

INVESTIGATING TEACHERS' RESPONSES TO STUDENT MISCONCEPTIONS

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As part of a study investigating primary/elementary teachers' pedagogical content knowledge teachers were asked to describe how they would respond to some hypothetical situations involving student misconceptions and errors. The nature of the task in which the error arose influenced teachers' emphasis on procedural or conceptual aspects, and teachers' responses revealed aspects of their pedagogical content knowledge. The usefulness of a questionnaire and interview approach for investigating these issues is also discussed.

BACKGROUND TO THE STUDY

Teachers' mathematical pedagogical content knowledge (PCK)—or knowledge for teaching mathematics—has received considerable attention since the mid-1980s (e.g., Shulman, 1986). One of the key components of PCK is knowledge about student misconceptions. As part of a larger study investigating PCK and its influence on instructional practice and student learning, this paper examines the self-reported practices of year 5 and 6 teachers in addressing certain typical misconceptions. It focuses on their strategies and the pedagogical and mathematical knowledge revealed in their responses. It also considers the effectiveness of using a questionnaire and interview to investigate PCK.

Dealing with student misconceptions

An understanding of common student misconceptions, and effective strategies to help students avoid them, is an important aspect of mathematical PCK (Graeber, 1999). In addition to trying to teach in such a way that students avoid misconceptions, teachers must also have approaches for dealing with those that inevitably arise. Once the misconception is recognised teachers must decide what strategies they can use. If reteaching occurs then decisions must be made about what to emphasise and how. Cognitive conflict is another strategy, in which students encounter a situation that contradicts their initial understanding in the hope they will then re-evaluate those beliefs (see Watson, 2002 for a good overview of the literature).

In addressing student misconceptions teachers' approaches may focus on procedural or conceptual aspects. Hiebert and Lefevre (1986) distinguish conceptual knowledge as being rich in relationships, with procedural knowledge having an emphasis on symbolic representation and algorithms. Most curriculum documents place an emphasis on both. One of the challenges in teaching is to address both aspects, a problem exacerbated if teachers lack conceptual fluency in key topics (Chick, 2003).

Investigating teachers' pedagogical content knowledge

Investigating teachers' PCK and the ways in which it is used by them has proved challenging, because PCK comes into its own in the classroom. Since classrooms vary from moment to moment and are thus different from one another in a myriad ways (as shown by Lampert, 2001) observing two teachers in their own classrooms can show their PCK only for the situations that arise for them, and so comparisons are not easily made. For large scale studies the resources required for an extensive examination of PCK are significant (see, e.g., McDonough & Clarke, 2002, who look at 6 teachers on a single topic). Some researchers have tried to set up a single consistent scenario (e.g., van der Valk & Broekman, 1999), by asking teachers to prepare a lesson plan on a particular topic. This at least makes it possible to investigate many teachers' PCK for that topic, but requires a considerable investment for multiple topics. So, in addition to examining the way teachers address student misconceptions and PCK *per se*, this study also asks if there is a way to investigate teachers' PCK for multiple teachers over multiple topics.

METHODOLOGY

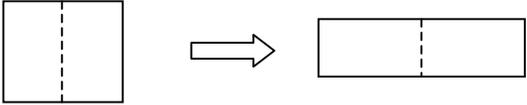
Participants and Procedure

Nine Australian teachers took part in the study. At the time of the study these teachers were teaching Grade 5 or 6 (students 10-12 years old). They had 2 to 22 years of teaching experience, but not all of it in Grade 5 or 6 classrooms.

As part of the larger study on pedagogical content knowledge participants completed a questionnaire and were then interviewed about their written responses. The questionnaire comprised 17 items examining mathematics teaching situations and beliefs, and participants completed it without restrictions on time or resources. The completed questionnaire was returned to the researchers who prepared questions for the interview, partly to clarify ambiguities or omissions in the written responses. Interviews thus varied among teachers, although there were also some common questions. Teachers' responses to four of the items from the questionnaire, together with their follow-up comments from the interview, are the focus of this paper.

