THE COMPETENT USE OF THE ANALYTIC METHOD IN THE SOLUTION OF ALGEBRAIC WORD PROBLEMS. A didactical model based on a numerical approach with junior high students

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The study proves that a didactical model based in a method to solve word problems of increasing complexity which uses a numerical approach was essential to develop the analytical ability and the competent use of the algebraic language with students from three different performance levels in elementary algebra. It is shown that before using the analysis ("numerical analysis") comprised in the method, a "preparatory analysis" is required. It was observed that when the use of this analytical process is given sense, the student advances in his/her ability to establish the relationships between the elements of the problem, which central aspect is the numerical equivalence between two quantities that mean the same in the problem. The study also revealed some of the obstacles that obstruct the analytic development.

INTRODUCTION

In the previous works [see, e.g., Filloy, Rubio, 1993; Rubio, 2002], it has been proven that the use of a didactical model based in the analytical method of numerical exploration makes possible the unleash of analytical processes that allow the student to symbolize arithmetic-algebraic word problems with one equation, where the numerical approach plays a mediating role between the arithmetic and algebraic methods. The evidences presented hereby are linked to the performance of 14-15 year-old in tests, clinic interviews and their work in the classroom by using the analytical method of numerical exploration, but having now as central objective to clarify the relationship between the development of the analytical ability of junior high students to solve new algebraic word problems of increasing complexity [see, e.g. Bednarz, Dufour-Janvier, 1994, Bednarz, 2001] and the evolution they acquire in the competent use of the algebraic language.

The empirical research shows that the use of the analysis ("numerical analysis") contained in the phases 2 and 3 of the analytical method of numerical exploration fosters the development of the student's ability to establish and produce meanings for: a) the numerical relationships between the unknowns; b) the relationships between them and the data; and; c) the comparison between two quantities which represent the same in the problem, that is, that they are equivalent regarding their meaning. The study shows that such numerical comparison is essential for the student to transit not only towards the symbolization of the problem with one equation, but also to be able to give sense to the equivalence between the two algebraic expressions comprising it which will be essential for the student to detach from the concrete

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model and, at the same time, to build the meanings that will allow him/her to give sense to the algebraic method to solve problems. Finally, the research revealed obstacles (cognitive tendencies, Filloy, 1991) that obstruct the student to start or continue with the analytical process to solve some families of problems never tackled before.

REFERENTIAL AND THEORETICAL FRAMEWORK

In [Rubio, 2002], it was said that the analytical method of numerical exploration has as an historical paradigm the method of the false position (*Regula Falsi*), which can be found in the Babilonians and in the Medieval treatises [Radford, 1996], as well as in text books of the 19^{th} and 20^{th} Centuries [Rubio, 2002]. The most important matter we have taken from this antique method to solve problems is the analytical intention its first steps have, where the use of the analysis is comprised, that is, the "assumption that the problem is solved" [Chabornneau, 1996)], but instead of using a literal as a solution of the problem, as it is used in the Cartesian Method, a hypothetical numerical quantity is designated to one of the unknowns of the problem. Both methods try to facilitate the analysis of the problems and to treat all of them in a similar way, as if they were the same problem (different from the arithmetic method where each problem is analyzed case by case). However, both proposals finally separate from each other because they have different projects to solve a problem; in the case of the (*Regula Falsi*), through one proportion and in the Cartesian Method with an equation.

The analytical method of numerical exploration used in the study tried to pick up in its Phases the use of the analysis included in both projects, but using their "numerical" aspect to obtain the equation that symbolizes the problem and to give sense to the use of the algebraic expressions comprising the same. The numerical approach is also framed within the historical perspective of the algebra development where the analysis is considered as a central process to solve problems algebraically where the hypothesis is its nucleus [Chabornneau, 1996]. Finally, it must be said that in the construction of the analytical method of numerical exploration the stages established by Piaget (1979) were also taken into account in relation to the assimilation process of the real facts to the mathematical-logical structures in the development of theories constructed from the physical experiences verified by such theories [see, e.g. Rubio, 1994].

