GIRLS JOURNEY TOWARDS PROPORTIONAL REASONING

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This study focused on 26 girls' development of proportional reasoning in two fifth-grade classrooms in Iceland. The students were used to instructional practices that encouraged them to devise their own solutions to mathematical problems. The results supported four levels of proportional reasoning. Level 1, girls showed limited ratio knowledge. Level 2, they perceived the given ratio as an indivisible unit. Level 3, students conceived of the given ratio as a reducible unit. And at Level 4 students no longer thought of ratios exclusively as unit quantities, but understood the proportion in terms of multiplicative relations. The results suggest that students can reach level 3 reasoning with less struggle than it takes to achieve level 4, which suggests that the knowledge needed to operate on level 3 was within their reach.

OBJECTIVES

This study investigates the developmental of proportional reasoning of girls in two fifth-grade classes in Iceland. The purposes of this study was to further investigate four levels of proportional reasoning identified in a pilot study that the author conducted in collaboration prior to the study reported here (Carpenter et al. 1999). In particular, do the four levels describe the pathway of a population of Icelandic girls before, during, and after they have engaged in a unit focused on proportional reasoning? Secondly, what evidence is there for the existence of Level 2, Level 3, and Level 4 ways of reasoning in students' verbal protocols? And finally how does instruction that is focused on students' reasoning help students make the transition from level to level?

BACKGROUND AND THEORETICAL ORIENTATION

Proportional reasoning represents a cornerstone in the development of children's mathematical thinking (Inhelder & Piaget, 1958; Resnick & Singer, 1993). Ratio and proportion are critical ideas for students to understand; however, although young children demonstrate foundations for proportional reasoning, students are slow to attain mastery of these concepts.

Many studies on children's proportional reasoning provide evidence of various influences on students' thinking about proportion. Among these influential factors are the problem numerical structures[1]. The number structure refers to the multiplicative relationship within and between ratios in a proportional setting. A "within"

[1] Another term commonly used is "numerical structure". I will use the term "number structure" or "numerical relationship" when talking about the multiplicative relationship that is presented in the problem.

relationship is the multiplicative relationship between elements in the same ratio[2] whereas a "between" relationship is the multiplicative relationship between the corresponding parts of the two ratios.

The multiplicative relationship can be integer or noninteger. For example, the problem $\frac{2}{4}=\frac{12}{x}$ has integer multiples both within the given ratio ($2\times 2=4$) and between ratios ($2\times 6=12$). In a noninteger ratio, on the other hand, occur when at least one of the multiplicative relationships (within the given ratio or between the two ratios) is not an integer (Freudenthal, 1983; Karplus, Pulos, & Stage, 1983). For example, the problem $\frac{8}{5}=\frac{48}{x}$ has an integer multiple between the two ratios ($8\times 6=48$) but the within-ratio relationship is noninteger ($8\times 5/8=5$ or $5\times 1\,3/5=8$) (Abromowitz, 1975; Freudenthal, 1983; Karplus et al., 1983; Tourniaire & Pulos, 1985).

From Qualitative to Multiplicative Reasoning

Researchers have hypothesized that students' learning of proportional reasoning can be described as a learning trajectory[3] (Carpenter et al., 1999; Inhelder & Piaget, 1958; Karplus et al., 1983). The literature on proportional reasoning reveals a broad consensus that proportional reasoning develops from qualitative thinking to multiplicative reasoning (Abromowitz, 1975; Behr, Harel, Post, & Lesh, 1992; Confrey, 1995; Inhelder, & Piaget, 1958; Kaput & West, 1994; Karplus et al., 1983; Kieren, 1993; Noelting, 1980a, 1980b; Resnick & Singer, 1993; Vergnaud, 1983).

Studies of individual cognition and the development of proportional reasoning have identified three categories of strategies that students use in reasoning about proportional relationships: qualitative, additive, and multiplicative (Behr, Harel, Post, & Lesh, 1992; Inhelder & Piaget, 1958; Karplus et al., 1983; Kieren, 1993; Resnick & Singer, 1993). These strategies represent different levels of sophistication in thinking about proportions.

