

# **GOOD CAS WRITTEN RECORDS: INSIGHT FROM TEACHERS**

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*Availability of a computer algebra system (CAS) provides a catalyst for teachers to reflect on long-standing practices of teaching mathematics, including how solutions to mathematical problems should be presented. This paper reports on how four teachers implementing CAS for the first time thought about this issue over a school year. The paper analyses their contributions to group discussions about their teaching practices at the beginning and end of the school year. New practice is needed to accommodate lack of intermediate steps available when CAS is used, and a cluster of issues relating to the use of CAS syntax. Their comments show considerable reflection about personal practice, the dominance of the external examination, and concern that new expectations might favour users of some brands of CAS over others.*

## **INTRODUCTION**

New availability of a computer algebra system (CAS) in the mathematics classroom and for formal assessment provides a catalyst for teachers to reflect on beliefs and long-accepted practices about teaching mathematics. This paper will report on how a group of teachers reflected on and reconsidered their long-standing practices of how to present written solutions to mathematical problems. They were prompted to reconsider this socio-mathematical norm (Krummheuer, 1995) by the perturbation to normal practice of working with our research team to implement the first mathematics subject permitting the use of CAS in secondary schools in their region. That having a complex calculator in the classroom perturbs normal practice is a common finding in the research literature (see, for example Artigue, 2002; Guin & Trouche, 1999; Stacey, 2003), and many aspects of this have been investigated.

The four teachers, Ken, Lucy, Neil and Meg (not their real names) were participants in the CAS-CAT project (CAS-CAT, n.d.), which researched curriculum, assessment and teaching using CAS in three secondary schools. A new subject, Mathematical Methods (CAS), was accredited for the state examination system for years 11 and 12 mathematics. In MMCAS, CAS could be used for all mathematical work, including in examinations, at the teacher's or student's discretion. Further descriptions and outcomes of the project are described in Stacey (2003), Ball (2003), Flynn and Asp (2002), and VCAA (2002). Previously, only graphics calculators without a symbolic facility had been permitted. The three project schools each used a different brand of CAS, with Lucy's and Ken's classes which were at the same school using the same machines.

The project tracked the progress of the teachers and first cohort of students through the first two years of implementation – year 11 in 2001 and year 12 in 2002. At the end of 2002, the Year 12 students sat for the first externally-set state examinations in

the new subject. Their results contributed to their university entrance scores, and were regarded as very important by the teachers, students and schools. Teachers were always concerned with preparing their students well for examinations - it was a very high priority for them at all times. Ball (2003) and Ball and Stacey (2004) report on the way in which students' recorded their solutions in the 2002 examinations.

During 2001 and 2002, the project team provided extensive teaching material to assist teachers and students, training in the use of CAS and discussions about pedagogy. Consequently, teachers had considered implementation issues prior to the discussions reported in this paper in 2002. The data for this paper is from two meetings. The first was held at the beginning of the school year (February 2002) and involved all teachers and the researchers. The second meeting was at the end of the school year (November 2002) and, at the teachers' request, involved the teachers only. Both meetings were audiotaped and transcribed by the researchers.

## HOW AND WHY DOES CAS CHANGE WRITTEN RECORDS?

Early in the planning and implementation of MMCAS, it was evident to the research team, the teachers and also the state-appointed examination setters, that the use of CAS might require changes in the normal way in which students write solutions, and the way in which written solutions are assessed by examiners. The major reason is that, in the phrase of Flynn and Asp (2002), CAS "gobbles up" intermediate working. Figure 1a shows in TI89 syntax, how a CAS can solve simultaneous equations using one input 'solve( $x+y=7$  and  $2x-y=5$ , { $x,y$ })'. The input line is second from the bottom (above MAIN) and the calculator display above is a restatement of the input followed by the answer  $x = 4$  and  $y = 3$ . Figure 1b shows how multiple CAS steps can often be combined into one 'nested' procedure with one line of CAS syntax and one output. The expression  $\sin(x)\cos(x)$  was differentiated with respect to  $x$  using syntax  $d(\sin(x)*\cos(x),x)$  and then  $x=\pi$  substituted into the derivative to give the result 1. Note in particular, that the symbolic derivative is not outputted. In examinations, this intermediate step of finding a correct derivative would often have been awarded a mark, even if the derivative is not explicitly requested.