METHODOLOGY

In this stage of our research, videotaped clinical interviews were carried out to six 14 to 15 year-old junior high students who had received teaching in elementary algebra topics in their previous courses. These students were chosen out of 45 students through a classification in the classrooms based on their performance with two diagnostic tests on arithmetical-algebraic problems and the solution of equations. The investigation used as a driving factor the scheme proposed in [Filloy, Rojano, Solares, 2002] to implement a controlled teaching system, within which the population to be studied is chosen.

THE EMPIRICAL STUDY: Development of the Analytical Ability

Due to the lack of space, we will show in this document that using of the numerical exploration method were the didactical proposal is based in, only with Berenice's case, which was the lowest level performance case. Through some episodes of three out of the nine videotaped interviews carried out to this case we will show the development of the analytical ability to solve new arithmetic-algebraic word problems of increasing complexity, the progressive creation of meanings for the algebraic expressions contained in the equations symbolizing such problems and some of the obstacles obstructing the solution of a problem. It was evidenced that it is only when such obstacles are eliminated that it is possible to continue with the production of meanings for algebraic expressions, making possible to advance in the competent use of the algebraic language.

The Analytic Method of Numerical Exploration [see, e.g. Rubio, 2002]. The method consists in seven phases, with which it is tried to: i) Clarify the unknowns of the problem (writing them separately) and the relationship between them (Phase 1); ii) Assume the problem as solved by designating a numerical hypothetical quantity to one of the unknowns, obtaining as of it, the numerical quantity of the other unknowns (from both the main and the secondary) (Phase 2); iii) To compare two numerical quantities meaning the same in the context of the problem (Phase 3), which is a essential background so the student can build the sense of use of the equivalence between two algebraic expressions of the equation symbolizing a problem (Phases 4 and 5). The Phases 6 and 7 of the analytical method of numerical exploration are linked to the syntactical part of the didactical model which is yet unpublished [Rubio, 1995], where a method to solve families of progressive complexity equations is used.

I. The Preparatory Analysis. The 1st interview shows us that the use of the "numerical analysis" included in the numerical approach, Phases 2 and 3, requires a previous analytical process, which we will call "preparatory analysis". Here the student must write the unknowns separately (make the unknowns explicit) and understand what is the relationship between them. The following episode shows that student Berenice can use the analysis ("numerical analysis") successively and be able to symbolize the problem with one equation until she is able to establish clearly the relationship between the unknowns. The interview begins when Berenice is posed with the problem of the belt which says: *"For $4.80 a belt and its buckle was bought. If the belt costs $4 more than the buckle, how much does each thing cost separately?.* The student reads the problem and writes the unknowns down as a list: price of the belt and price of the buckle (Phase 1). Then, she goes to Phase 2, which she does not obey since she does not designate an arbitrary numerical amount to one of the unknowns, but she wrongly establishes that the price of the belt is 4 dollars. Let us see:

Interviewer: …Why do you write the price of the belt there... which is four?

Berenice: Because it says here that it costs 4 dollars… [Later, she is told:

Interviewer: ...Then, why do they ask you how much does the belt and the buckle cost? …Where does it say… the belt cost this much?

Berenice: The belt costs four dollars more than the buckle. [Berenice realizes that there is a comparison relationship between the unknowns]

Interviewer: If, for example, your shirt costs $4 more than my sock…? Have explained myself?

Berenice: Then it will be how much does each thing cost!

This evidences that it is only until Berenice realizes that there is a comparison relationships between the unknowns of the problem that she can carry out the "preparatory analysis" and pass to the use of the "numerical analysis" (Phases 2 and 3) and finally symbolize the problem with the equation (Phases 4 and 5): w+w+4=4.80, where w is the value of the buckle.

