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Newton-Krylov Domain Decomposition Solvers for Adaptive hp Approximations of the Steady Incompressible Navier-Stokes Equations with Discontinuous Pressure Fields

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1 Introduction

In this paper, we extend non-overlapping domain decomposition techniques, previously developed for second order elliptic problems and the Stokes operator [LP96], to the solution of incompressible flow problems governed by the Navier-Stokes equations. In this approach, we will reduce the original problem to a problem set on the subspace of divergence free functions, and apply existing domain decomposition techniques to the resulting sub-problem. The advantage of this approach is to greatly reduce the size of the algebraic systems that have to be solved.

Adaptive hp finite elements, in which the spectral order and element size are independently varied over the whole domain, are capable of delivering solution accuracies far superior to classical $h-$ or $p-$version finite element methods, for a given discretization size. Several researchers [BS87, DORH89, ROD89] have, in fact, shown that the reduction in discretization error with respect to number of unknowns can be exponential for general classes of elliptic boundary value problems, as opposed to the asymptotic algebraic rates observed for h or p-version finite element methods. Together with multiprocessor computing, these methods thus offer the possibility of orders-of-magnitude improvement in computing efficiency over existing finite element models.

The principal computational cost in any finite element solution is encountered in the solver. For the nonlinear Navier-Stokes equations, solved using a Newton iteration scheme, the major computational cost is in the linear solve in each iterate. If time

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stepping is used to linearize the problem, the use of implicit methods, leads to a similar situation. In a parallel computing environment, conventional direct solvers based on some variant of Gauss elimination are usually inefficient for the irregular sparse linear systems generated by adaptive *hp* discretizations. Further, these linear systems are often very poorly conditioned, ruling out most standard iterative solvers. Thus, efficient solvers, meeting the twin criteria of being parallelizable and controlling the conditioning of the system, need to be used.

In this paper, we discuss a practical and efficient parallel iterative solver that meets the above criteria. The solver combines nonlinear Newton iterations with iterative substructuring and coarse grid preconditioning. The inner solver for the linearized problem at each Newton iterate can be thought of as a combination of multiple direct solvers at the subdomain level together with a preconditioned iterative solver, to handle the interface problem efficiently in parallel. The iterative solver of choice is the GMRES algorithm.

In the remaining sections we introduce the steady state incompressible Navier-Stokes equations and its weak formulation, its finite element discretization and describe a domain decomposition iterative solver and present some numerical results.

2 The Steady Navier-Stokes Equations

We define the spaces $V = (H_0^1(\Omega))^2$ and $Q = L_0^2(\Omega)$ and a domain $\Omega \subset \mathbb{R}^2$ with boundaries $\partial\Omega$ that are assumed to be locally Lipschitz . The Navier-Stokes problem on $\Omega \subset \mathbb{R}^2$ consists of finding a velocity, pressure pair $(u, P) \in V \times Q$ satisfying

$$\begin{aligned} (u \cdot \nabla)u - \nu\Delta \cdot u + \nabla P &= f \quad in\ \Omega \\ \nabla \cdot u &= 0 \quad in\ \Omega \\ u &= g \quad on\ \partial\Omega \end{aligned}$$

where f is a body force, ν is the kinematic viscosity.

Basic Formulation and LBB condition

Let us consider the following mixed finite element approximation of the Navier-Stokes problem:

Find $u_h \in V_h$, $p_h \in W_h$ such that

$$c(u_h, u_h, v_h) + a(u_h, v_h) + \quad b(v_h, p_h) = L(v_h) \qquad \forall v_h \in V_h \quad (2.1)$$

$$b(u_h, q_h) = -b(\bar{u}, q_h) \qquad \forall q_h \in W_{h0} \quad (2.2)$$

where $u_h + \bar{u}$ is the approximate velocity field and p_h is the approximate pressure field inside an incompressible viscous fluid flowing through a given domain $\Omega \subset \mathbb{R}^2$ with imposed velocity $\bar{u}$ at the boundary $\partial\Omega$ of Ω. The domain Ω is partitioned into finite elements such that $\bar{\Omega} = \cup_e K_e$, and the finite element spaces V_h and W_{h0} are conforming finite dimensional approximations of $H_0^1(\Omega, \mathbb{R}^2)$ and $L_0^2(\Omega)$ given by

$$
\begin{array}{ll}
V_h & = \{v_h \in C(\bar{\Omega}, \mathbb{R}^2), v_{h|K_e} \in Q_k(K_e), \forall e, v_h = 0 \text{ on } \partial\Omega\}, \\
W_h & = \{q_h \in L^2(\Omega), q_{h|K_e} \in Q_l(K_e), \forall e\}, \text{ and } W_{h0} = \{q_h \in W_h, \int_\Omega q_h d\Omega = 0\}.
\end{array}
$$

