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Overlapping Domain Decomposition with Non-matching Grids

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1 Introduction

In this paper we consider two topics. In Section 2 we introduce a new macro-hybrid formulation based on overlapping domain decomposition for linear elliptic equations with symmetric positive definite operators. The problem is discretized by the mortar element method using non-matching grids on the interfaces between subdomains. An iterative method of an optimal order of arithmetical complexity is proposed for solving the arising algebraic systems in the case of regular quasiuniform hierarchical grids. An example of such a formulation was originally given in [Kuz95b]. The approach proposed here has many common points with the decentralization methods studied more than twenty years ago in [BLT74, Lem74]. In these papers the authors used splittings of bilinear forms between different subdomains to decompose a variational problem.

The second important topic is presented in Section 4 where we consider an extension of results from [Kuz95a, Kuz95b] to the case of overlapping subdomains. Here we present several results which mainly concern the construction of the interface preconditioner.

In Section 5 results of numerical experiments with 2D- and 3D-overlapping subdomains are given.

2 Macro-hybrid based on Overlapping Domain Decomposition

Let us consider a model elliptic problem

$$\begin{aligned} -\Delta u + cu &= f \quad \text{in } \Omega \\ \frac{\partial u}{\partial \mathbf{n}} &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.1}$$

Ninth International Conference on Domain Decomposition Methods
Editor Petter E. Bjørstad, Magne S. Espedal and David E. Keyes

where $f \in L_2(\Omega)$ is a given function, $c \equiv \text{const} \in (0; 1]$, $\partial\Omega$ is the boundary of a domain Ω and $\mathbf{n}$ is the outer unit normal vector to $\partial\Omega$. For the sake of simplicity we assume that Ω is a polygon in $\mathbf{R}^p$, $p = 2, 3$ with $\operatorname{diam}(\Omega) \sim O(1)$, and all further subdomains of Ω are also polygons with diameters $O(1)$.

The classical weak formulation of (2.1) is: find $u \in H^1(\Omega)$ such that

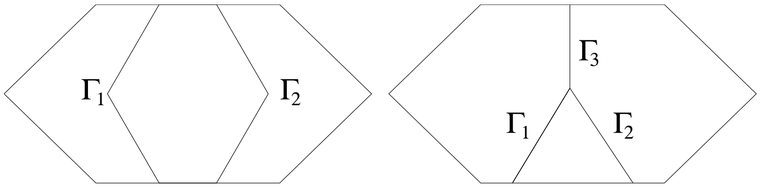
$$\Phi(u) = \min_{v \in H^1(\Omega)} \Phi(v), \tag{2.2}$$

where

$$\Phi(v) = \int_\Omega \left[|\nabla v|^2 + cv^2 - 2fv \right] d\Omega. \tag{2.3}$$

Let Ω_1 and Ω_2 be two overlapping subdomains of Ω ($\Omega_1 \cap \Omega_2 \neq \emptyset$) such that $\overline{\Omega_1 \cup \Omega_2} = \overline{\Omega}$. We assume that subdomains Ω_1 and Ω_2 are regularly shaped. Examples of such a partitioning of Ω into two subdomains are given in Fig. 1 (Ω_1 is located on the left, i.e. $\Gamma_2 \subset \partial\Omega_1 \cap \Omega$).

Figure 1 a) Overlapping and b) overlapping / nonoverlapping domain decomposition



We denote the intersection of Ω_1 and Ω_2 by Ω_{12} and define two bilinear forms

$$a_k(u, v) = \int_{\Omega_k} [a_k \nabla v \cdot \nabla u + c_k uv] \, d\Omega, \qquad k = 1, 2, \tag{2.4}$$

two linear forms

$$l_k(v) = \int_{\Omega_k} f_k v d\Omega, \qquad k = 1, 2, \tag{2.5}$$

and two quadratic functionals

$$\psi_k(v) = a_k(v, v) - 2l_k(v), \qquad k = 1, 2. \tag{2.6}$$

The coefficients a_k, c_k and functions f_k are defined by

$$a_k = \begin{cases} 1 & \text{in } \Omega_k \setminus \Omega_{12} \\ q_k & \text{in } \Omega_{12} \end{cases} \qquad c_k = \begin{cases} c & \text{in } \Omega_k \setminus \Omega_{12} \\ q_k c & \text{in } \Omega_{12} \end{cases}$$
$$f_k = \begin{cases} f & \text{in } \Omega_k \setminus \Omega_{12} \\ q_k f & \text{in } \Omega_{12} \end{cases} \tag{2.7}$$