The four items were designed to explore how teachers respond to student errors or misconceptions, with a focus on the subtraction algorithm, the division algorithm, fraction addition, and the relationship between area and perimeter. Each item showed work from a hypothetical student, and invited teachers to indicate what they would do in response. The subtraction and the area/perimeter items are shown in Figures 1 and 2. In the division item the hypothetical student disregarded the place value associated with any zeros in the dividend. For the fraction item ($7/10 + 2/5$) the student correctly obtained an equivalent fraction for $2/5$, but the resulting addition of $7/10 + 4/10$ was carried out by adding numerators and adding denominators. For this item there were direct questions about what teachers thought the student did and did not understand. The researchers regarded the misconceptions as being archetypal, and expected that

most of the teachers would have encountered at least some of them either in teacher training or in actual classroom practice.

<p>You notice a student working on these subtraction problems:</p> $\begin{array}{r} 438 \\ -172 \\ \hline 346 \end{array} \qquad \begin{array}{r} 5819 \\ -2673 \\ \hline 3266 \end{array}$ <p>What would you do to help this student?</p>	<p>A student cuts the following shape in half to make a new shape, saying that the two shapes have the same area and perimeter:</p>  <p>What would you say to this student?</p>
<p>Figure 1. The subtraction item.</p>	<p>Figure 2. The area/perimeter item.</p>

Analysis

The data were analysed to identify what teachers said they would do in response to the students' work, to identify whether their responses to students were predominantly procedurally or conceptually based, and finally to identify differences in displayed pedagogical content knowledge. The written responses were analysed first, with additional data provided from the interviews.

Strategies used by teachers in their responses to students are shown in Table 1; these categories were refined as the researchers identified common themes from the data.

Category	Definition
Re-explain	Explaining or re-explaining any part of either the concept or procedure.
Cognitive Conflict	Setting up a situation in which the student might identify a fundamental mathematical contradiction between the original response and the new situation, thus encouraging the student to re-evaluate the erroneous approach.
Probes student thinking	Asking the student to explain working or thinking, either to discover what the student is thinking to help the teacher decide what to do next, or to get the student to see the error. [It was not always possible from the data to establish which of these the teacher intended, so no distinction was made.]
Other	Any strategy not clearly in the above categories, e.g., "use simpler examples".

Table 1. Categories of teacher strategies in response to student misconceptions.

Where teachers gave an explanation of how to do the student's problem correctly, this explanation was further categorised as conceptual or procedural or both. To be judged as conceptual the response had to have clear reference to underlying mathematical principles, as opposed to merely giving a recipe for a procedure without justification. Finally, the participants' responses were examined more closely to identify aspects of pedagogical content knowledge evident in their explanations. Teachers' names have been changed for reporting results.

RESULTS

Teachers' approaches to dealing with misconceptions

Table 2 shows the strategies used by teachers in response to the student misconceptions. Re-explain was the most common strategy, although less common for the area/perimeter item. Teachers only suggested probing student thinking in response to the subtraction and division items, usually asking students to say their procedure aloud. Cognitive conflict was a strategy for all items, but the methods teachers used to evoke cognitive conflict varied between the items. For example, using diagrams or materials to compare the sizes of the fractions and demonstrate the unreasonableness of the student's result was common in the fraction addition item, while using inverse operations or a calculator to check answers was common for the subtraction and division items. In the area and perimeter item five teachers said they would ask students to "measure and check" the perimeters of the two shapes to establish the cognitive conflict. A general strategy proposed across items to address misconceptions was to consider simpler examples.

Item	Re-explain [*]	Cognitive conflict [*]	Probes student thinking [*]	Other [*]
Subtraction	9	2 (2)	1 (3)	1 (1)
Division	8	5 (2)	1 (2)	2 (1)
Fraction addition	7	4 (1)	0	1
Area/perimeter [†]	4	5 (2)	0	1

*Numbers in brackets refer to responses added in the interview that were different from the original questionnaire response.

† One teacher believed the misconception, so did not offer any strategy for this item.

Table 2. Numbers of teachers using particular strategies in response to misconceptions. Teachers may have used more than one approach.