In this framework the trilinear form c, the bilinear forms a and b and the linear form L are defined by

$$
\begin{array}{lll}
c(u,u,v) & = \int_\Omega (u \cdot \nabla) u . v d\Omega, & \forall u, v \in H^1(\Omega, \mathbb{R}^2) \\
a(u,v) & = \int_\Omega \nu \nabla u . \nabla v d\Omega, & \forall u, v \in H^1(\Omega, \mathbb{R}^2) \\
b(v,q) & = \int_\Omega q \operatorname{div} v d\Omega, & \forall v \in H^1(\Omega, \mathbb{R}^2) \text{ and } \forall q \in L^2(\Omega), \\
L(v) & = \int_\Omega f . v d\Omega - a(\bar{u}, \bar{v}) - c(\bar{u}, \bar{u}, v) &
\end{array}
$$

We now introduce a non-overlapping partition of Ω into a finite number of subdomains such that $\bar{\Omega} = \cup_{i=1}^N \bar{\Omega}_i$ The interface $\Gamma_i = \partial\Omega_i - \partial\Omega$ is supposed to coincide with interelement boundaries, and we suppose that the finite element spaces V_h and W_h satisfy the following LBB compatibility condition on each subdomain(including $\Omega_0 = \Omega$).

Assumption 1 There exists a constant $\beta(k)$ independent of the mesh size h, but possibly dependent on the local degree k of the finite elements, such that

$$
\inf_{q \in L_0^2(\Omega_i) \cap W_h} \sup_{v \in H_0^1(\Omega_i) \cap V_h} \left[\frac{\int_{\Omega_i} q \operatorname{div} v \, d\Omega}{||q||_{0,\Omega_i} ||v||_{1,\Omega_i}} \right] \geq \beta(k) \; \forall i = 0, ..., N \tag{2.3}
$$

3 Choice of Compatible Spaces for Adaptive *hp* FEM

We begin the discussion by defining appropriate polynomial spaces. Let $L_i(t)$ denote the Legendre polynomial of degree i. Now define $U_i(x) = \int_{-1}^x L_i(t)dt$ the integrated Legendre polynomial. Note that $U(\pm 1) = 0$. Further define the "volumetric" and "lateral" polynomial combinations over the master element $\hat{K} = \hat{I}_x \times \hat{I}_y = [-1,1] \times [-1,1]$ as

$$
J_k(\hat{K}) = \{ \sum_{i,j=1}^{k-1} a_{ij} U_i(x) U_j(y), a_{ij} \in \mathbb{R}\} \text{ and } E_k(\hat{K}) = P_1(\hat{I}_x) P_k(\hat{I}_y) \cup P_k(\hat{I}_x) P_1(\hat{I}_x)
$$

where $P_k(\hat{I}_x)$ (resp. $P_k(\hat{I}_y)$) denotes the space of polynomials of maximum degree k in x (resp. y). Different choices of finite elements for an *hp* approximation may now be defined by appropriate combinations of E_k and J_k for each element. For example, one may choose $Q_k(\hat{K}) = E_k(\hat{K}) \oplus J_k(\hat{K})$.

In the context of p version finite element methods, Stenberg and Suri [SS95] have recently proposed systematic ways of constructing compatible higher order approximations for the velocity and pressure spaces such that Assumption 1 is satisfied automatically. One such construct is $W_k(\hat{K}) = Q_{k-1}(\hat{K})$, with corresponding velocity space $V_k(\hat{K}) = Q_{k+1}(\hat{K}) \; \forall k \geq 2$.

In adaptive p and *hp* methods, the polynomial order may change from element to element with C^0 continuity of the velocity approximation being maintained by extending the higher order function on the shared edge into the lower order element. The space $E_k(\hat{K})$ must be redefined to reflect this.