One step solving (Fig. 1a)

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F1+ Tools F2+ Algebra F3+ Calc F4+ Other F5+ PrgmD F6+ Clean Up
■ solve(x + y = 7 and 2 · x - y = 5, {x, y})
x = 4 and y = 3
...x+y=7 and 2*x-y=5, {x, y}
MAIN RAD AUTO FUNC 1/30

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Nested procedure (Fig. 1b)

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F1+ Tools F2+ Algebra F3+ Calc F4+ Other F5+ PrgmD F6+ Clean Up
■ d/dx(sin(x) · cos(x)) | x = π 1
d(sin(x)*cos(x),x)|x=π
MAIN RAD AUTO FUNC 1/30

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Figure 1. Examples of CAS procedures which do not provide intermediate results.

The state authority, the VCAA, were concerned that partial credit should be able to be awarded for extended response questions and approved the instructions in Figure 2 to appear on the examination papers. The third dot point is relevant here. Throughout

2002, teachers had to decide what ‘appropriate working’ meant for written records produced in the CAS context.

In order to provide guidelines for the teachers, the research team had devised the RIPA rubric, described with examples in Ball and Stacey (2003), to help establish good practice for the written communication of mathematical solutions. The rubric suggested that students should make sure that their written solutions to problems make the plan of the overall solution (P) clear, specify what was inputted (I) to the calculator although not in calculator syntax, and provide reasons (R). However, they need record only selected answers (A) – there is no reason to transfer to the written record all of the intermediate outputs of the calculator to paper. The February meeting which provided data for this paper, began with researchers initiating a group discussion of how students should be trained to record their solutions in MMCAS classes, during which the RIPA ideas were raised. RIPA promoted much discussion among the researchers and teachers which continued throughout the year. Some teachers found RIPA helpful to share with their class, and others did not.

- Answer all questions in the spaces provided.
- A decimal approximation will not be accepted if an exact answer is required to a question.
- Appropriate working must be shown if more than one mark is available.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Figure 2. Instructions for MMCAS examination 2002 (VCAA, 2002).

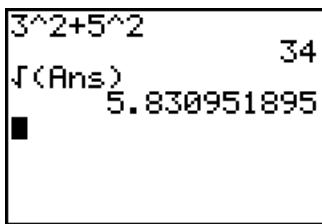
### PYTHAGORAS EXAMPLE

The February discussion on teaching students to record solutions in a CAS environment began with examples such as those in Figure 1 but the teachers, planning how to raise these issues with students, wanted to discuss simple examples where students would not find the mathematics challenging. They suggested finding the hypotenuse of a right-angled triangle with sides 3cm and 5cm, and talked about various written solutions such as those in Figure 3, which also shows associated calculator screendumps. Comparing Figures 3a-3c shows that students might be using quite different syntax and calculator methods to solve even basic problems; an illustration of the explosion of methods observed in other studies (e.g. Artigue, 2002).

Figure 3a is a typical solution using a scientific or graphics calculator: the inputs are not symbolic and it is not possible to obtain a surd answer. Ken noted, as a teaching difficulty, that his students (especially the less able students) would often include too many intermediate steps (e.g.  $c^2 = 9 + 25$ ) which were unnecessary to show in senior work, because they could be reasonably taken-as-shared. He attributed this to teachers of more junior classes not adjusting their expectations for written work to the presence of even scientific calculators. “*And some junior teachers actually would make [students] write all of that because they’re not used to using technology*”. This, and other comments by the teachers, indicated that the impact of scientific and graphics calculators on written work has not been thoroughly considered in schools.

Figure 3b shows a CAS solution using the TI89. The first input is the equation  $c^2 = 3^2 + 5^2$ ; possible as a symbolic calculator is being used. The next step in the solution is to take the square root of both sides of the equation. The TI89 actually inputs and shows this as an operation on the equation as one entity  $\sqrt{(c^2 = 34)}$ : a move that is *not* part of standard mathematics. It is important for teachers to make sure that their students are aware of syntax such as this which is used by the CAS but is certainly not standard for mathematical written records. The result of this command (right hand side of line) is  $|c| = \sqrt{34}$ , a statement which looks unusual to students, who may not immediately deduce that  $c = \sqrt{34}$ ; instead they would expect  $c = \pm\sqrt{34}$ . Dealing with unexpected output is another issue with which teachers using CAS need to assist students. The solution to this point has been worked with CAS in “exact mode”. Obtaining an approximate answer for  $c$  is not entirely trivial. Taking the square root of the equation has to be repeated in approximate mode, accessed in this case by pressing the “green diamond” button before ENTER, giving the output shown.