II. Progressive Construction of Meanings for Algebraic Expressions. In the 2nd interview, Berenice was posed two problems of the same type, but of greater complexity. It could be seen that she advanced in her analytical development since she carried out a correct "preparatory analysis" and used the analysis ("numerical analysis") properly. However, once she symbolized the following problem (Phases 4 and 5): *"21400 will be distributed among three persons in such a way that the first person has half of the second and the third what the other two have together. How much does every person received?"* with the equation: x+x/2+x+x/2=21,400, where "x" is what the 2nd person has, she had problems expressing which was the meaning of the algebraic expression x/2. Let us see:

Interviewer:… What does the x/2 is?

Berenice: Half of the x value

Interviewer: And in the problem, what does it represent?

Berenice: The money of the second person is half of the first one [she is only translating]

Interviewer: Then, what does x/2 is? ... the numbers can help and tell you what it is..

Berenice: …the money of the first person.

Once Berenice gives the meaning of x/2, she says that: x+x/2 is the money of the 3rd person and that "x+x/2+x+x/2" is the total of the money distributed.

III. The Analytical Development in a New but Simple Problem. In the 3rd interview, Berenice was presented with the problem: "There are chickens and rabbits in a stock-yard, the heads are counted and are 460; the feet are counted and are 1492. How many chickens and how many rabbits are there in the stock-yard?" which has a different scheme from the others she has faced until now. She shows her analytical development by using the "preparatory analysis" (Phase 1) and the "numerical analysis" (Phases 2 and 3) achieving the symbolization with one equation (Phases 4 and 5): 2(460-X)+ 4X=1492. She is able to identify the meaning of each term of the equation and what is the most relevant for her algebraic development is that she gives a sense of algebraic use to the equal sign when she says that: 2(460-X)+ 4X is equivalent to 1492 because both represent the total number of feet.

IV. A New Problem Where an Obstacle Arises and Which Impedes to Continue with the Analytical Development. In another part of the interview, Berenice faced the *Problem of the guided tour: One group of students has to do a collection to pay a guided tour. If each one of them gave $62, they would be lacking $200. If each one gave $82, then they will have $1000 in excess. How many students are there in the group?*, which is totally new for her since at until this moment of the interview she had only faced problems symbolized with one equation, where in one of its sides is a number and this problem is symbolized with an algebraic equation (of the type ax+b=cx+b). This and other similar problems served to evidence the presence of an obstacle of semantic order (linked to the construction of meanings for the equivalence of two "quantities" that represent the same in the context of the problem) which impedes Berenice to continue with the analysis and symbolization of the problem using an equation and stopping her analytical development and algebraic. The above makes us think that problems of this kind will comprise a "didactical cut" (Filloy, Rojano, 1989) of semantic order, since it is created before the student had obtained the equation.

A) The Analytical Process Before the Obstacle Arises. Berenice follows Phases 1, 2 and 3 of the analytical method of numerical exploration to solve the "problem of the guided visit" carrying out the "preparatory analysis", where she makes the unknown "number of students" explicit. Once this is done, she uses the "numerical analysis" (assuming that 80 students is the solution) to obtain the relationships of the problem, establishing the comparison between two numerical quantities that mean the same in the problem (the total of the money paid), which is written as follows: $5160=$5560? Although Berenice manages to obtain this comparison, she is not able to give sense to its use yet. It must be said that the latter is not only the culminant aspect of the analytical process, but also what allows the advancement in the analytical and algebraic development, since as of such comparison the equation symbolizing the problem can be obtained, as well as to give sense to the equivalence of two different algebraic expressions.

B) The Interruption of the Analytical Process When a Obstacle Arises. The following episode shows that a obstacle arises in Phase 4 (which specifies to work backwards as of the comparison, in this case: $5160=$5560?, recovering the operations made to obtain the quantities comprising the comparison). Berenice writes under 5160 the addition: 4960+200, but when writing the operation where the 5560 came from, she erases what she just wrote. This indicates the presence of an obstacle, linked with negative cognitive tendencies [Filloy, 1991], that impede her to continue with the recovery process of operations in both sides of the comparison. Let us see:

Interviewer: Why did you erase?

