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Iterative Substructuring Preconditioners for the Mortar Finite Element Method

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1 The Mortar Element Method

The mortar element method, first introduced by C.Bernardi, Y.Maday and A.T.Patera in [BMP90], has the advantage of allowing non-matching nonoverlapping grids at the interfaces between subdomains. Therefore, the method permits implementation of different approximations on each subdomain, which means that grids can be built completely independently. It is designed to provide an efficient parallelizable evaluation and solution framework. In our case, we use the finite element method for each subdomain. In [RLJK96], we describe more fully the space of Lagrange multipliers chosen.

Let us consider the model elliptic problem in Ω: Find u in $H_0^1(\Omega)$ such that

$$\forall v \in H_0^1(\Omega), \int_\Omega \nabla u . \nabla v \ dx + c \int_\Omega uv \ dx = \int_\Omega fv \ dx \tag{1.1}$$

This formulation is very handy for introducing nonoverlapping domain decompositions. Indeed, assume that Ω is partitioned into nonoverlapping (Lipschitz) subdomains $\overline{\Omega} = \bigcup_{k=1}^K \overline{\Omega}^k, \quad \Omega^k \cap \Omega^\ell = \emptyset$ if $k \neq \ell$

Problem (1.1) can be rewritten as follows: *Find $u \in H_0^1(\Omega)$ such that*

$$\forall v \in H_0^1(\Omega), \quad \sum_{k=1}^K \int_{\Omega^k} \nabla(u_{|\Omega^k}) \nabla(v_{|\Omega^k}) dx + c \sum_{k=1}^K \int_{\Omega^k} u_{|\Omega^k} v_{|\Omega^k} \ dx = \sum_{k=1}^K \int_{\Omega^k} f_{|\Omega^k} v_{|\Omega^k} dx$$

Instead of searching an element u defined globally over Ω, it is more convenient, especially when local discretizations are to be used, to search for a K-uple $u^* = (u_1, \ldots, u_K)$. The space V spanned by these restrictions

$$V = \{v^* = (v_1, \ldots, v_K), \quad \exists v \in H_0^1(\Omega), \forall k, 1 \leq k \leq K, \ v_k = v_{|\Omega^k}\}$$

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can be conveniently rewritten as an aggregate of the local spaces

$$X_k = \{v_k \in H^1(\Omega^k), \quad v_k = 0 \text{ over } \partial\Omega^k \cap \partial\Omega\}$$

as follows

$$V^* = \{v^* = (v_1, \ldots, v_K) \in \Pi_{k=1}^K X_k, \quad \forall k, \ell,\ 1 \le k, \ell \le K,\ v_k = v_\ell \text{ over } \partial\Omega^k \cap \partial\Omega^\ell\}.$$

This leads naturally to introduce the notation $\Gamma_{k,\ell} = \partial\Omega^k \cap \partial\Omega^\ell$. The constraint across the interface $\Gamma_{k,\ell}$ can be relaxed by inducing the definition of a Lagrange multiplier in the Euler equation. The Lagrange multiplier belongs to a closed subspace M of $\Pi_{1\le k<\ell\le K} H^{-1/2}(\Gamma_{k,\ell})$. The problem (1.1) is equivalent to the following one : *Find* $u^* \in V^*$ *such that*

$$\forall v^* \in V^*, \quad \sum_{k=1}^K \int_{\Omega^k} \nabla u_k \nabla v_k dx + c \sum_{k=1}^K \int_{\Omega^k} u_k v_k dx = \sum_{k=1}^K \int_{\Omega^k} f_k v_k dx \tag{1.2}$$

Discretization

We discretize the problem by the Galerkin method. Let us consider a parameter h standing for a discretization parameter. For any value of h, for any k, $1 \le k \le K$, we introduce a finite dimensional subspace X_h^k of $X_k \cap C^0(\overline{\Omega}^k)$. For any k, $1 \le k \le K$, $\Gamma^{k,j}$, $1 \le j \le j(k)$ stand for the (eventually curved) segments which coincide with the edges of Ω^k, ($j(k)$ denote the number of edges of Ω^k). We then define the skeleton S as the union of all edges of all subdomains: $S = \bigcup_{k=1}^K \bigcup_{j=1}^{j(k)} \overline{\Gamma}^{k,j}$. Finally, we choose a finite set $\mathcal{M}$ of pairs $m = (k,j)$ such that the $\Gamma^{k,j}$ are disjoint from each other. We denote by γ^m, and we call mortars, these $\Gamma^{k,j}$. To describe the discrete space, we begin by defining trace spaces.

