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Hierarchical Boundary Element Preconditioners in Domain Decomposition Methods

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1 Introduction

For a non–overlapping domain decomposition of a bounded domain $\Omega \subset I\!R^n$ $(n = 2, 3)$ we consider a variational problem to find $u \in V$ such that

$$a(u, v) \;=\; f(v) \tag{1.1}$$

holds for all test functions $v \in V$. This formulation corresponds to a mixed boundary value problem for a self–adjoint and elliptic partial differential operator. The Hilbert space V is given by all functions $u \in H^{1/2}(\Gamma_S)$ vanishing on the Dirichlet boundary Γ_D; Γ_S denotes the skeleton of the domain decomposition and $f(\cdot)$ is a given bounded linear form. The symmetric and V–elliptic bilinear form in (1.1) is given by

$$a(u, v) \;=\; \sum_{i=1}^{p} \int_{\Gamma_i} (S_i u_{|\Gamma_i})(x) \cdot v_{|\Gamma_i}(x)\, ds_x \;, \tag{1.2}$$

where S_i denotes the locally defined Steklov–Poincaré operators mapping the local Dirichlet data $u_{|\Gamma_i}$ onto the Neumann data t_i. This Dirichlet–Neumann map can be expressed explicitly by boundary integral operators in terms of the boundary integral equations

$$\left.\begin{array}{rcl} (V_i t_i)(x) & = & (\frac{1}{2}I + K_i)u_{|\Gamma_i}(x) - (N_0^i f)(x), \\ (D_i u_{|\Gamma_i})(x) & = & (\frac{1}{2}I - K_i')t_i(x) - (N_1^i f)(x) \end{array}\right\} \text{for } x \in \Gamma_i. \tag{1.3}$$

The mapping properties of all operators introduced above are well known [Cos88]. The symmetric representation of S_i follows immediately from (1.3),

$$(S_i u_{|\Gamma_i})(x) \;=\; \left[D_i + \tilde{K}_i' V_i^{-1} \tilde{K}_i \right]\; u_{|\Gamma_i}(x) + N_1^i f - \tilde{K}_i' V_i^{-1} N_0^i f \,, \tag{1.4}$$

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where

$$(\tilde{K}_i u_{|\Gamma_i})(x) \;=\; \left\{ \begin{array}{cl} (K_i u_{|\Gamma_i})(x) & \text{for } x \in \Gamma_D, \\ (\frac{1}{2}I + K_i)u_{|\Gamma_i}(x) & \text{elsewhere.} \end{array} \right.$$

Let $V_h \subset V$ be a finite dimensional subspace, then the Galerkin–Bubnov discretization of (1.1) based on the symmetric formulation (1.4) leads to the algebraic system of linear equations

$$\sum_{i=1}^{p} A_i^\top \left(D_{h,i} + \tilde{K}_{h,i}^\top V_{h,i}^{-1} \tilde{K}_{h,i} \right) A_i \underline{u} \;=\; \underline{f}\,. \tag{1.5}$$

The local stiffness matrices are given by

$$\begin{array}{rcl} D_{h,i}[\ell, k] &=& \langle D_i {}'^{\mu}_{k}, {}'^{\mu}_{\ell} \rangle_{L^2(\Gamma_i)}\,, \\ \tilde{K}_{h,i}[s, k] &=& \langle \tilde{K}_i {}'^{\mu}_{k}, {}'^{\nu}_{s} \rangle_{L^2(\Gamma_i}\,, \\ V_{h,i}[s, r] &=& \langle V_i {}'^{\nu}_{r}, {}'^{\nu}_{s} \rangle_{L^2(\Gamma_i)} \end{array}$$

for $k, \ell = 1, \ldots, N_i,\; r, s = 1, \ldots, M_i$ and where A_i denotes Boolean matrices describing the transformation of the global numbering into the local one. The ${}'^{\mu}_{k}$ and ${}'^{\nu}_{r}$ are appropriate trial functions, e.g. smoothest piecewise polynomial B–splines of degree μ and ν, respectively. To solve the symmetric and positive definite system (1.5) by the conjugate gradient iteration scheme we need an optimal preconditioner to keep the numerical amount of work as low as possible. The construction of a preconditioner is essential for domain decomposition algorithms based either on a finite element or a boundary element discretization of the original problem as well as on coupling both. There are numerous different approaches to solve the corresponding finite element equations by using hierarchical [BPS87, SBG96, Wid88] or of Neumann–Neumann type preconditioners [LeT94] or [HW92] with boundary elements. However, the resulting spectral condition number of the preconditioned system matrix often depends on mesh and material parameters of the model. Here we give a general technique to construct optimal preconditioners independent of these bad parameters by using the symmetric representation of the local Steklov–Poincaré operators and its spectral equivalence to the Galerkin discretization of the hypersingular integral operator.