where q_k are positive constants, $k = 1, 2$ such that $q_1 + q_2 = 1$. It is important that

$$\psi_k(v) = q_k\Phi(v), \quad \forall v \in H^1(\Omega),\ \operatorname{supp} v \in \Omega_{12},\ k = 1, 2. \tag{2.8}$$

To introduce and to analyze macro-hybrid formulations of elliptic problems we have to deal with interfaces between subdomains. To this end we introduce the following notation:

$$\begin{aligned} \gamma_g &= (\partial\Omega_1 \cap \Omega) \cup (\partial\Omega_2 \cap \Omega), \\ \gamma_{in} &= \partial\Omega_1 \cap \partial\Omega_2, \\ \partial\gamma_{in} &= \bar{\gamma}_{in} \setminus \overset{\circ}{\gamma}_{in}\ . \end{aligned} \tag{2.9}$$

Here $\overset{\circ}{\gamma}_{in}$ is the interior part of γ_{in} with respect to $(p-1)$-dimensional topology, γ_g is said to be the "global" interface (with respect to the macro-hybrid formulation to be presented), and $\partial\gamma_{in}$ is the set of cross points in the case $p = 2$ and the set of interedges in the case $p = 3$.

The set $\gamma_g \setminus \partial\gamma_{in}$ can be presented as the union of nonoverlapping open subsets Γ_s, $s = 1, \ldots, s_g$ such that each Γ_s is a piece-wise linear curve in the case of $p = 2$ and a piece-wise linear surface in the case of $p = 3$. It is obvious that this partitioning γ_g into nonoverlapping open subsets is unique.

For examples given in Fig. 1 we have $s_g = 2$ in the case a) and $s_g = 3$ in the case b). Respectively, $\partial\gamma_{in}$ consists of four points both in the cases a) and b).

Now we introduce the space $V = H^1(\Omega_1) \times H^1(\Omega_2)$, the space

$$W = \left\{\bar{v} = (v_1, v_2) :\ \bar{v} \in V,\ \int\limits_{\Gamma_s} (v_1 - v_2)\mu\, ds = 0,\ \forall \mu \in H^{-1/2}(\Gamma_s),\ s = 1, \ldots, s_g\right\} \tag{2.10}$$

and the quadratic functional

$$\psi(\bar{v}) = \psi_1(v_1) + \psi_2(v_2), \quad \bar{v} \in V. \tag{2.11}$$

It can be shown (see, for instance [BF91]) that under the assumptions made the following macro-hybrid formulation of problem (2.1):

$$\bar{u} \in V :\ \psi(\bar{u}) = \min_{\bar{v} \in W} \psi(\bar{v}) \tag{2.12}$$

has a unique solution and is equivalent to problem (2.2). We understand the equivalence in the sense that

$$u(x) = u_k(x) \quad \forall x \in \Omega_k, \tag{2.13}$$

where u is the solution function to (2.2).

Problem (2.12) has also an equivalent formulation in terms of Lagrange multipliers. For instance, in the case of example in Fig. 1b it can be presented in the following form: find $(\bar{u}, \bar{\lambda}) \in V \times \Lambda$ such that

$$
\begin{aligned}
a_1(u_1, v_1) + \int\limits_{\Gamma_1} \lambda_1 v_1 \, ds + \int\limits_{\Gamma_2} \lambda_2 v_1 \, ds + \int\limits_{\Gamma_3} \lambda_3 v_1 \, ds &= l_1(v_1), \\
a_2(u_2, v_2) - \int\limits_{\Gamma_1} \lambda_1 v_2 \, ds - \int\limits_{\Gamma_2} \lambda_2 v_2 \, ds - \int\limits_{\Gamma_3} \lambda_3 v_2 \, ds &= l_2(v_2), \\
\int\limits_{\Gamma_1} (u_1 - u_2)\mu_1 \, ds &= 0, \\
\int\limits_{\Gamma_2} (u_1 - u_2)\mu_2 \, ds &= 0, \\
\int\limits_{\Gamma_3} (u_1 - u_2)\mu_3 \, ds &= 0,
\end{aligned}
\tag{2.14}
$$