Nature of teachers' explanations

Table 3 shows the number of teachers whose explanations were categorised as procedural or conceptual in response to student misconceptions. The type of response seemed to depend on the item. Teachers were more likely to respond to the subtraction and division items with a purely procedural explanation, as evident in this response to the subtraction item: "Break this problem down. Treat it as 3 separate or 4 separate subtractions to begin with" [Brian]. In contrast, purely conceptual explanations were only given in response to the fraction addition and, as shown here, the area/perimeter items: "For the first shape the 2 rectangles share a long side, but in the second only a small side and therefore the distance around the outside would be different" [Clare]. Most explanations combined procedural and conceptual aspects, such as the following response to the subtraction item, which emphasised the importance of conceptual understanding in supporting the procedure:

[After demonstrating “borrow[ing] a ten”, and saying that “1 ten and 2 ones ... is the same as 12 ones”] We will also look at our MAB equation and note that now we only have three tens. I will ask if we still have the number 42, albeit represented differently ... During this initial equation I would not ask the students to write the equation down; the focus would be on making, discussing and solving the equation. [Amy]

Item	Procedural explanation	Conceptual explanation	Procedural and conceptual explanation	Not clear/ no explanation
Subtraction	4	0	4	1
Division	6	0	3	0
Fraction addition	0	2	7	0
Area/perimeter	1	2	4	2

Table 3. Numbers of teachers giving procedural or conceptual explanations in response to misconceptions.

Differences in PCK evident in responses

Teachers’ PCK can be examined within the framework described above. Although differences in the approach that teachers took for the different items may have been affected by the nature of the item (as discussed later), a comparison of responses to the same item, by teachers who adopted a similar strategy, is revealing about the PCK held by those teachers.

A closer look at three responses to the fraction item demonstrates this. Brian, Erin and Amy all responded to this item by re-explaining subtraction to the student. Brian and Erin decided to break the first subtraction into three parts (for ones, tens, and hundreds), and all three mentioned using MAB (base 10 arithmetic/Dienes’ blocks). However, there are some important differences. Brian suggested breaking the problem into parts, but did not discuss how to bring the parts together again to complete the problem. While he described using MAB to demonstrate that “you can’t take 7 from 3”, he did not explain what he would do next. Erin’s response was similar, but she went on to describe using MAB to help “take from” the hundreds column when there are not enough tens. Amy’s response to this problem was noticeably more detailed, and she made an additional decision to begin with a simpler (two digit) problem. She described going through each step in the subtraction algorithm with the MAB, frequently highlighting important aspects for the student, such as the equivalence of the original number, 42, to its new form, $30 + 12$. She emphasised the importance of students understanding this process first, and only then “we would look at how we represent our trades on paper”.

The differences in the choices made and in the detail and coherence of the three responses reveal differences in knowledge about teaching the subtraction algorithm, and about this misconception. Brian has identified the problem, and actions he might

take, but in a limited way, and he has not connected his ideas into a logically sequenced response to the misconception. Erin's response is more detailed and connected, but still contains many gaps, while Amy's response is detailed and coherent, and reveals more explicit awareness of the reasons for her decisions.

Teachers' responses to the division item can be examined similarly. While most responses characterised as "re-explain" mentioned ideas like the importance of place value, zero as a "place holder", and the use of multiplication in checking answers, a deeper examination of responses reveals more aspects of the PCK of the teacher. The focus of Hilary's response was to remind students to "put a zero down" in certain cases, without justifying why this is the case. Irene did the same thing, but added that "just because a zero is there doesn't mean that it's nothing, it just happens to be no tens." She also discussed the meaning of the different columns in terms of place value, and was very specific about why she feels students need to understand this.

DISCUSSION AND CONCLUSION

A number of factors are evident in the teachers' responses. It appears that the nature of the items themselves and the topics covered in each may influence how teachers said they would respond, particularly with reference to the emphasis on procedural and/or conceptual explanations. Although not apparent in the data presented here, there was also evidence that some teachers may be inclined towards one more than the other. Other teacher-based differences are shown in the closer analysis of the teachers' PCK revealed in their responses. These issues, and the question of the effectiveness of the questionnaire/interview approach, are discussed in detail below.