2 Spectral Equivalence Inequalities

To construct an optimal preconditioner C_S for the assembled stiffness matrix

$$S_h \;=\; \sum_{i=1}^{p} A_i^\top S_{h,i} A_i \tag{2.6}$$

we first consider the local matrices

$$S_{h,i} \;=\; D_{h,i} + \tilde{K}_{h,i}^\top V_{h,i}^{-1} \tilde{K}_{h,i}\,. \tag{2.7}$$

Obviously, we have the lower spectral equivalence inequality

$$(D_{h,i} \underline{v}_i, \underline{v}_i) \;\leq\; (S_{h,i} \underline{v}_i, \underline{v}_i) \tag{2.8}$$

for all $\underline{v}_i \in I\!R^{N_i}$ due to the $H^{-1/2}(\Gamma_i)$–ellipticity of the single layer potential operators. For two–dimensional problems suppose $\mathrm{diam}(\Omega_i) < 1$ [HW77]. Since V_i is a self–adjoint pseudodifferential operator, the operator $T_i = \tilde{K}_i' V_i^{-1} \tilde{K}_i$ is self–adjoint and $H^{1/2}(\Gamma_i)$–elliptic, too. If we denote by $T_{h,i}$ the Galerkin discretization with the matrix entries

$$T_{h,i}[\ell, k] \;=\; \langle T_i \varphi_k^\mu, \varphi_\ell^\mu \rangle_{L^2(\Gamma_i)}$$

for $k, \ell = 1, \ldots, N_i$, we get the upper spectral equivalence inequality [Ste96]

$$(\tilde{K}_{h,i}^\top V_{h,i}^{-1} \tilde{K}_{h,i} \underline{v}_i, \underline{v}_i) \;\leq\; (T_{h,i} \underline{v}_i, \underline{v}_i) \tag{2.9}$$

due to the ellipticity of V_i, and by adding $(D_{h,i} \underline{v}_i, \underline{v}_i)$,

$$(S_{h,i} \underline{v}_i, \underline{v}_i) \;\leq\; ((D_{h,i} + T_{h,i}) \underline{v}_i, \underline{v}_i) \tag{2.10}$$

for all $\underline{v}_i \in I\!R^{N_i}$. This means, that the discrete Steklov–Poincaré operator $S_{h,i}$ is bounded by the Galerkin discretization of the continuous Steklov–Poincaré operator. Since S_i and D_i are both $H^{1/2}(\Gamma_S)$ semi–definite and bounded, the discrete Steklov–Poincaré operator $S_{h,i}$ is spectrally equivalent either to $D_{h,i} + T_{h,i}$ or to $D_{h,i}$. This result holds independent of the dimension and the discretization, i.e. of the mesh and trial functions used. Altogether we have the spectral equivalence inequalities

$$\sum_{i=1}^{p} (D_{h,i} \underline{v}_i, \underline{v}_i) \;\leq\; \sum_{i=1}^{p} (S_{h,i} \underline{v}_i, \underline{v}_i) \;\leq\; \sum_{i=1}^{p} ((D_{h,i} + T_{h,i}) \underline{v}_i, \underline{v}_i) \tag{2.11}$$

with $\underline{v}_i = A_i \underline{v}$. Employing the isomorphism $\underline{v} \in I\!R^N \leftrightarrow v_h \in H^{1/2}(\Gamma_S)$, this is also equivalent to

$$c_1 \cdot \sum_{i=1}^{p} ||v_h||^2_{H^{1/2}(\Gamma_i)} \;\leq\; \sum_{i=1}^{p} (S_{h,i} \underline{v}_i, \underline{v}_i) \;\leq\; c_2 \cdot \sum_{i=1}^{p} ||v_h||^2_{H^{1/2}(\Gamma_i)}$$

where the constants are independent of the discretization parameters.