Research with preadolescent students indicates that their representation of situations that involve ratio and proportion occurs on an informal basis long before they are capable of treating the topic quantitatively. A qualitative reasoning strategy is based on an informal or intuitive knowledge of relationships without numerical quantification (Kieren, 1993). Next is additive reasoning, which requires quantification of the ratio relationships. The process of additive reasoning is often referred to as a buildup strategy. For example, consider the following problem:

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[2] Here I define ratio as the relationship between two quantities that have two different measure units.

[3] By learning trajectory, I am referring to the path that student reasoning travels as students' understanding of proportion develops. As students reasoning develops, so too does student ability to solve increasingly complex problems. Corresponding to their increasing ability to solve difficult problems, students' strategies for solving problems also get more complex and more mathematically sophisticated.

It is lunchtime at the Humane Society. The staff has found that 8 cats eat 5 large cans of cat food. How many large cans of cat food would the staff members need to feed 48 cats? (In an algebraic equation, $\frac{8}{5} = \frac{48}{x}$.)

A student might use a buildup strategy to arrive at the solution of 30 cans (Figure 1).

Cats	8	16	24	32	40	48
Cans	5	10	15	20	25	30

Figure 1. Buildup strategy in the form of a ratio table for the problem $\frac{8}{5} = \frac{48}{x}$

Buildup strategies are often observed during childhood and adolescence and appear to be the dominant strategy for many students these ages (Kaput & West, 1994; Tourniaire & Pulos, 1985). Although the buildup strategy can be used successfully in many cases, developing more sophisticated reasoning is crucial for solving more complex problems and understanding the complexity of the multiplicative relationship (Tourniaire & Pulos, 1985). Proportional reasoning is multiplicative and therefore the transition from buildup strategies to multiplicative strategies is considered to be a benchmark of development (Inhelder & Piaget, 1958; Karplus et al., 1983; Noelting, 1980a, 1980b).

When solving simple proportion problems, two types of multiplicative strategies have been identified: "within ratio" and "between ratios" (Karplus et al., 1983; Noelting, 1980a; Vergnaud, 1983). The within-ratio strategy is based on applying the multiplicative relationship within one ratio to the second ratio to produce equal ratios. The between-ratio strategy is based on determining the multiplicative relationship between corresponding parts of the two ratios to create equal ratios. For example, consider the following problem:

A hiking group is organizing a field trip, and they estimate that it will take 3 hours to walk 9 km. How long will it take the group to walk 33 km? (In an algebraic equation, $\frac{3}{9} = \frac{x}{33}$.)

Either ratio strategy—within or between—can be used to find the answer. A student using the within-ratio strategy would notice that the distance is 3 times the hours ($3 \times 3 = 9$) and therefore the same should apply to the target ratio, which would result in 11 hours ($11 \times 3 = 33$). A student using the between-ratios strategy would look for the multiplicative relationship between 9 km and 33 km, realize that $33 = 9 \times 3\frac{2}{3}$, and multiply $3 \times 3\frac{2}{3} = 11$ to get the answer.

While earlier research on students' reasoning relied on within-ratio and between-ratios strategies to analyze students' thinking (Abramowitz, 1975; Karplus et al., 1983; Vergnaud, 1983), Lamon (1993; 1994; 1995) offered a different lens through which to understand students' development of proportional reasoning. Lamon proposed two processes, unitizing and norming, as central to the development of proportional reasoning. Unitizing involves the construction of a reference unit from a given ratio relationship. Norming refers to the reinterpretation of another ratio in terms of that reference unit (Lamon, 1994; 1995). For example, consider the previous problem about the Humane Society (in an algebraic equation, $\frac{8}{5}=\frac{48}{x}$).

Using norming and unitizing, a student might interpret the target ratio as a multiple of the given ratio $\frac{8}{5}$. Therefore, she needs six groups of the 8-to-5 ratio unit in order to get an answer for 48 cats. For their calculations, students might use methods such as buildup strategies or direct multiplication. A student using a between strategy, on the other hand, would consider a single quantity in the given ratio and operate on that quantity, recognizing that the same operation must apply to the second quantity. Referring to the same equation, $\frac{8}{5}=\frac{48}{x}$, the student multiplies 8×6 to get 48 then multiplies 5×6 to get the answer, 30. When unitizing and norming, the student thinks of the ratio as a complex unit. The student can operate on the unit $\frac{8}{5}$ by adding, multiplying, or reducing—but each operation is interpreted as creating a new unit that preserves the relationship within the given ratio. In other words, when unitizing or norming, the operation is performed on the ratio as a unit instead of individual terms in the ratio.