For $k1, k2, k3, k4 \geq 2$

$$\begin{aligned} E_{\mathbf{k}}(\hat{K}) &= E_{k1}(\gamma_1) \oplus E_{k2}(\gamma_2) \oplus E_{k3}(\gamma_3) \oplus E_{k4}(\gamma_4) \\ &= \mathcal{P}_{k1}(\gamma_1)\mathcal{P}_1(y) \oplus \mathcal{P}_{k2}(\gamma_2)\mathcal{P}_1(x)\mathcal{P}_{k3}(\gamma_3)\mathcal{P}_1(y) \oplus \mathcal{P}_{k4}(\gamma_4)\mathcal{P}_1(x) \end{aligned}$$

where $\mathbf{k} = \{k_1, k_2, k_3, k_4\}$ and E_{ki} are the edge spaces of polynomial order ki defined on the edges γ_i. Let $k_m = max\{k_1, k_2, k_3, k_4\}$. Now the spaces for the adaptive hp-version can be redefined for this situation as

$$\begin{aligned} W_k(\hat{K}) &= E_{k1-1}(\gamma_1) \oplus E_{k2-1}(\gamma_2) \oplus E_{k3-1}(\gamma_3) \oplus E_{k4-1}(\gamma_4) \oplus J_{k_m-1}(\hat{\omega}) \\ V_k(\hat{K}) &= E_{\mathbf{k}} \oplus J_{k_m+1}(\hat{\omega}). \end{aligned}$$

This is easily implemented in the code by augmenting the order of the bubble functions for velocity to one more than the maximum of all the edge polynomial orders. The pressure shape functions are then constructed as one order less than the velocity functions on the edges and two orders less in the bubble function. As show in [LP96] Assumption 1 can be established for this construction of the spaces.

4 Solution Algorithm

The Nonlinear Iteration

For flows characterized by low Reynolds numbers ($Re = 1/\nu \approx 100$) it is often easy and possible to solve the nonlinear system of equations arising from an adaptive hp discretization of (2.1,2.2) using Newton's method. However, for higher Reynolds numbers this technique does not appear to perform well. Newton's method generates a sequence of iterates of the form $u_h^k, k = 0, 1, 2, ...$ Given u_h^{k-1} we find u_h^k by solving

$$\left. \begin{array}{lll} c(u_h^k, u_h^{k-1}, v_h) + & c(u_h^{k-1}, u_h^k, v_h) & + a(u_h^k, v_h) + b(v_h, p_h^k) = (f, v_h) + \\ & & c(u_h^{k-1}, u_h^{k-1}, v_h) \ \forall v_h \in V_h \\ b(u_h^k, q_h) & & = 0 \qquad\qquad \forall q_h \in W_{h0} \end{array} \right\} \tag{4.4}$$

Thus at each stage we need to solve a large irregularly sparse linear system. We will now discuss an efficient parallel solver for this system. The solver uses iterative substructuring with coarse grid preconditioning of the type discussed in [OPF94, LP96].

Reduction to Interface Problem

We use fast local subdomain based sparse solvers and the discontinuity of the pressure field to reduce the global problem to one posed purely in terms of the interface velocities. We start by using the structure in the p version to decompose the velocity space at the element level into the three subspaces of vertex functions (V), edge functions (E) and bubble functions (B), and augment it with a pressure space(P). The element stiffness can then be written as:

$$\mathbf{K_{elt}} = \begin{bmatrix} VV_{elt} & VE_{elt} & VP_{elt} & VB_{elt} \\ EV_{elt} & EE_{elt} & EP_{elt} & EB_{elt} \\ PV_{elt} & PE_{elt} & 0 & PB_{elt} \\ BV_{elt} & BE_{elt} & BP_{elt} & BB_{elt} \end{bmatrix} \begin{Bmatrix} u_V \\ u_E \\ P \\ u_B \end{Bmatrix}$$

We also compute a vector corresponding to $b(u_h, 1) = \bar{B}_{el}$ for subsequent use in the preconditioning form.

The bubble functions have support only inside an element, so they may be immediately eliminated using a static condensation procedure. This modifies the rest of $\mathbf{K_{elt}}$ and $\bar{B}_{el}$. The zero on the diagonal corresponding to the pressure degrees of freedom is now replaced by $\widehat{PP} = -(PB)(BB)^{-1}(BP)$. If the pressure is assumed to be continuous in each subdomain and discontinuous across interfaces (Γ_i) we obtain a subdomain stiffness matrix of the form:

$$\mathbf{K_I} = \begin{bmatrix} WW_I & WF_I & WI_I \\ FW_I & FF_I & FI_I \\ IW_I & IF_I & II_I \end{bmatrix} \qquad \mathbf{II_I} = \begin{bmatrix} \widetilde{VV_I} & \widetilde{VE_I} & \widetilde{VP_I} \\ \widetilde{EV_I} & \widetilde{EE_I} & \widetilde{EP_I} \\ \widetilde{PV_I} & \widetilde{PE_I} & \widetilde{PP_I} \end{bmatrix}$$

where WW_I and FF_I denote the vertex and edge degrees of freedom associated with subdomain interfaces on the I^{th} subdomain and II_I denotes those on the interior.