Let us denote by v_I^i the piecewise linear interpolant of a function $v^i \in H^{1/2}(\Gamma_i)$ with $v_I^i(x_C) = v^i(x_C)$ for all coarse grid nodes x_C associated with a mesh size H_i. Then from Sobolev's imbedding theorem, one obtains the error estimate

$$||v^i - v_I^i||^2_{H^{1/2}(\Gamma_i)} \;\leq\; c \cdot L(s) \cdot H_i^{2s-1} \cdot ||v^i||^2_{H^s(\Gamma_i)} \tag{2.12}$$

for $s > \frac{n-1}{2}$, where $L(s) = (2s - n + 1)^{1-n}$. For a function $v_h \in V_h$ and $v_h^i = v_{h|\Gamma_i}$, this inequality, together with the inverse inequality in V_h, implies the stability condition

$$||v_h^i - v_I^i||^2_{H^{1/2}(\Gamma_i)} \;\leq\; c \cdot K(h_i, H_i) \cdot ||v_h^i||^2_{H^{1/2}(\Gamma_i)} \tag{2.13}$$

where

$$K(h_i, H_i) = \gamma^{n-2} \cdot (\log \gamma)^{n-1}, \; \gamma = \frac{H_i}{h_i} \, .$$

For the bilinear form

$$c(v, v) \;=\; \sum_{i=1}^{p} \left\{ ||v - v_I||^2_{H^{1/2}(\Gamma_i)} + ||v_I||^2_{H^{1/2}(\Gamma_i)} \right\} \tag{2.14}$$

then we find the spectral equivalence inequalities

$$\frac{c_1}{(1+K(h,H))} \cdot c(v_h,v_h) \;\leq\; \sum_{i=1}^{p} (S_{h,i}\underline{v}_i, \underline{v}_i) \;\leq\; c_2 \cdot c(v_h,v_h) \;, \tag{2.15}$$

which correspond to the spectral equivalence inequalities for finite element preconditioners of hierarchical type, c.f. [BPS87, Wid88]. According to (2.14) and the mapping properties of the hypersingular integral operators, the preconditioning bilinear form is given by

$$\tilde{c}(u,v) \;=\; \sum_{i=1}^{p} \left\{ \langle D_i(u-u_I), v-v_I \rangle_{L^2(\Gamma_i)} \;+\; \langle D_i u_I, v_I \rangle_{L^2(\Gamma_i)} \right\} \,. \tag{2.16}$$

3 Preconditioners

For a given function $v_h \in V_h \leftrightarrow \underline{v} \in I\!R^N$ the splitting

$$v_h(x) \;=\; \tilde{v}_h + v_I(x), \quad \tilde{v}_h(x) = v_h(x) - v_I(x)$$

corresponds to the basis transformation

$$\begin{pmatrix} \underline{v}_H \\ \underline{\tilde{v}} \end{pmatrix} = \begin{pmatrix} I_q & 0 \\ -I_h & I_{N-q} \end{pmatrix} \underline{v} \,. \tag{3.17}$$

Here, I_q is the identity matrix of dimension q corresponding to the number of unknown coarse grid nodes; I_{N-q} is the identity matrix for all remaining fine grid nodes and I_h is the discrete counterpart of the linear interpolation. The Galerkin discretization of the preconditioning form (2.16) now leads to the matrix representation

$$C_{S,1} \;=\; \begin{pmatrix} I_q & -I_h^\top \\ 0 & I_{N-q} \end{pmatrix} \begin{pmatrix} D_{HH} & 0 \\ 0 & D_{hh} \end{pmatrix} \begin{pmatrix} I_q & 0 \\ -I_h & I_{N-q} \end{pmatrix}, \tag{3.18}$$

where D_{HH} and D_{hh} are the assembled stiffness matrices for the coarse and fine grid trial functions, respectively. In general, this preconditioner corresponds to the BPS preconditioner [BPS87], which was also used for elasticity problems in [SBG96]. However, the diagonal matrix in (3.18) may be computed exactly by using the hypersingular boundary integral operator. If we have given a fine grid preconditioner C_h, which is spectrally equivalent to D_{hh}, then the resulting hierarchical preconditioner is given by

$$C_{hier}^{-1} \;=\; \begin{pmatrix} I_q & 0 \\ I_h & I_{N-q} \end{pmatrix} \begin{pmatrix} D_{HH}^{-1} & 0 \\ 0 & C_h^{-1} \end{pmatrix} \begin{pmatrix} I_q & I_h^\top \\ 0 & I_{N-q} \end{pmatrix}. \tag{3.19}$$

Due to the spectral equivalence inequalities (2.15), the spectral condition number of the preconditioned system is bounded by

$$\kappa(C_{hier}^{-1} S_h) \;\leq\; c \cdot (1 + K(h,H)) \,,$$

which depends on the discretization parameters, i.e. on the relation of the coarse to the fine grid mesh sizes.