- First, for any k, $1 \le k \le K$ and for any j, $1 \le j \le j(k)$, we set $W_h^{k,j}$
$$W_h^{k,j} = \{v_{|\Gamma^{k,j}}, v \in X_h^k\} \quad .$$
- Next, for any $m^* = (k,j)$ not in $\mathcal{M}$, we choose a space $\tilde{W}_h^m$ of discrete functions on the non-mortar sides. The product of all these spaces provides a global discretization $\tilde{W}_h$ of the functions on the skeleton S by $\tilde{W}_h = \Pi_{m\notin\mathcal{M}} \tilde{W}_h^m$.

For any $m \in \mathcal{M}$, we denote by W_h^m the space $W_h^{k(m),j(m)}$. The mortar space is defined by $W_h = \{\varphi; \varphi_{|\gamma^m} \in W_h^m, m \in \mathcal{M}\}$. The discrete space X_h is the space of functions v_h on Ω such that:

- For any k, $1 \le k \le K$, $v_{h,k} = v_{h|\Omega^k} \in X_h^k$.
- there exists a function $\varphi \in W_h$ such that:
  If $\Gamma^{k,j}$ is a mortar, $v_{h,k|\Gamma^{k,j}} = \varphi$
  If $\Gamma^{k,j}$ is not a mortar
$$\forall \psi \in \tilde{W}_h^{k,j}, \int_{\Gamma^{k,j}} (v_{h,k|\Gamma^{k,j}} - \varphi)\psi \, d\tau \quad = \quad 0 \quad .$$

The discretized variational formulation is: *Find* $u_h \in X_h$ *such that*

$$\forall v_h \in X_h,\ \sum_{k=1}^{K} \int_{\Omega^k} \nabla u_{h,k}.\nabla v_{h,k}\ dx + c\sum_{k=1}^{K} \int_{\Omega^k} u_{h,k} v_{h,k}\ dx = \sum_{k=1}^{K} \int_{\Omega^k} f v_{h,k}\ dx \quad . \tag{1.3}$$

The problem can be reformulated into a saddle point problem.
Let a_h be the symmetric bilinear form on $X_h \times X_h$:

$$a_h(u_h, v_h) = \sum_{k=1}^{K} \int_{\Omega^k} \nabla u_{h,k}.\nabla v_{h,k}\ dx + c\sum_{k=1}^{K} \int_{\Omega^k} u_{h,k} v_{h,k}\ dx \quad ,$$

and b_h the bilinear form on $X_h \times \tilde{W}_h$:

$$b_h(v_h, \mu_h) = \sum_{1 \le k < \ell \le K} \int_{\Gamma_{k,\ell}} (v_{h,k} - v_{h,\ell})\mu_h \quad .$$

We can associate to a_h the linear operator A_h and to b_h the linear operator B_h such that $a_h(u_h, v_h) = (A_h u_h, v_h)$ and $b_h(v_h, \mu_h) = (B_h v_h, \mu_h)$. Therefore, the problem (1.3) admits a following saddle-point formulation: Find the pair (u_h, λ_h) in $X_h \times \tilde{W}_h$ such that

$$\begin{aligned} A_h u_h + B_h^t \lambda_h &= f_h \\ B_h u_h &= 0 \quad . \end{aligned} \tag{1.4}$$

2 Extension of the Dual Schur Method Preconditioner

Conforming Case

Let us consider a subdomain Ω_i. We number its degrees of freedom beginning by those lying inside Ω_i and finishing by those lying on the interfaces between Ω_i and the others subdomains. With this numbering, the stiffness matrix of Ω_i has the following block representation:

$$A_i = \begin{pmatrix} A_{ii} & A_{if} \\ A_{fi} & A_{ff} \end{pmatrix} \tag{2.5}$$

The restriction of the matrix B on interface is the matrix which makes the correspondence between the degrees of freedom on interface and the degrees of freedom of the Lagrange multipliers.
The dual operator on each subdomain is given by $D^{(i)} = B_i A_i^{-1} B_i^t$

The interpretation of the preconditioner is to find a matrix M which is a good approximation of D^{-1} so as to apply the conjugate gradient method on a well conditioned problem. The preconditioner chosen in [Rou89] is $\bar{M} = \sum_i B_i A_i B_i^t$.

Nonconforming Case

The interface matrix is given by the bilinear form associated to the trace operator seen in (1.4).

The interface matrix is written as $B_i = P_i R_i$ where P_i is a projection matrix and R_i is the restriction matrix on the interface.

Therefore, the dual operator matrix D is given by

$$D = \sum_i B_i A_i^+ B_i^t = \sum_i P_i R_i A_i^+ R_i^t P_i^t.$$

The preconditioner chosen for the nonconforming case is

$$M2 = \sum_i (P_i P_i^t)^{-1} (P_i R_i A_i R_i^t P_i^t)(P_i P_i^t)^{-1}$$

where $(P_i P_i^t)^{-1}$ are matrix terms.