Berenice: Because for me ... I believe … the comparison of these two results are different and I don not know exactly if this [she points out the amount of 5560 pesos which is at the right side of the comparison] ...well, this is not the correct one … and it is what I was going to write as the result.

It can be observed that the obstacle is connected to the cognitive tension she still has between the arithmetic and equivalence uses (algebraic) of the equal sign being noticed that the first one prevails in spite that moments before she had used this notion in Phase 3 by comparing two numerical "quantities" (5160 = 5560?), which mean the same in the problem's context. She interprets the numerical quantities of such comparison as two different and isolated results, where the 5560 is considered as the result of what is in the left side. This represents a problem for Berenice since she knows it is not correct due that 5560 was obtained from numerical operations different to those in the left side of the equal sign. She solves the question incorrectly by saying that 5560 is not the correct value and erasing all what she had done.

The obstacle also impeded Berenice to resume the Phases of the analytical method of numerical exploration and give sense to the comparison and recover the operations:

Interviewer: …What have you written in the comparison?

Berenice: The total of money of… [she points out to the amounts 5160 and 5560]

Interviewer: …And, is not it what you want?

Berenice: No…

Interviewer: Don't you want to compare the total of money?

Berenice: No.

In order to overcome the obstacles that have arose, Berenice is reminded that the sense of phase 3 is to compare two quantities that mean the same in the problem and when recovering operations she has to forget about the meanings of the operations, which she can resume later when solving the problem. Then she says with a smile:

Berenice: Yes, but because Phase 4 says that we have to recover the operations, then I have to recover the two operations I made.

Finally, the student can detach from the obstacles giving sense to the comparison: 5160 = 5560? and to the recovery of operations of both numerical quantities. As of this, she achieves giving sense to the equivalence between the two algebraic expressions of the equation 62X+200=80X-1000 that she obtained by following Phases 4 and 5 of the numerical exploration method. Let us see:

Interviewer: What is all this? [Pointing out to 62X+200]

Berenice: is all they collect to go to the guided tour

Interviewer: What is all this? [Pointing out to 82X – 1000]

Berenice: The money they needed to go to the guided visit

Interviewer: What are we going to find upon the solution of the equation?

Berenice: The number….of students.

CONCLUSIONS

As it can be observed in the episodes of Berenice's interviews, the use of the numerical exploration method to solve algebraic-arithmetic word problems was determining to develop the analytical ability of the student. The empirical evidences obtained in the experimental work (more than 20 hours of videotaped interviews with six students from three stratums of knowledge are kept) show that such development was fundamental so the students of the study were able to progress in the symbolization of new problems of progressive complexity and in the creation of meanings for algebraic expressions contained in the equations that represent the problems. The study also revealed two different and complementary analytical moments in the symbolization process of a problem: i) the "preliminary analysis", which is the moment when the unknowns must be make explicit and the relationships between them (if there is more than one unknown) must be understood, and ii) the use of the analysis ("numerical analysis") which begins with the assumption that the problem is solved and has its culminant point when the comparison between two quantities having the same meaning in the context of the problem is establish and that it ends when acquiring the equation symbolizing such problem.

The investigation also evidenced that when using the analysis ("Analysis is al the Heart of Algebra", Chabornneau, 1996) within a stratum language more concrete than the algebraic one, as the one comprised in the analytic method of numerical exploration, allows the student to produce meanings with which he/she can give sense, first, to the equivalence of two "quantities" that represent the same in the problem and, as of this, to the equivalence between the two algebraic expressions contained in the equation symbolizing the problem. It was observed that the development of these skills resulted as fundamental to symbolize and solve an arithmetic-algebraic word problem and to advance in the competent use of the algebraic language. Finally, the empirical research allowed to prove the presence of obstacles that obstruct the continuation of the analytical process in order to symbolize a problem. An obstacle of this kind, linked to the tension between the arithmetic use and the equivalence use of the equal sign [Kieran, 1981] arise when the students are presented for the first time a problem that is symbolized with an algebraic equation. It is thought that this obstacle presents a "didactical cut", of semantic order, in the transit of the arithmetic thought to the algebraic one.

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