Since this preconditioner (3.19) is not optimal, we consider a second one. For the matrix

$$S_{h,2} = \sum_{i=1}^{p} A_i^\top D_{h,i} A_i \tag{3.20}$$

we conclude from (2.11), that the spectral condition number of the preconditioned system $C_{S,2}^{-1} S_h$ is bounded by a constant, i.e.

$$\kappa(C_{S,2}^{-1} S_h) \leq c,$$

where c does not depend on the discretization parameters and not on the domain decomposition considered, either. On the other hand, we have to realize the matrix multiplication with $C_{S,2}^{-1}$ in an efficient manner. Since the matrix $C_{S,2}$ is given explicitly by the locally stored matrices $D_{h,i}$, the matrix times vector multiplications can be executed in parallel. Therefore we can use iterative schemes to realize $C_{S,2}^{-1}$, e.g. multigrid methods [CKL96] or a conjugate gradient iteration using the BPS type preconditioner described above.

4 Numerical Results

In this section we compare our proposed preconditioner $C_{S,2} = CG(D_h)$ with the known BPS preconditioner in the case of the Laplace equation. We further show, that this technique can be used also in the case of linear elasticity. We compare the number of cg iterations and the corresponding computing times to get a relative error reduction of 10^{-6}. All computations were made on an Intel Paragon.

For the simple model problem of the Laplace equation in the unit square and a domain decomposition into 64 subdomains we get the results shown in Table 1.1:

Table 1 Numerical results for the Laplace equation

	BPS		$CG(D_h)$	
N	Iter	sec	Iter	sec
64	24	22.52	11	41.28
128	24	26.55	12	47.01
256	25	45.47	13	62.11
512	27	127.39	13	107.80
1024	28	489.38	14	326.77

The number of iterations for the BPS preconditioner is twice the number of our proposed technique, where, on the other hand, the costs to realize the preconditioner are more expensive. Since we can bound the spectral condition number independent of the mesh size, and since the costs of the preconditioners are to set in relation to the matrix times vector multiplication itself, i.e. the solution of a mixed boundary value problem per global iteration step for the realization of the Steklov–Poincaré operator, our new preconditioner seems to be optimal, in agreement with our theory.

In Table 1.2 we present the results of our proposed preconditioner by solving a mixed boundary value problem in linear elasticity with up to 32 subdomains. As one can see, the number of iterations is nearly the same as for the Laplace equation, i.e. we have independence of the underlying partial differential equation.

Table 2 Numerical results in linear elasticity

	p=2		p=8		p=32	
N	Iter	sec	Iter	sec	Iter	sec
64	11	3.49	13	11.78	14	111.82
128	11	9.87	14	26.08	15	139.82
256	11	33.56	14	73.43	15	205.59
512	11	129.22	14	263.63	15	438.44

5 Conclusions

The proposed preconditioning technique is based on the Galerkin discretization of the hypersingular boundary integral operator; and therefore is well suited for the symmetric formulation of boundary element methods. We note, that the Galerkin discretization of the hypersingular integral operator can be reduced to the computation of weakly singular integral operators by partial integration [Ned82]. Because of the spectral equivalence of the discrete Steklov–Poincaré operator and the discrete hypersingular operator, from the latter one can derive other preconditioners of algebraic type, as e.g. multigrid methods or structured matrices like block circulant matrices, which, in turn, can be inverted by the fast Fourier transformation. The realization of the matrix times vector multiplication with the discrete Steklov–Poincaré operator requires the solution of local mixed boundary value problems. For the iterative solution of these problems one can use the concept of pseudodifferential operators of dual order [SW96], i.e. the discrete hypersingular integral operator can be used as a preconditioner of the single layer potential and vice versa. The proposed preconditioning technique is almost independent of the underlying partial differential equation and is also well suited for coupled boundary and finite element methods [Lan94].

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