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Preconditioners for Mixed Spectral Element Methods for Elasticity and Stokes Problems

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1 Introduction: Linear Elasticity and Stokes Systems

We introduce and analyze some preconditioned iterative methods for the large indefinite linear systems arising from mixed spectral element discretizations of the linear elasticity and Stokes systems in three dimensions. For other approaches to the iterative solution of spectral element methods for Stokes and Navier-Stokes problems, see Maday, Patera and Rønquist [MPR92], Fischer and Rønquist [FR94], Rønquist [Røn96], Casarin [Cas96] and the references therein. For p-version finite element preconditioners for elasticity, see Mandel [Man96].

Let $\Omega \subset R^3$ be a polyhedral domain and Γ_0 a subset of its boundary. Let $\mathbf{V}$ be the Sobolev space $\mathbf{V} = \{\mathbf{v} \in H^1(\Omega)^3 : \mathbf{v}|_{\Gamma_0} = 0\}$. The linear elasticity problem consists in finding the displacement $\mathbf{u} \in \mathbf{V}$ of the domain Ω, fixed along Γ_0, subject to a surface force of density $\mathbf{g}$ along $\Gamma_1 = \partial\Omega - \Gamma_0$ and subject to an external force $\mathbf{f}$:

$$2\mu \int_\Omega \epsilon(\mathbf{u}) : \epsilon(\mathbf{v})\ dx + \lambda \int_\Omega div\mathbf{u}\ div\mathbf{v}\ dx \ = <\mathbf{F}, \mathbf{v}> \quad \forall \mathbf{v} \in \mathbf{V}. \tag{1}$$

Here λ and μ are the Lamé constants, $\epsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ is the linearized stress tensor, and the inner products are defined as $\epsilon(\mathbf{u}) : \epsilon(\mathbf{v}) = \sum_{i=1}^3 \sum_{j=1}^3 \epsilon_{ij}(\mathbf{u})\epsilon_{ij}(\mathbf{v})$, $< \mathbf{F}, \mathbf{v} >= \int_\Omega \sum_{i=1}^3 f_i v_i\ dx + \int_{\Gamma_1} \sum_{i=1}^3 g_i v_i\ ds$. When λ approaches infinity, this pure displacement model describes materials that are almost incompressible. In terms of the Poisson ratio $\nu = \frac{\lambda}{2(\lambda+\mu)}$, these materials are characterized by ν close to $1/2$. It is well known that when low order h-version finite elements are used in the discretization of (1), the locking phenomenon causes a deterioration of the convergence rate as $h \to 0$; see Babuška and Suri [BS92]. If the p-version is used instead, locking in $\mathbf{u}$ is eliminated, but it could still be present in quantities of interest such as $\lambda div\mathbf{u}$. Moreover, the stiffness matrix obtained by discretizing the pure displacement model (1) has a condition number that goes

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to infinity when $\nu \to 1/2$. Therefore, the convergence rate of iterative methods deteriorates rapidly as the material becomes almost incompressible. Locking problems are eliminated altogether by introducing the new variable $p = -\lambda div\mathbf{u} \in L^2(\Omega) = W$ and by rewriting the pure displacement problem in a mixed formulation (see Brezzi and Fortin [BF96]): Find $(\mathbf{u}, p) \in \mathbf{V} \times W$ such that

$$\left\{ \begin{array}{rcrcll} a(\mathbf{u},\mathbf{v}) & + & b(\mathbf{v},p) & = & <\mathbf{F},\mathbf{v}> & \forall \mathbf{v} \in \mathbf{V} \\ b(\mathbf{u},q) & - & \frac{1}{\lambda}c(p,q) & = & 0 & \forall q \in W, \end{array} \right. \tag{2}$$