Using Lamon's (1994; 1995) operation of unitizing and norming, the author and team of colleagues identified 4 levels of reasoning in a pilot study conducted in the US in one classroom over a 2 weeks period (Carpenter, et. al. 1999). At Level 1, students showed limited ratio knowledge. The most common strategy was finding the additive differences within and between the ratios. Level 2 is characterized by the perception of the ratio as an indivisible unit. Students at this level are able to combine the ratio units together by repeated addition of the same ratio to itself or by multiplying that ratio by a whole number, but they cannot solve proportion problems in which the given ratio has to be partitioned such as, problems in which the target ratio is a noninteger multiple of the given ratio (e.g., $\frac{8}{12}=\frac{42}{x}$ or $\frac{8}{3}=\frac{2}{x}$). At Level 3, the given ratio is thought of as a reducible unit. Therefore, students at Level 3 can scale the ratio by nonintegers. An example of a Level 3 strategy combines the reduction of the given ratio with a buildup strategy by using either addition or multiplication. Students at Level 4 think of ratios as mere than just as unit quantities. They recognize the relation within the terms of each ratio and between the corresponding terms of the ratios.

METHOD AND ANALYSIS

The subjects of this study are the 26 fifth-grade[4] girls in two classrooms at one of Reykjavik's public schools. I observed every math class throughout the course of the study, taking on the role of "participant observer". During data collection, students worked on 24 problems that were created during 10 weeks of instruction. Each set of problems was composed of three problems with the same contextual structure but with different multiplicative relationships in the proportion. The numbers were chosen to further students' understanding of proportion and to aid their recognition of

[4] Fifth grade in Iceland refers to children that turn 10 years old in the year they start 5th grade.

the multiplicative relationships in the two ratios in the proportion. By varying the multiplicative relationship in the problems, sets of problems were created to distinguish between Level 2 and Level 3 students and between Level 3 and Level 4 students

The pretest and the posttest were created using the same criteria as the instructional problems in regard to the number structure of the problems. The pretest comprised of 18 problems in three sets with different multiplicative relationships. The posttest comprised 12 problems. Students' problems solutions strategies were collected. During instructions students worked both individually and in groups on their problems. All the written work the students produced and artifacts from their work were collected. Also all whole-classroom discussions were videotaped and transcribed.

The criteria for determining what each student's level of reasoning was based on upon which problems she could solve and which problem she could not. Level 1 students use incorrect strategies. The most common strategy is to find the additive difference within the given ratio or between the ratios and apply that difference to the target ratio.

At Level 2 students perceive the given ratio as an indivisible unit and interpret it as a unit whole. They are able to solve only problems that have an integer relationship between the ratios, such as $\frac{2}{6}=\frac{x}{36}$. The numbers in the target ratio have to be bigger than the numbers in the given ratio. At Level 3 students perceive the given ratio as a divisible unit. The given ratio is interpreted as a unit whole. Level 3 students are able to solve problems that involve both an integer and a noninteger relationship, such as $\frac{6}{12}=\frac{15}{x}$. Finally at Level 4 students no longer think of ratios exclusively as unit quantities. They can take into account both the within and between relationships and choose the one relationship that is easier to calculate. They are not limited to building up or partitioning the given unit.

RESULTS

Question 1. The four-level model of proportional reasoning identified in the pilot study proved to be a beneficial tool to analyze their work. Analyzing the pretest the classification of students' solutions resulted in the creation of a transitional level "emerging Level 3". On both pre- and posttest the results show a perfect fit; students on Level 2 were not able to solve any of more complex problems that emerging Level 3 students were able to solve successfully, nor were the emerging Level 3 students able to solve any of the most complex problems that Level 3 students were able to solve with success.

The problems were structured to discriminate between students at different levels of reasoning. Problems that could be solved by students reasoning on Level 2 had an integer relationship between the ratios and involved enlarging (e.g., $\frac{2}{8}=\frac{x}{24}$). Students reasoning on Level 3 could solve problems that were previously mentioned as well as problems that have a noninteger relationship between the ratios ($\frac{5}{6}=\frac{x}{21}, \frac{15}{10}=\frac{6}{x}$).