Scientific/graphics (Fig. 3a)    CAS solution 1 (Fig. 3b)    CAS solution 2 (Fig. 3c)



$3^2+5^2$   
 $\sqrt{Ans}$  34  
 ■ 5.830951895

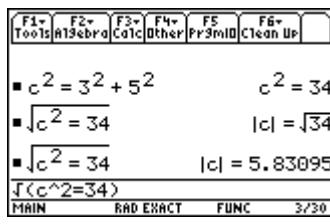
Using Pythagoras

$$c^2 = 3^2 + 5^2$$

$$c^2 = 34$$

$$c = \sqrt{34} \text{ since } c > 0$$

$$c = 5.83$$



$c^2 = 3^2 + 5^2$        $c^2 = 34$   
 $\sqrt{c^2} = 34$        $|c| = \sqrt{34}$   
 $\sqrt{c^2} = 34$        $|c| = 5.83095$   
 $\sqrt{(c^2=34)}$   
 MAIN RAD EXACT FUNC 3/30

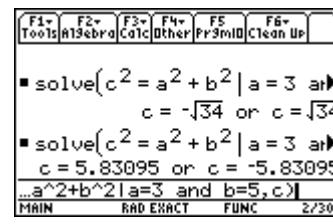
Using Pythagoras

$$c^2 = 3^2 + 5^2$$

Square root of both sides

$$c = \pm\sqrt{34}$$

$$c = \sqrt{34} \text{ since } c > 0$$



$\text{solve}(c^2 = a^2 + b^2 \mid a = 3 \text{ and } b = 5, c)$   
 $c = -\sqrt{34} \text{ or } c = \sqrt{34}$   
 $\text{solve}(c^2 = a^2 + b^2 \mid a = 3 \text{ and } b = 5, c)$   
 $c = 5.83095 \text{ or } c = -5.83095$   
 $a^2 + b^2 \mid a = 3 \text{ and } b = 5, c)$   
 MAIN RAD EXACT FUNC 2/30

Solve ( $c^2 = a^2 + b^2 \mid$

$$a=3 \text{ and } b=5, c)$$

$$c = -\sqrt{34} \text{ or } c = \sqrt{34}$$

$$c = \sqrt{34} \text{ since } c > 0$$

$$c \approx 5.83$$

Figure 3. Several written solutions for Pythagoras example, with calculator output.

On the Casio calculator, obtaining an approximate answer is embedded deeper within the menus, requiring the syntax TRNS (F1) then ALPHA then B (or TRNS (F1) then log button). Students became adept at those button sequences which they often used, but this created two issues for teachers. Firstly, students need to commit to memory sequences of button pushes which are not highly visible from the menu structure, and naturally begin to think in these terms. Neil commented “*but my kids use language like TRNS ALPHA B APPROX*” and was concerned that these might appear in their written records. Secondly, different brands of CAS use syntax that familiar users come to regard as intelligible and “standard” but which are very different to other brands. For students who are learning in a CAS classroom, how are they to distinguish between standard and non-standard notation?

Figure 3c shows a different CAS solution to the Pythagoras problem, using the TI89 solve and substitute commands:  $\text{Solve } (c^2=a^2+b^2 \mid a=3 \text{ and } b=5, c)$ . This shows no intermediate working at all, although the plan of the solution seems clear. Should the teacher accept this written record which (a) shows no intermediate steps and (b) uses the calculator syntax directly? If this calculator syntax is permissible, is the illogical  $\text{Solve } (c^2=a^2+b^2 \mid a=3 \text{ and } b=5, x)$ , which produces the same output, also permissible?

The Pythagoras example, simple as it is, shows that in addition to the problem of which steps should be shown in the written record, there are also a second set of issues arising for teachers related to the use of calculator syntax: what syntax can be accepted in written records, how will students know what is special to their learning environment and what is standard. Moreover, there are different problems arising for different brands of calculator. In the rest of this paper, we report on the teachers' thinking on these two sets of issues. As we shall see, there are two aspects to this thinking – what is good mathematical practice and what is necessary to write in order to score marks on the important end-of-year external examination.