where $a(\mathbf{u},\mathbf{v}) = 2\mu \int_\Omega \epsilon(\mathbf{u}) : \epsilon(\mathbf{v}) dx$, $b(\mathbf{v},q) = -\int_\Omega div\mathbf{v} q dx$, $c(p,q) = \int_\Omega pq dx$. When $\lambda \to \infty$ (or, equivalently, $\nu \to 1/2$), we obtain from (2) the limiting problem for incompressible elasticity. In case of homogeneous Dirichlet boundary conditions on the whole boundary $\partial\Omega$, problem (2) is equivalent to a generalized Stokes problem, with $a(\cdot,\cdot)$ replaced by $\overline{a}(\mathbf{u},\mathbf{v}) = \mu \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} dx$ and with $c(\cdot,\cdot)$ scaled by $\lambda + \mu$ instead of λ. In this case, the pressure will have zero mean value, so we define $W = L_0^2(\Omega)$. When $\lambda \to \infty$ we obtain the classical Stokes system describing the velocity $\mathbf{u}$ and pressure p of a fluid of viscosity μ: Find $(\mathbf{u}_0, p_0) \in \mathbf{V} \times W$ such that

$$\left\{ \begin{array}{rcrcll} \overline{a}(\mathbf{u}_0,\mathbf{v}) & + & b(\mathbf{v},p_0) & = & <\mathbf{F},\mathbf{v}> & \forall \mathbf{v} \in \mathbf{V} \\ b(\mathbf{u}_0,q) & & & = & 0 & \forall q \in W. \end{array} \right. \tag{3}$$

2 Mixed Spectral Element Methods

Let Ω_{ref} be the reference cube $[-1,1]^3$, $Q_n(\Omega_{ref})$ be the set of polynomials on Ω_{ref} of degree n in each variable and $P_n(\Omega_{ref})$ be the set of polynomials on Ω_{ref} of total degree n. Let the domain Ω be decomposed into a finite element triangulation $\bigcup_{i=1}^N \Omega_i$ of nonoverlapping elements. Each Ω_i is the affine image of the reference cube $\Omega_i = F_i(\Omega_{ref})$, where F_i is an affine mapping. We discretize each displacement component by conforming spectral elements, i.e. by continuous, piecewise polynomials of degree n:

$$\mathbf{V}^n = \{\mathbf{v} \in \mathbf{V} : v_k|_{\Omega_i} \circ F_i \in Q_n(\Omega_{ref}),\ i = 1, \cdots, N, \quad k = 1,2,3\}.$$

We consider two choices for the discrete pressure space W^n:

$$\begin{array}{rcl} W_1^n & = & \{q \in W : q_i \circ F_i \in Q_{n-2}(\Omega_{ref}),\ i = 1, \cdots, N\}, \\ W_2^n & = & \{q \in W : q_i \circ F_i \in P_{n-1}(\Omega_{ref}),\ i = 1, \cdots, N\}. \end{array}$$

The first choice gives us the $Q_n - Q_{n-2}$ method that Maday, Patera and Rønquist [MPR92] proposed for the Stokes system. A very convenient basis for W_1^n consists of the tensor-product Lagrangian interpolants associated with the internal Gauss-Lobatto-Legendre (GLL) nodes, described in the next section. The second choice gives us the Method 2 analyzed in Stenberg and Suri [SS96]. For this space standard p-version bases can be used. We will call this method $Q_n - P_{n-1}$.

Gauss-Lobatto-Legendre (GLL) Quadrature and the Discrete Problem

Denote by $\{\xi_i, \xi_j, \xi_k\}_{i,j,k=0}^n$ the set of GLL points on $Q_n(\Omega_{ref})$, and by σ_i the weight associated with ξ_i . Let $l_i(x)$ be the Lagrange interpolating polynomial vanishing at all the GLL nodes except at ξ_i, where it equals one. The basis functions on the reference cube are then defined by a tensor product as $l_i(x)l_j(y)l_k(z)$, $0 \leq i,j,k \leq n$. This is a nodal basis, since every polynomial in $Q_n(\Omega_{ref})$ can be written as $u(x,y,z) = \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} u(\xi_i,\xi_j,\xi_k)l_i(x)l_j(y)l_k(z)$. We then replace each integral of the continuous model (2) by GLL quadrature sums:

$$(u,v)_{Q,\Omega} = \sum_{s=1}^{N}\sum_{i,j,k=0}^{n}(u\circ F_s)(v\circ F_s)|J_s|(\xi_i,\xi_j,\xi_k)\sigma_i\sigma_j\sigma_k,$$

where $|J_s|$ is the determinant of the Jacobian of F_s. The analysis of this discretization technique can be found in Bernardi and Maday [BM92] and Maday, Patera and Rønquist [MPR92]. Applying this spectral element discretization to (2), we obtain the following discrete elasticity problem: Find $(\mathbf{u},p) \in \mathbf{V}^n \times W^n$ such that

$$\left\{ \begin{array}{lclcll} a_Q(\mathbf{u},\mathbf{v}) & + & b_Q(\mathbf{v},p) & = & <\mathbf{F},\mathbf{v}>_{Q,\Omega} & \forall \mathbf{v}\in\mathbf{V}^n \\ b_Q(\mathbf{u},q) & - & \frac{1}{\lambda}c_Q(p,q) & = & 0 & \forall q \in W^n, \end{array}\right. \tag{4}$$

where $a_Q(\mathbf{u},\mathbf{v}) = 2\mu(\epsilon(\mathbf{u}):\epsilon(\mathbf{v}))_{Q,\Omega}$, $b_Q(\mathbf{v},q) = -(div\mathbf{v},q)_{Q,\Omega}$, $c(p,q) = (p,q)_{Q,\Omega}$. This is a saddle point problem with a penalty term and has the following matrix form:

$$Kx = \left[\begin{array}{cc} A & B^T \\ B & -\frac{1}{\lambda}C \end{array}\right] x = b\ . \tag{5}$$

The stiffness matrix K is symmetric and indefinite. It is less sparse than the one obtained by low-order finite elements, but is still well-structured. In the incompressible case, the C block is zero. For the Stokes problem, the discretization of the equivalent formulations (3) leads to an analogous block structure, with A consisting of three uncoupled discrete Laplacians.

The inf-sup Constant for Spectral Elements

The convergence of mixed methods depends not only on the approximation properties of the discrete spaces $\mathbf{V}^n$ and W^n, but also on a stability condition known as the inf-sup (or LBB) condition; see Brezzi and Fortin [BF96]. While many important h-version finite elements for Stokes problems satisfy the inf-sup condition with a constant independent of h, the important spectral elements proposed for Stokes problems, such as the $Q_n - Q_{n-2}$ and $Q_n - P_{n-1}$ methods, have an inf-sup constant that approaches zero as $n^{-(d-1)/2}$ ($d = 2,3$). This result has been proven for the $Q_n - Q_{n-2}$ method by Maday, Patera and Rønquist [MPR92], where an example is constructed showing that this estimate is sharp. Stenberg and Suri [SS96] proved the following, more general, result covering both methods.

Theorem 1. (Stenberg and Suri [SS96]) Let the spaces $\mathbf{V}^n$ and W^n satisfy assumptions (A1)-(A4) of [SS96] (satisfied by both our methods). Then for $d = 2,3$

$$\sup_{\mathbf{v}\in\mathbf{V}^n\setminus\{0\}} \frac{(div\mathbf{v},q)}{\|\mathbf{v}\|_{H^1}} \geq Cn^{-(\frac{d-1}{2})}\|q\|_{L^2} \quad \forall q\in W^n,$$

where the constant C is independent of n, N and q.