$\forall (\bar{v}, \bar{\mu}) \in V \times \Lambda$. Here $\Lambda = \prod\limits_{s=1}^{3} H^{-1/2}(\Gamma_s)$. It can be easily shown that

$$
\lambda_1 = q_2 \frac{\partial u_2}{\partial \mathbf{n}_2} \text{ on } \Gamma_1, \quad \lambda_2 = -q_1 \frac{\partial u_1}{\partial \mathbf{n}_1} \text{ on } \Gamma_2, \quad \lambda_3 = -\frac{\partial u_1}{\partial \mathbf{n}_1} \text{ on } \Gamma_3,
\tag{2.15}
$$

where $\mathbf{n_1}$ and $\mathbf{n_2}$ are the outer normal vectors to $\partial\Omega_1$ and $\partial\Omega_2$, respectively. Recall that $u_1 \equiv u$ in Ω_1 and $u_2 \equiv u$ in Ω_2.

In a compact form (2.14) can be presented [GW88, BF91] by: find $(\bar{u}, \bar{\lambda}) \in V \times \Lambda$ such that

$$
\begin{aligned}
\hat{a}(\bar{u}, \bar{v}) + b(\bar{\lambda}, \bar{v}) &= \hat{l}(\bar{v}), \\
b(\bar{\mu}, \bar{u}) &= 0, \qquad \forall (\bar{v}, \bar{\mu}) \in V \times \Lambda.
\end{aligned}
\tag{2.16}
$$

Here

$$
\begin{gathered}
V = \prod_{k=1}^{m} V_k, \qquad V_k = H^1(\Omega_k), \ k = 1, \ldots, m, \\
\Lambda = \prod_{s=1}^{s_g} \Lambda_s, \qquad \Lambda_s = H^{-1/2}(\Gamma_s), \ s = 1, \ldots, s_g, \\
\hat{a}(\bar{u}, \bar{v}) = \sum_{k=1}^{m} a_k(u, v), \quad \hat{l}(\bar{v}) = \sum_{k=1}^{m} l_k(v), \\
b(\bar{\mu}, \bar{u}) = \sum_{s=1}^{3} \int (u_1 - u_2)\mu_s \, ds,
\end{gathered}
\tag{2.17}
$$

where $m = 2$ and $s_g = 3$.

Remark 1 *Generalization to a larger number of subdomains is straightforward. For instance, if we use the formulation (2.16)-(2.17) we have to assume that for any simply connected subdomain $G \subset \Omega \setminus \gamma_g$ a positive constant q_G exists such that*

$$
a_k(u, v) = q_G a(u, v), \quad l_k(v) = q_G l(v), \quad \forall u, v \in H^1(\Omega), \quad \operatorname{supp} v \subset G, \quad k = \overline{1, m}.
\tag{2.18}
$$

Remark 2 *If $\int\limits_\Omega f\,d\Omega = 0$ and $c \ll 1$ then problem (2.1) can be considered as a singular perturbation of the Neumann problem*

$$\begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega \\ \dfrac{\partial u}{\partial \mathbf{n}} & = & 0 & \text{on } \partial\Omega. \end{array} \tag{2.19}$$

3 The Mortar Element Method and Algebraic Systems

We consider the only case when Ω_{kh} are conforming triangular, $p = 2$, or tetrahedral, $p = 3$, partitions of Ω_k, $k = 1, \dots, m$, and γ_g does not intersect the interiors of the grid cells. Then V_{kh} are the standard piecewise linear finite element subspaces of $V_k \equiv H^1(\Omega_k)$, $k = 1, \dots, m$. The finite element subspaces $\Lambda_{sh} \subset \Lambda \equiv H^{-\frac{1}{2}}(\Gamma_s)$, $s = 1, \dots, s_g$ are chosen using the mortar element technique from [BMP89, BM94, Kuz95b].