Differences among teachers' responses

The data in Tables 2 and 3, and the detailed responses, indicate differences in approach among teachers and items. The subtraction and division items attracted many responses that were purely procedural. This may be because they involved student tasks with an algorithmic calculation, and that while the procedure in the algorithms and the overall concept involved (subtraction or division) are linked, it can be a difficult link to maintain while focussing on the mechanics of the procedure. Indeed, what was noticeable in the teachers' responses was the lack of conceptual support for teaching the procedure, such as modelling the algorithm using MAB.

In contrast, responses to the area/perimeter item were more often conceptual or both conceptual and procedural, with only one purely procedural response. There is no complex algorithm associated with this item, and the procedure involved in measuring perimeter mimics the concept more clearly than in the algorithm items. Teachers' responses often indicated that the *meaning* of area and perimeter would be revisited; and where the procedure was stated it was supported by some conceptual explanation. Moreover, five teachers indicated that they would pursue the concepts further through an exploration of shapes with the same area but different perimeter.

The fraction addition item lies somewhere between these two: the link between the procedure and the concept may not be evident to the student, but is readily demonstrated by the teacher. The number of responses that combined procedure and concept offers support to this idea. Three teachers drew a diagram in their questionnaire, and all the teachers at least made mention of drawing a diagram or using physical materials to model the problem. Such a diagram, of two wholes with seven tenths shaded in the first and four tenths in the other, mimics the written problem quite closely.

For the more algorithmically-based items, the prevalence of “re-explain” as a strategy in teachers’ responses is unsurprising. The way in which cognitive conflict was used varied across items. The idea of considering the inverse operations to evoke cognitive conflict for the subtraction and division items was rare. Many teachers sensibly suggested that students estimate their answer for the division item; the cognitive conflict should be dramatic in this case, whereas for the subtraction item the estimated result may not sufficiently differ from the student’s erroneous answer. Finally, for the perimeter item, teachers seemed to believe that the student’s misconception was because the student had generalised some idea of “conservation”, rather than a lack of understanding of the procedure of measuring the perimeter; hence their suggestion that the student should “measure and check”.

Investigating teachers’ pedagogical content knowledge

Some quite significant aspects of PCK were revealed using the questionnaire and interview approach. When two teachers asserted that they would adopt a particular approach in response to the same student misconception, the differences in the details offered by those teachers often revealed evidence of differences in PCK, or at least in the application of their PCK. For some teachers, the careful articulation of a well-developed sequence of ideas seemed to reflect deep understanding of concepts and strategies for making those ideas meaningful for students. In other cases, the concepts were not well-linked, nor were concepts well-supported pedagogically.

It should be noted that the responses in the questionnaire alone were informative, but the follow up interview gave additional insights and allowed teachers to reveal more of their understanding and repertoire of strategies. It certainly seems, perhaps not surprisingly, that investigating PCK is intrinsically labour-intensive. Nevertheless, this approach allows investigation of a wide range of topics, recalling that there were more items on the questionnaire than the four reported here.

Limitations

There are a number of limitations to this study. With such a small number of teachers we cannot quantitatively generalise about the likely responses of other upper primary teachers, although the study has given some qualitative insights. It should also be noted that the questionnaire and interview approach still does not provide legitimate evidence for what teachers may do in a real-life teaching situation, in response to situations they have more control over and with students they know (further insights

into this are evident in the work of Lampert, 2001). Furthermore, the absence of a strategy or explanation in a teacher's response does not necessarily imply that it is not part of his or her teacher knowledge.

Conclusion

The study suggests that teachers respond to student misconceptions in a variety of ways. Their responses to the items appeared to be influenced by the nature of the student task, and revealed aspects of their PCK and the emphasis they place on procedural and conceptual aspects of mathematical understanding. Although a framework for investigating teachers' PCK with this method has not yet been developed, it appears from initial explorations that the questionnaire and interview have the capacity to reveal subtle differences between teachers' responses, which may be attributed to differences in knowledge. This will form a background for the larger study, where PCK will be examined in the context of the classroom. It may also highlight aspects of PCK, and conceptual and procedural knowledge, that are important to emphasise for pre-service teachers.

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