In matrix form, the inf-sup condition becomes $q^tBA^{-1}B^tq \geq \beta_0^2 q^tCq$, $\forall q \in W^n$,

Table 1 Substructuring preconditioner: local condition numbers of $\hat{S}^{-1}S$

n	ν						
	0.3	0.4	0.49	0.499	0.4999	0.49999	0.499999
8	59.6995	64.5997	122.126	176.323	187.449	188.659	188.781
ν	n						
	2	3	4	5	6	7	8
0.499999	-	60.3303	89.1704	112.048	137.546	162.999	188.781

where $\beta_0 = Cn^{-(\frac{d-1}{2})}$ is the inf-sup constant of the method. Therefore β_0^2 scales as $\lambda_{min}(C^{-1}BA^{-1}B^t)$. Numerical experiments by Maday, Patera and Rønquist [MPR92], have shown that for the $Q_n - Q_{n-2}$ method, for practical values of n (e.g. $n \leq 16$), the dependence of β_0 on n is much weaker. Our numerical experiments show that the situation is even better for the $Q_n - P_{n-1}$ method. The trade-off in this case is the loss of a tensorial basis.

3 Preconditioned Iterative Methods

We will consider three classes of preconditioners: a) block-diagonal and b) triangular preconditioners for the whole indefinite system $Kx = b$; and c) substructuring methods for the Schur complement S of K associated with the interface variables. a) and b) are based on recent work by Klawonn [Kla96] on standard h-version finite elements, while c) is based on the wire basket spectral element methods introduced by Pavarino and Widlund [PW96] for the scalar case.

Block-diagonal Preconditioners

Consider the block-diagonal preconditioner with positive definite blocks $\hat{A}$ and $\hat{C}$:

$$\hat{D} = \begin{bmatrix} \hat{A} & 0 \\ 0 & \hat{C} \end{bmatrix} . \tag{6}$$

$\hat{A}$ and $\hat{C}$ are assumed to be good preconditioners for A and C respectively:
i) $\exists a_0, a_1 > 0$ such that $a_0^2 \mathbf{v}^t \hat{A} \mathbf{v} \leq \mathbf{v}^t A \mathbf{v} \leq a_1^2 \mathbf{v}^t \hat{A} \mathbf{v}, \quad \forall \mathbf{v} \in \mathbf{V}^n$;
ii) $\exists c_0, c_1 > 0$ such that $c_0^2 q^t \hat{C} q \leq q^t C q \leq c_1^2 q^t \hat{C} q, \quad \forall q \in W^n$. Interesting choices for $\hat{A}$ are given by h-version finite element discretizations on the GLL mesh or by substructuring domain decomposition methods, where a_0 and a_1 have a polylogarithmic dependence on the spectral degree n (for the scalar case, see Pavarino and Widlund [PW96] and Casarin [Cas96]). Since the resulting preconditioned system is symmetric, we can use the Preconditioned Conjugate Residual Method (PCR); see Hackbusch [Hac94]. Combining Klawonn's result ([Kla96], pp. 46-47) and Theorem 1, we obtain the following convergence result.
Theorem 2. If K is the stiffness matrix of the discrete system (4) obtained with either the $Q_n - Q_{n-2}$ or the $Q_n - P_{n-1}$ method and $\hat{D}$ is the block-diagonal preconditioner

Table 2 Exact block-diagonal preconditioner: iteration counts for $Q_n - Q_{n-2}$ on one element (in brackets are the iterations counts with the inexact Q_1 **u**-block and exact p-block)

n	ν							
	0.3	0.4	0.49	0.499	0.4999	0.49999	0.499999	0.5
9	11 (57)	15 (72)	31 (139)	39 (173)	41 (179)	41 (179)	41 (179)	41 (179)
ν	n							
	2	3	4	5	6	7	8	9
0.5	1 (1)	7 (13)	21 (48)	31 (84)	35 (111)	37 (134)	39 (158)	41 (179)

(6), then

$$cond(\hat{D}^{-1}K) \leq C\beta_0^{-1} = Cn^{(\frac{d-1}{2})}, \quad d = 2, 3.$$

This implies that the number of iterations of our algorithm is bounded by $Cn^{(\frac{d-1}{2})}$.