The mortar finite element discretization of (2.16)–(2.17) is defined by: find $(\bar{u}_h, \bar{\lambda}_h) \subset V_h \times \Lambda_h$ such that

$$\begin{array}{lcl} \hat{a}(\bar{u}_h, \bar{v}) + b(\bar{\lambda}_h, \bar{u}_h) & = & \hat{l}(\bar{v}), \\ b(\bar{\mu}, \bar{u}_h) & = & 0, \end{array}$$

$\forall\, (\bar{v}, \bar{\mu}) \in V_h \times \Lambda_h$ where $V_h = \prod\limits_{k=1}^{m} V_{kh}$ and $\Lambda_h = \prod\limits_{s=1}^{s_g} \Lambda_{sh}$. Problem (3.20) leads to an algebraic system

$$\mathcal{A}x = y \tag{3.20}$$

with a saddle-point matrix

$$\mathcal{A} = \begin{pmatrix} A & B^T \\ B & O \end{pmatrix} \tag{3.21}$$

and vectors

$$x = \begin{pmatrix} u \\ \lambda \end{pmatrix}, \qquad y = \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{3.22}$$

Here A is a symmetric positive definite matrix and $\operatorname{Ker} B^T = 0$. It follows immediately that $\det \mathcal{A} \neq 0$.

For further analysis we need a more detailed description of A and B in block forms. The simplest block representations of A and B^T are:

$$A = \begin{pmatrix} A_1 & O & O \\ O & \ddots & O \\ O & O & A_m \end{pmatrix}, \qquad B^T = \begin{pmatrix} B_1^T \\ \vdots \\ B_m^T \end{pmatrix} \tag{3.23}$$

Here the kth block corresponds to the degrees of freedom of the finite element space V_{kh}, $k = 1, \dots, m$.

For each subdomain Ω_k we partition degrees of freedom (grid nodes) into two groups. In the second group denoted by γ we collect the degrees of freedom which correspond to the grid nodes belonging to $\gamma_k = \gamma_g \cap \bar{\Omega}_k$. All other degrees of freedom we collect in the first group denoted by I. These partitionings induce the following block representations:

$$A_k = \begin{pmatrix} A_{kI} & A_{kI\gamma} \\ A_{k\gamma I} & A_{k\gamma} \end{pmatrix}, \qquad B_k^T = \begin{pmatrix} O \\ B_{k\gamma}^T \end{pmatrix}. \tag{3.24}$$

Let $\mathcal{B}$ be a symmetric positive definite matrix and $\mathcal{H} = \mathcal{B}^{-1}$. Since $\mathcal{A} = \mathcal{A}^T$ the preconditioned Lanczos [MK74, Kuz95b] can be used to solve system (3.20). In this paper we also recommend the preconditioned conjugate method based on the $\mathcal{B}$-norm of minimal errors [MK74]:

$$\begin{aligned} \hat{p}_l &= \begin{cases} \mathcal{H}\xi^0, & l = 1, \\ \mathcal{H}\xi^{l-1} - \alpha_l \hat{p}_{l-1}, & l > 1, \end{cases} \\ p_l &= \mathcal{H}\mathcal{A}\hat{p}_l, \\ x^l &= x^{l-1} - \beta_l p_l, \end{aligned} \tag{3.25}$$

$$\alpha_l = \frac{(\xi^{l-1}, \mathcal{A}\hat{p}_{l-1})_{\mathcal{H}}}{(\mathcal{A}\hat{p}_{l-1}, \mathcal{A}\hat{p}_{l-1})_{\mathcal{H}}}, \quad \beta_l = \frac{(\xi^{l-1}, \hat{p}_l)}{(\mathcal{A}\hat{p}_l, \mathcal{A}\hat{p}_l)_{\mathcal{H}}},$$

where $\xi^l = \mathcal{A}x^l - y$ are the residual vectors, $l = 1, 2, \ldots$ Assume that the eigenvalues of $\mathcal{H}\mathcal{A}$ belong to the union of segments $[d_1; d_2]$ and $[d_3; d_4]$ with $d_1 \le d_2 < 0 < d_3 \le d_4$. Then the convergence estimate

$$\|x^l - x\|_{\mathcal{H}} \le 2q^l \|x^0 - x\|_{\mathcal{H}}, \quad l \ge 1, \tag{3.26}$$

holds [MK74] where $q = \dfrac{\hat{d} - \check{d}}{\hat{d} + \check{d}}$, $\hat{d} = \max\{d_4; |d_1|\}$, and $\check{d} = \min\{d_3; |d_2|\}$.