If the pressure field is assumed to be discontinuous across elements then, the pressure degrees of freedom may be eliminated at the element level with respect to an average element pressure. This elimination is carried out using a procedure identical to the static condensation used on the bubble nodes after setting one of the pressure nodes to a value of zero. This is necessary to make $\widehat{PP}$ invertible. The pressures can be computed consistently by requiring $p_{ih} \in L_0^2(\Omega_i)$. If the actual pressure on this node is denoted p, then the actual pressure is $p = \bar{P_{el}} + P_{rel}$ where, $P_{rel} \in L_0^2(\Omega_i)$ is the relative pressure computed. The value $\bar{P_{el}}$ of the pressure on the remaining node can be computed from the velocities at the subdomain level and corresponds therefore to a subdomain internal degree of freedom. It will be associated to the the local constraint of volume conservation $\int_{elt}$ div $u_{ih} = 0$. These values can be eliminated by treating this local constraint by a penalty approach on all subdomain elements except one.

The static condensation process can now be used (irrespective of the pressure approximation) at the subdomain level to obtain

$$\widetilde{\mathbf{K_I}} = \begin{bmatrix} \widetilde{WW}_I & \widetilde{WF}_I & 0 \\ \widetilde{FW}_I & \widetilde{FF}_I & 0 \\ 0 & 0 & II_I \end{bmatrix} \qquad \mathbf{S_I} = \begin{bmatrix} \widetilde{WW}_I & \widetilde{WF}_I \\ \widetilde{FW}_I & \widetilde{FF}_I \end{bmatrix}$$

Note that the same modifications are also carried out on $\bar{B}_I$.

The matrix S_I is the contribution of each subdomain to the interface operator S. The vectors B_I assemble into $\bar{B}$ described in conjunction with the solution of the interface operator. Procedures for parallel iterative solution for S are discussed next.

Interface Solver using GMRES and Divergence-Free Search Vectors

The interface operator S is non-symmetric and the iterative solver of choice is usually GMRES, a method that minimizes the residual over a Krylov space. The basic GMRES algorithm however suffers from the the drawback that the work grows quadratically and storage grows linearly with the number of iterations. The restarted version of the algorithm alleviates this difficulty to some extent, and is the one used in this study. The preconditioned version of the GMRES includes in each iteration, a solve $\mathbf{C}$ $G^n = R^n$ where C is a preconditioning operator, R^n is the residual in the n^{th} iterate and G^n is the computed search vector.

Further, to satisfy the incompressibility condition on the interface velocities, we restrict our choice of search directions to divergence free vectors. This is accomplished by modifying the preconditioning step $\mathbf{C}$ $G^n = R^n$ to

$$\begin{aligned} CG^n + \quad \bar{B}^T\bar{p} &= \quad g \\ \bar{B}G^n &= \quad 0 \end{aligned}$$

where

$$\bar{B} = \int_{\Omega_i} \nabla v_h.1 \; d\Omega_i = \int_{\Gamma_i} v_h.n \; d\Gamma$$

and $\bar{p}$ is a vector of average pressure per subdomain. This computation reduces to one coarse solve of a problem of dimension equal to the number of subdomains per application of the preconditioner, and the initial cost of setting up and factoring $\bar{B}C^{-1}\bar{B}^T$.

Choice of Preconditioner

As described in [OPF94, LP96] matrix S is naturally blocked into a small portion $(\widetilde{WW})$ corresponding to the nodes on the interface and the larger portion corresponding to the unknowns associated with the edges$(\widetilde{FF})$ and their interactions $\widetilde{WF}$ and $\widetilde{FW}$. As a preconditioner C, we explore two choices, denoted C_1 and C_2, analogous to the choices in [LP96]. These are a) the $\widetilde{WW}$ block and the diagonals of $\widetilde{FF}$ and b) the $\widetilde{WW}$ block and the block diagonals of $\widetilde{FF}$. Block diagonals correspond to the degrees of freedom associated with a particular edge. In matrix notation these are:

$$\mathbf{C_1} = \begin{bmatrix} \widetilde{WW}_I & 0 \\ 0 & diag(\widetilde{FF}_I) \end{bmatrix} \qquad \mathbf{C_2} = \begin{bmatrix} \widetilde{WW}_I & 0 \\ 0 & diagB(\widetilde{FF}_I) \end{bmatrix}$$

where $diag(\widetilde{FF}_I)$ and $diagB(\widetilde{FF}_I)$ denote the diagonal and block diagonal respectively.