Triangular Preconditioners

Consider the lower and upper triangular preconditioners

$$\hat{T}_L = \begin{bmatrix} \hat{A} & 0 \\ B & \hat{C} \end{bmatrix}, \qquad \hat{T}_U = \begin{bmatrix} \hat{A} & B^T \\ 0 & \hat{C} \end{bmatrix}, \tag{7}$$

where $\hat{A}$ and $\hat{C}$ are positive definite matrices. We will denote by T_L and T_U the case with exact blocks $\hat{A} = A$ and $\hat{C} = C$. Since the resulting preconditioned system is no longer symmetric or positive definite, we need to use Krylov methods for general nonsymmetric systems. We will consider three relatively recent methods: GMRES, Bi-CGSTAB and QMR; see Freund, Golub and Nachtigal [FGN92]. We remark that each application of the inverse of the triangular preconditioners $\hat{T}_L$ or $\hat{T}_U$ is only marginally more expensive than the block-diagonal preconditioner, because in addition to the solution of a system for $\hat{A}$ and one for $\hat{C}$, it requires only one application of B (or B^t). Klawonn ([Kla96], p. 56) proved that the spectrum of $T^{-1}K$ is real and positive. Combining Klawonn's result and Theorem 1, we obtain the following result.

Theorem 3. If K is the stiffness matrix of the discrete system (4) obtained with either $Q_n - Q_{n-2}$ or $Q_n - P_{n-1}$ spectral elements and T is the lower or upper triangular preconditioner (7) with exact blocks , then

$$cond(T^{-1}K) \leq C\beta_0^{-2} = Cn^{(d-1)}, \quad d = 2, 3.$$

The case of a triangular preconditioner with inexact blocks is studied in Theorem 5.2 in Klawonn [Kla96], pg. 59, under the standard assumptions i) and ii) of the previous section. The estimate provided is analog to the case with exact blocks, but it is more complicated and we refer to [Kla96] for the details.

A Substructuring Preconditioner

For scalar elliptic problems, a complete study of substructuring methods for h-version finite elements can be found in Dryja, Smith and Widlund [DSW94]. For the spectral

Table 3 Exact block-diagonal preconditioner: iteration counts for $Q_n - Q_{n-2}$ on many elements

n	$N = N_x \times N_y \times N_z$	ν							
		0.3	0.4	0.49	0.499	0.4999	0.49999	0.499999	0.5
2	$8 = 2 \times 2 \times 2$	6	7	7	7	7	7	7	7
2	$64 = 4 \times 4 \times 4$	10	13	19	21	21	21	21	21
2	$216 = 6 \times 6 \times 6$	10	13	21	23	23	23	23	23

Table 4 Exact lower-triangular preconditioner: iteration counts for $Q_n - Q_{n-2}$ on one element; G=GMRES, B=Bi-CGSTAB, Q=QMR

n	ν							
	0.3	0.4	0.49	0.499	0.4999	0.49999	0.499999	0.5
	G B Q	G B Q	G B Q	G B Q	G B Q	G B Q	G B Q	G B Q
11	7 4 6	9 5 8	21 14 18	28 23 24	29 32 25	29 25 25	29 23 25	29 23 25

ν	n							
	4	5	6	7	8	9	10	11
	G B Q	G B Q	G B Q	G B Q	G B Q	G B Q	G B Q	G B Q
0.5	11 10 12	19 14 18	21 15 20	22 20 21	24 20 22	26 17 23	28 20 25	29 23 25

element case, see Pavarino and Widlund [PW96] and Casarin [Cas96]. If we order first all interior variables and then all the interface variables, $(\mathbf{u}_I, p, \mathbf{u}_B)$ (we recall that pressure unknowns are only interior), the stiffness matrix K can be reordered in the block form