### CAS “GOBBLES” UP INTERMEDIATE STEPS

When CAS “gobbles” up intermediate steps, what is the key information that should be recorded to show working? This question can be considered from the point of view of good mathematical practice which motivated the RIPA suggestions, or from the point of view of how marks will be allocated in examinations. It was the latter which dominated teachers’ comments on this issue in both the documented meetings, since they were always very concerned with maximizing students’ performance on the end of year high stakes examinations. Ken, for example, commented on one RIPA example: “*But in an exam you would get maybe credit here and credit here and anything else here you’re doing for yourself, not for any marks*”. It was probably because RIPA did not directly address the examination question that some teachers did not find it very useful.

In November, Meg commented that her students were still unsure of the validity of a written record that just described the CAS steps, rather than showing intermediate algebraic manipulation. Meg believed that practice in previous years, requiring every step of by-hand working to be shown, made it difficult for her students to accept written records without all the steps that would be necessary in by-hand work.

“... And my kids had a real problem showing the steps of the working. [They asked]: “If I just write down the process that I have to follow, the mathematical equation and write down that I need to solve for that equation and I need to do this and do that and then just use the calculator, is that enough?”

In February, Ken had also commented that his students, especially the less able, wanted to record too many steps, even when working by-hand. In November Neil believed that it was his more capable students who were recording too many steps, inserting algebraic manipulation. By the end of year 12 it might have been expected that students and teachers would feel more confident working in parallel with CAS

and pen-and-paper but it remained an issue, more for some students than the teachers. Here we see a struggle to establish a new socio-mathematical norm.

The instruction on the examination paper (Figure 2) “Appropriate working must be shown if more than one mark is available” remained problematic throughout the year. Neil stated “*Well, what does that [mean]? That's meaningless now [with CAS]*”. One teacher responded that a ‘bit of work’ should be shown, and Neil responded “*Show a bit of work? Show the work that you value? I mean, we don't show, even in year 7, all the work. Even year 7's can skip steps*”. There are decisions to be made everyday of what can be taken-as-shared. Lucy suggested that the focus for students might profitably shift from a command to “*show your working*” to a request “*can you let us know what you're doing*”, which could be interpreted more broadly.

Lucy clearly supported the idea that CAS might promote more of an overview of solution processes (reflected in a more condensed written record) and she had observed this in her classes.

“... that's surprised me a bit; just how good some of the kids can get at saying ‘Oh, I can see that what I need to do here is [solve] two equations in two unknowns’. They're much clue-ier seeing that [a type of problem exists] inside a problem. [They might say]: ‘So now I know that I've got to solve, I've got to define this function that way and I'm going to solve it for this, for this and for that’ ...”

Lucy believed her students had made progress in that they could focus on solving at a macro level and were content that the details, essentially routine procedures, were to be performed by CAS. If, as Lucy suggests, students see an overview of a solution and they can articulate the processes being used to solve then maybe they are going to be able to produce good written records to describe these solutions, without worrying too much about whether they should include detail within those processes.

## WHAT CAS SYNTAX IS ACCEPTABLE IN WRITTEN RECORDS?

The teachers generally discouraged their students from using CAS syntax in their written records, but this was still an issue for them in November. Teachers acknowledged that students had started to use CAS words to communicate mathematical thinking. What represented accepted or CAS specific language was of concern to them in both meetings. Meg explained that she encouraged students to write a description of the process to be used: “*I told them ... to write down the procedure, what they were going to [use]; write it down, [such as] solve for x, solve for k, solve for whatever.*” Neil was concerned with the “different feel” of the calculator used at his school. This included the different ways of approaching problems which it encouraged (space precludes examples of this interesting point) and also the different syntax. Neil dealt with this by explicitly instructing his students not to use CAS language:

“Write down what a mathematician could understand. Write down a logical sequence but don't use the calculator [language]’ ... So just before the kids went into the exams I just said to them ‘Remember don't say CALC-DIFF’...and they [asked] ‘what will we say?’

And someone else in the group would say ‘[write] differentiate’ and then they’d say ‘Can we use the word SOLVE?’ I said, ‘Well, yes, because it’s a mathematical word.’”