Figure 1 Driven Cavity Flow for low Reynolds numbers. Solution obtained using two level Newton-Krylov domain decomposition solver.

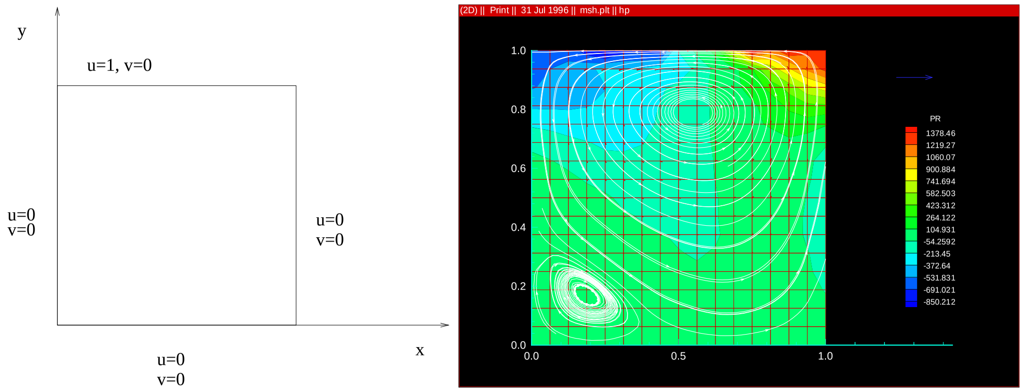
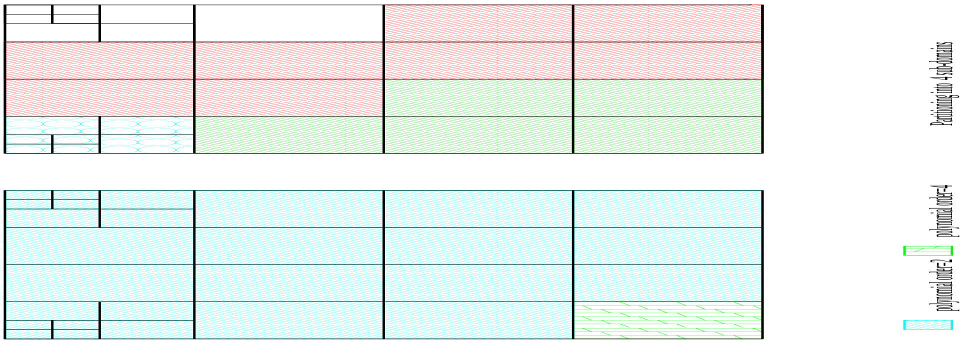


Figure 2 Sample hp adapted mesh for the problem and partitioning into 4 subdomains. Solver converged in 9 preconditioned GMRES and 6 Newton iterations.

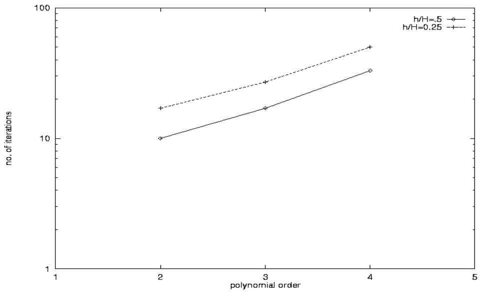


5 Numerical Results

The algorithm described in this paper is new and only validation studies for low Reynolds number ($\frac{1}{\nu} \approx 100$) are currently available. Numerical experience indicates a strong dependence on the highest polynomial order used and a weaker dependence on the smallest element size used. Fig. 1 shows the test problem and results obtained on a uniform discretization of 64 quadratic elements using this solver. Fig. 2 shows a sample hp adaptive mesh and its partitioning into 4 subdomains, also used on the same problem. The GMRES algorithm used in the inner loop for the linearized problem shows good convergence (see Fig. 3) for polynomial orders $p \leq 4$ and values of $h/H \geq .125$, where h/H is the ratio of mesh size to subdomain size.

The use of the two level iterative scheme permits us to use inexact solves in the linear solver and still obtain fairly rapid convergence of the overall solution algorithm. This option needs further testing to establish minimum levels of accuracy in the inner loop to maintain convergence rates in the outer loop. Experience seems to indicate that between 10 and 15 GMRES iterations are adequate.

Figure 3 Convergence rates obtained for inner loop GMRES solver using C_1 preconditioner, for different choices of h/H (ratio of minimum element size to subdomain size) and polynomial order p.



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