
- *There exists a C^1-diffeomorphism φ which acts on $B_p^{1/p,p}$ but not on $\dot{B}_p^{1/p,p}$.*

Proof. We first consider the diffeomorphism φ such that

$$\varphi'(x) = \min\left(1, \frac{1}{\sqrt{|x|}}\right);$$

it is easy to see that φ is quasi-conformal—indeed, φ' belongs to Muckenhoupt's A_∞ class—and then defines a bounded operator on $\dot{B}_p^{1/p,p}$; on the other hand, one has

$$\varphi^{-1}(x + 1) - \varphi^{-1}(x) = \frac{x}{2} + \frac{3}{4} \quad (x \geq 1),$$

so φ does not satisfy (11) and T_φ is not bounded on $B_p^{1/p,p}$.

Next we consider the odd function φ such that

$$\varphi(x) = e^x - 1, \quad x \geq 0.$$

One has $(\varphi^{-1})' \in L^\infty$ then, a fortiori, (12); moreover, it is easily seen that φ satisfies the condition (14), hence leads to a bounded operator on $B_p^{1/p,p}$. On the other hand, we have

$$\frac{\varphi(2x) - \varphi(x)}{\varphi(x) - \varphi(0)} = e^x - 1 \quad (x > 0),$$

which implies that φ is not quasi-conformal. □

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