This shows that Neil categorised SOLVE as appropriate to record in contrast to CALC-DIFF (the first and second menu items required to carry out a differentiation) which he saw as brand specific CAS syntax. Here we see that some students were unsure of what was standard mathematical language and what was a specific CAS word right until the examination. Ken helped us to see this issue from another point of view. Ken was new to teaching senior mathematics, and so he was not as firmly enculturated into this world as the other teachers. For example, to him, the notation for solving simultaneous equations in Figure 1 was standard: “*You could even put a and b in [curly] brackets .... That's accepted notation, isn't it.*” Ken made comments such as this throughout the year. When Neil said that his students used the language TRNS ALPHA B APPROX, Ken observed that his brand of CAS “...*doesn't have that [nonstandard] calculator language... You don't run into that problem.*”, although later he agreed that the symbol ! for substitute was an example of calculator language. Ken showed how CAS-specific language had become taken-as-shared in his classroom and highlighted the difficulties that novices may have in distinguishing standard from calculator-oriented practice. Ken also commented: “*I used to say 'Let y equal f(x)', but now you say 'DEFINE f(x) equal..*” and Meg and Lucy both agreed with this, implying that this would be good practice in the examinations. In this case, DEFINE is a command used by the brands of calculators in their two schools. This claim by others that DEFINE should be used, perturbed Neil as he saw it as brand specific language (STORE has a somewhat similar function on his calculator).

Fundamental to this is the question of what is considered ‘standard notation’ and what is ‘syntax’. Comments by teachers suggest that the line between these may be blurring and that some CAS language or syntax had become standard in these classrooms. Student examination scripts in fact showed some use of CAS syntax (Ball, 2003). This suggests that some commands that might be considered syntax by teachers may be standard mathematical practice from the perspective of students. This is not unexpected when CAS is the normal technology in the classroom.

## CONCLUDING REMARKS

From February to November, teachers came to moderately comfortable personal positions about the advice they gave to their students. The first main issue was the conflict between expectations that students will show working, and the fact that CAS does not report intermediate results. However, as experience of CAS syntax and the differences between brands grew during the year, they became more aware that students needed explicit guidance about what calculator language was acceptable in written work. The teachers’ concern for students’ welfare meant that the demands of the examination dominated their actions. To differing extents, they also looked beyond this. Discussion with their students about written records and use of CAS seemed to be a key factor to helping students develop good practices. This involves

deciding what can be taken-as-shared and what are acceptable warrants for mathematical explanation, as teachers and students grapple to establish new socio-mathematical norms for the new environment. These teachers raised issues and worked towards agreed understanding about good practice.

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### References

- Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning*, 7(3), 245-274.
- Ball, L. (2003). Communication of mathematical thinking in examinations: Features of CAS and non-CAS student written records for a common year 12 examination question. *The International Journal of Computer Algebra in Mathematics Education*, 10(3), 183-194.
- Ball, L., & Stacey, K. (2004). A new practice evolving in learning mathematics: Differences in students' written records with CAS. In M. Johnsen Høines, & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, Bergen: PME. (Vol. 2, pp.87-94).
- Ball, L., & Stacey, K. (2003). What should students record when solving problems with CAS? Reasons, information, the plan and some answers. In J. T. Fey, A. Cuoco, C. Kieran, L. Mullin, & R. M. Zbiek (Eds.), *Computer Algebra Systems in Secondary School Mathematics Education* (pp. 289-303). Reston, VA: NCTM
- CAS-CAT Research Project. (n.d.). <http://extranet.edfac.unimelb.edu.au/DSME/CAS-CAT>. Retrieved 10 January 2005.
- Flynn, P., & Asp, G. (2002). Assessing the potential suitability of "show that" questions in CAS-permitted examinations. In B. Barton, K. C. Irwin, M. Pfannkuck, & M. O. J. Thomas (Eds.), *Mathematics Education in the South Pacific. Proceedings of the 25th Annual Conference of the MERGA* (pp. 252-259). Sydney: MERGA.
- Guin D., & Trouche, L. (1999). The complex process of converting tools into mathematical instruments: the case of calculators. *International Journal of Computers for Mathematical Learning*, 3, 195-227.
- Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: interaction in the classroom culture*. Hillsdale, NJ: Erlbaum.
- Stacey, K. (2003). Using computer algebra systems in secondary school mathematics: Issues of curriculum, assessment and teaching.. In W-C. Yang, S-C. Chu, T. de Alwis & M-G. Lee (Eds.), *Proceedings of the 8th Asian Technology Conference in Mathematics* (pp. 40-54). USA: ATCM.
- Victorian Curriculum and Assessment Authority (2002). *VCE Mathematical Methods (CAS) Pilot study: Written examination 2*. Melbourne: Author.