Problems that proved to be transition problems from Level 2 to Level 3 were the problems that had a scale-down number structure such as $\frac{8}{24}=\frac{2}{x}$. The difference between Level 2 and Level 3 reasoning is the need to scale down the given ratio. During the emerging Level 3 stage, students are able to scale down by whole numbers but they cannot use their knowledge of scaling down within other number structures. Strategies that students used to solve the problem distinguished between Level 3 and Lev el 4 reasoning.

On the pretest, 35 percent of the girls displayed Level 1 reasoning. Around 40 percent exhibited Level 2 reasoning. Twenty-three percent of the girls were emerging Level 3. One girl showed Level 3 reasoning on her pretest. On the posttest only 3 girls reached Level 4 thinking, whereas more than 80 percent reached Level 3 thinking. Therefore, it is evident that reaching Level 4 thinking involves a very complex thinking that most of the girls had not yet adopted.

Question 2. Throughout the course of the study, girls were thinking about the given unit as a single entity that they then operated on to reach their target number. The buildup strategy, the most common strategy, provides clear evidence of the ways in which students understand the given ratio as a single unit that they can then build up or build down. Common explanations from the girls were related to the idea that everything they did had to apply to both terms of the ratio.

Following is an example of a Level 2 girl's explanation of her strategy for the following problem to support that argument: It is lunch hour at the humane society. The staff members have found out that 8 cats need 5 large cans of cat food. How many large cans of cat food would they have to have if they were to feed 48 cats?

Student: I did it—like, here is 8 and then 5 cans of food, and then again—then there is 8 and 16 cans of food until I…reached 48 cats, and then the answer is 30 cans of cat food.

Teacher: How did you know that you should have 8 groups?

Student: Well, I did not know that because I did 8:5 and 8:5 and 8:5 and added the 8s together until I had 48.

She explained her strategy in terms of the unit as an entity. She operated on the unit of 8:5 until she reached her target number of 48. She did not think in advance about the number of groups she had to use; rather, as she is building her units, she is adding on until she know where to stop.

Question 3. Nina represents close to 30 percent of the students. She was a typical Level 1 student at the time of her pretest. In the beginning of the unit, Nina needed a little scaffolding to help her move away from her additive thinking. She quickly solved her first problem ($\frac{2}{6}=\frac{x}{36}$) by finding the additive difference between the ratios. Her first answer was 32. After only a few questions, she was able to get on the right track.

Teacher: What if you had 4 cans of food, how many cats could you feed?

Nina: 8 cats.

Teacher: We know that 2 cans of cat food can feed 6 cats. We get 2 more cans, and can they only feed 2 more cats?

Nina: No, 2 cans can feed 6 cats, not 2.

Teacher: What does that mean, then?

Nina: Well, it is like if 2 cans can feed 6 cats, then another 2 cans can feed another 6 cats.

Teacher: Think about that more and how you can solve your problem differently. I will come back to you.

The teacher left Nina to grapple with her new ideas about the problem. When it came to sharing time, Nina had not yet figured out how to go about solving the problem with her newfound knowledge. A couple of the strategies that were shared were buildup strategies in which students took the given unit $\frac{2}{6}$ and built it up unit-by-unit to reach the target number. Nina really liked that strategy and utilized it with success. When the teacher got to Nina, she had solved the problem by using a buildup strategy. When the teacher asked her to explain what she had done, it became clear that she understood clearly what the numbers in the buildup strategy stood for.

Nina: First there were 2 cans and 6 cats, then next there would be 4 cans for 12 cats and—

Teacher: And why is that?

Nina: It’s like first there were 2 cans and 6 cats, then there were 2 more cans and 6 more cats would eat that, and that is like having 4 cans and 12 cats.

When Nina started working on the second problem, she paused a little bit and thought hard before solving the problem with a buildup strategy. The scaffold from the teachers and from the discussion of different strategies provided a basis for girls to attain more advanced levels of proportional reasoning. The case of Nina shows how a student was afforded the opportunity to learn from her teacher and from other students using more advanced thinking. This example also illustrates how less advanced students may learn from listening to other students explain more efficient strategies than they commonly use and how a class may build on ideas that are distributed among members of the class.

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