3 Hierarchical Basis of the Lagrange Multipliers Space

Our motivation comes from the work of H. Yserentant, [Yse86]. In his paper, the condition number of the stiffness matrices arising in the discretization of selfadjoint and positive definite elliptic problems by finite element methods when using hierarchical basis of the finite element spaces instead of the usual nodal bases is analysed. It is showed in [BDY88] that the condition number of such a stiffness matrix behaves like $0((\log K)^2)$ where K is the condition number of the stiffness matrix with respect to a nodal basis. In case of a triangulation with uniform mesh size h this means that the stiffness matrix with respect to a hierarchical basis has a condition number behaving like $0((\log \frac{1}{h})^2)$ instead of $0((\frac{1}{h})^2)$ for a nodal basis.

Therefore, in the same idea, we consider the dual operator matrix with respect to a hierarchical basis not of the finite element space but of the Lagrange multipliers space.

We begin in this section by stating the basic methodology for the building of a hierarchical basis. We start with a coarse initial mesh $\mathcal{T}_1$. Beginning with this mesh, we construct a nested family $\{\mathcal{T}_\alpha\}$ of meshes. In this section, $\mathcal{T}_{\alpha+1}$ is obtained from $\mathcal{T}_\alpha$ by subdividing any elements of the mesh $\mathcal{T}_\alpha$.

The space $V(\Omega)$ is approached by the succession of the finite element spaces corresponding $\{V_\alpha(\Omega)\}$. $V_{\alpha+1}$ is obtained from V_α by adding the basis functions $\Phi_A^{\alpha+1}$ on the nodes introduced at this level of refinement and by not changing all the old basis functions. Obviously we have $V_1(\Omega) \subset V_2(\Omega) \subset \ldots \subset V_\alpha(\Omega) \subset V_{\alpha+1}(\Omega) \subset \ldots \subset V(\Omega)$ We have the relation $V_{\alpha+1} = V_\alpha \oplus \nu_{\alpha+1}$ where $\nu_{\alpha+1}$ is the subspace of $V_{\alpha+1}$ consisting of all finite element functions vanishing in the nodes of $\mathcal{T}_k$, of level l with $1 \leq l \leq k$. Therefore, that means the hierarchical basis of $V_{\alpha+1}$ is the direct sum of the hierarchical basis of V_α and the nodal basis of $\nu_{\alpha+1}$.

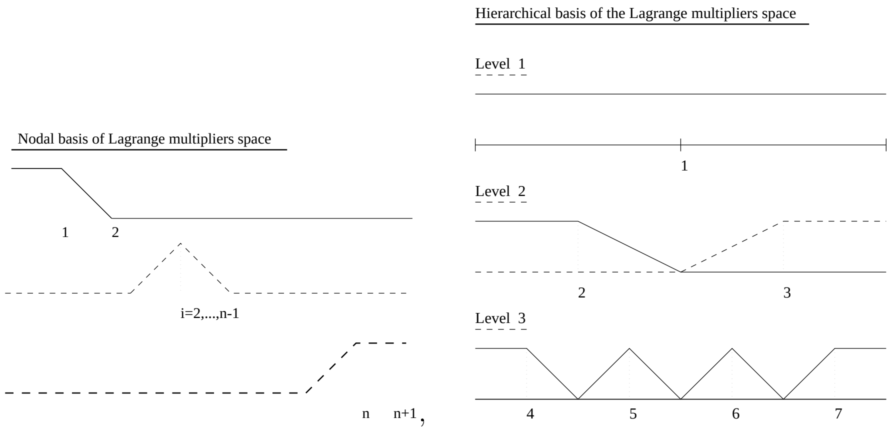
The next figure shows how we choose the Lagrange multipliers space.

4 A Block Diagonal Preconditioner

We remind that finally we arrive to an algebraic saddle point problem:

$$\begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} U \\ \lambda \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix} \tag{4.6}$$

Figure 1 The Lagrange multiplier space



where A is a block diagonal matrix and B is the interface matrix, the jump operator. The system is equivalent to

$$\mathcal{A}\begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \tag{4.7}$$

Let $\mathcal{B}$ be a symmetric and positive definite matrix of the same size as $\mathcal{A}$.
Suppose that eigenvalues of the spectral problem $\mathcal{A}x = \nu\mathcal{B}x$ belong to the union of the segments $[d_1; d_2] \cup [d_3; d_4]$ where $d_1 \leq d_2 < 0 < d_3 \leq d_4$. Then it is possible to implement the generalized Lanczos method of minimal iterations to solve the saddle point problem $\mathcal{A}x = y$.
In [Kuz95], to give the motivation of their choice for the preconditioner $\mathcal{B}$, the eigenvalue problem is considered:

$$\begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix}\begin{pmatrix} u \\ \lambda \end{pmatrix} = \nu \begin{pmatrix} A & 0 \\ 0 & BA^{-1}B^t \end{pmatrix}\begin{pmatrix} u \\ \lambda \end{pmatrix} \tag{4.8}$$