$$K = \begin{pmatrix} K_{II} & K_{IB} \\ K_{IB}^T & K_{BB} \end{pmatrix}.$$

Eliminating the interior variables, we are left with the solution of a linear system with the Schur complement $S = K_{BB} - K_{IB}^T K_{II}^{-1} K_{IB}$. Our substructuring method will define a preconditioner for S. We further subdivide the interface variables into face and wire basket variables $\mathbf{u}_B = (\mathbf{u}_{\mathcal{F}}, \mathbf{u}_{\mathcal{W}})$, so that S can be reordered in the block form

$$S = \begin{pmatrix} S_{\mathcal{F}\mathcal{F}} & S_{\mathcal{F}\mathcal{W}} \\ S_{\mathcal{F}\mathcal{W}}^T & S_{\mathcal{W}\mathcal{W}} \end{pmatrix}.$$

Our additive preconditioner $\hat{S}$ is built from independent solvers associated with each face $\mathcal{F}_i$ (local problems) and the wire basket $\mathcal{W}$ (coarse problem):

$$\hat{S}^{-1} = \sum_{faces \mathcal{F}_i} R_{\mathcal{F}_i}^T S_{\mathcal{F}_i\mathcal{F}_i}^{-1} R_{\mathcal{F}_i} + R_0^T S_{\mathcal{W}\mathcal{W}}^{-1} R_0,$$

where R_0 represent a change of basis for $\mathcal{W}$ and $R_{\mathcal{F}_i}$ are restrictions matrices. Each local solver $S_{\mathcal{F}_i\mathcal{F}_i}^{-1}$ and $S_{\mathcal{W}\mathcal{W}}^{-1}$ can be replaced by an appropriate approximate solver. In joint work with O. Widlund, we are in the process of analyzing this algorithm using the Schwarz framework and recent work by Casarin [Cas96] for Stokes problems.

Table 5 Exact lower-triangular preconditioner: iteration counts for $Q_n - Q_{n-2}$ on many elements; G=GMRES, B=Bi-CGSTAB, Q=QMR

n	N	ν							
		0.3	0.4	0.49	0.499	0.4999	0.49999	0.499999	0.5
		G B Q	G B Q	G B Q	G B Q	G B Q	G B Q	G B Q	G B Q
2	2^3	3 2 4	4 2 4	4 3 4	4 3 4	4 3 4	4 3 4	4 3 4	4 3 4
2	4^3	5 4 6	6 4 7	9 6 10	10 7 11	10 7 11	10 7 11	10 7 11	10 7 11
2	6^3	6 4 6	7 5 7	10 7 11	11 7 12	11 7 12	11 7 12	11 7 12	11 7 12

4 Numerical Results

All the computations were performed in MATLAB 4.2 on Sun SPARC stations. The model problem considered is (2) on the reference cube $[-1,1]^3$, discretized with the $Q_n - Q_{n-2}$ or $Q_n - P_{n-1}$ spectral element methods. The resulting discrete systems have a matrix structure as in (5). The iterative methods considered are PCR for the block-diagonal preconditioner and GMRES (without restart), Bi-CGSTAB and QMR for the triangular preconditioner. The initial guess is zero and the right-hand side consists of uniformly distributed random numbers in [-1,1]. The stopping criterion is $\|r_i\|_2/\|r_0\|_2 \leq 10^{-6}$, where r_i is the $i-$th residual. We considered mainly preconditioners with exact blocks, in order to study the algorithms under the best of circumstances (inexact **u**-blocks based on piecewise linear Q_1 finite elements on the GLL mesh are considered in Table 2). For brevity, we report only the results for the $Q_n - Q_{n-2}$ method. The $Q_n - P_{n-1}$ iteration counts were consistently better, thanks to a better inf-sup constant. More details for the block-diagonal and triangular preconditioners can be found in Pavarino [Pav96a], [Pav96b].

The results reported in the following tables agree with the theory: the convergence rate of the proposed methods is independent of ν and N but is mildly dependent on n (almost linearly for incompressible materials and Stokes problems) via the inf-sup constant.

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