Obviously this problem can have only three nontrivial solutions $\{\frac{1-\sqrt{5}}{2}; 1; \frac{1+\sqrt{5}}{2}\}$.
A preconditioner $\mathcal{B}$ is taken with $R_u \sim A$ and $R_\lambda \sim BA^{-1}B^t$:

$$\mathcal{B} = \begin{pmatrix} R_u & 0 \\ 0 & R_\lambda \end{pmatrix} \tag{4.9}$$

Our idea is to test

$$R_u = A \quad \text{and } R_\lambda^{-1} = M = \sum (P_i P_i^t)^{-1}(P_i R_i A_i R_i^t P_i^t)(P_i P_i^t)^{-1}$$

5 Numerical Results

The Schur dual interface matrix $D = \sum_{i=1}^{K} B_i A_i^+ B_i^t$ corresponds to the discretization of a compact operator. Because of this compactness, the eigenvalues of D accumulate towards 0 when h goes to 0, and the high end of the spectrum is less populated than the lower end. The spectral distribution of the interface problem has important consequences on the convergence rate of the conjugate gradient algorithm. During the first iterations, the conjugate gradient captures the eigenvalues corresponding to the low modes of the structure. Since D has only a few relatively high eigenvalues that correspond to the low physical modes of the structure, the CG algorithm applied to the solution of the dual problem gives quickly a good approximation of the displacement.

Figure 2 Spectral density with and without preconditioner, 2×2, $c = 1$, $h_1^{-1} = h_2^{-1} = 24, h_3^{-1} = h_4^{-1} = 32$

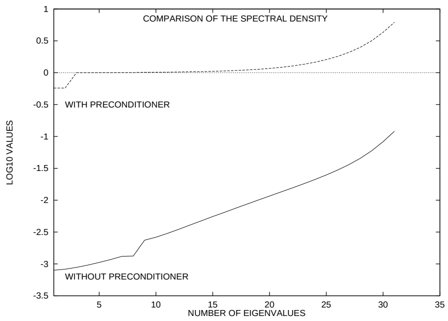


Figure 2 shows that the first preconditioner reduces the condition number of the cluster of small eigenvalues of the dual interface problem, and therefore, favors a superconvergence behavior of the CG algorithm. Figure 3 highlights the superconvergence effect.

Figure 3 Residual with hierarchical/nodal mortar space, 2×2, $c = 1$,$h_1^{-1} = h_2^{-1} = h_3^{-1} = h_4^{-1} = 33$

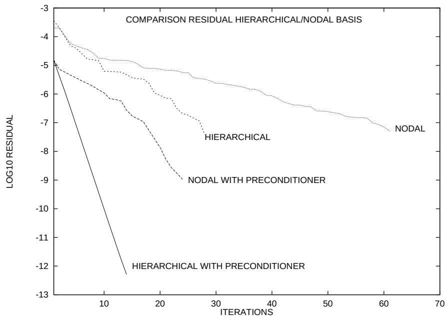
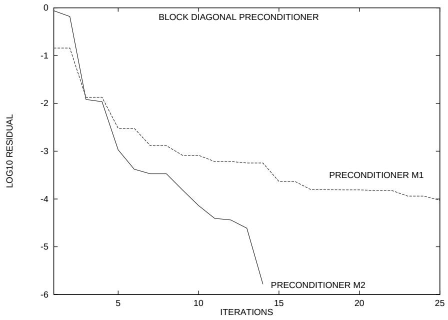


Figure 3 shows a comparison between the residual for the CG with and without preconditioner for the dual matrix written in its nodal and hierarchical basis. We have a very good convergence for the hierarchical matrix with the hierarchical preconditioner.

Figure 4 shows that the convergence is better with the preconditioner $R_\lambda^{-1} = M2 = \sum (P_i P_i^t)^{-1} (P_i R_i A_i R_i^t P_i^t)(P_i P_i^t)^{-1}$

Figure 4 Residual with $R_\lambda = M_1 = BA^{-1}B^t$ and
$R_\lambda^{-1} = M_2 = \sum (P_i P_i^t)^{-1}(P_i R_i A_i R_i^t P_i^t)(P_i P_i^t)^{-1}$, 2×2, $c = 1$,
$h_1^{-1} = h_2^{-1} = 16, h_3^{-1} = h_4^{-1} = 24$



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