4 Block Diagonal Preconditioning

We propose a preconditioner $\mathcal{H}$ as a block diagonal matrix:

$$\mathcal{H} = \begin{pmatrix} H_A & O \\ O & H_\lambda \end{pmatrix} \tag{4.27}$$

where H_A is also a block diagonal matrix:

$$H_A = \begin{pmatrix} H_1 & O & O \\ O & \ddots & O \\ O & O & H_m \end{pmatrix}. \tag{4.28}$$

All blocks are symmetric positive definite matrices. H_k are said to be the subdomain preconditioners, and H_λ is said to be the interface preconditioner.

If matrices H_k are spectrally equivalent to the matrices A_k^{-1} with constants independent of the value of the coefficient c, and if a matrix H_λ is spectrally equivalent to the matrix S_λ^{-1} with S_λ given by

$$S_\lambda = BA^{-1}B^T \equiv \sum_{k=1}^{m} B_{k\gamma} S_{k\gamma}^{-1} B_{k\gamma}^T \tag{4.29}$$

with the constants independent of the value of c then the values of $\hat{d}, \check{d}$ in (3.26) are positive constants [Kuz95a] also independent of c. Here

$$S_{k\gamma} = A_{k\gamma} - A_{k\gamma I} A_{kI}^{-1} A_{kI\gamma} \tag{4.30}$$

are the Schur complements. Our aim is to construct a preconditioner $\mathcal{H}$ spectrally equivalent [Kuz95a] to the matrix $\mathcal{A}^{-1}$ with constants independent of c.

Subdomain Preconditioners

Let us define matrices $\overset{\circ}{A}_k$ and M_k by:

$$\begin{aligned} (\overset{\circ}{A}_k\, v,\, w) &= \int\limits_{\Omega_k} \nabla v_h \cdot \nabla w_h d\Omega, \\ (M_k v,\, w) &= \int\limits_{\Omega_k} v_h w_h d\Omega \end{aligned} \tag{4.31}$$

$\forall\, v_h, w_h \in V_{kh},\ k = 1, \dots, m$. Thus, matrices $\overset{\circ}{A}_k$ are the stiffness matrices for the operator $-\Delta$ with the Neumann boundary conditions, and M_k are the corresponding mass matrices. It can be easily shown [Kuz95b] that

$$A_k^{-1} \sim \left(\overset{\circ}{A}_k + M_k\right)^{-1} + \frac{1}{c} P_k \tag{4.32}$$

where $P_k M_k$ is the M_k-orthogonal projector onto Ker $\overset{\circ}{A}_k$ and the sign "$\sim$" denotes the spectral equivalence. Moreover, the constants of the spectral equivalence in (4.32) are independent of the value of c.

Suppose that a matrix $\overset{\circ}{H}_k$ is spectrally equivalent to the matrix $\left(\overset{\circ}{A}_k + M_k\right)^{-1}$. Then the matrix

$$H_k = \overset{\circ}{H}_k + \frac{1}{c} P_k \tag{4.33}$$

is spectrally equivalent to matrix A_k^{-1} with constants independent of the value of c.

We have plenty of choices for $\overset{\circ}{H}_k$, $k = 1, \dots, m$.

Interface Preconditioner

We can easily show [Kuz95b] that

$$S_{k\gamma}^{-1} \sim \tilde{S}_{k\gamma}^{-1} + \frac{1}{c} P_{k\gamma} \tag{4.34}$$

where $\tilde{S}_{k\gamma}^{-1}$ is the Schur complement for the matrix $\overset{\circ}{A}_k + M_k$ and $P_{k\gamma} M_{k\gamma}$ is the $M_{k\gamma}$ orthogonal projector onto $\operatorname{Ker} S_{k\gamma}$ in the case $c = 0$. Moreover, the constants of equivalence in (4.34) are independent of the value of c. Here $M_{k\gamma}$ is the interface mass matrix defined by:

$$(M_{k\gamma} v,\, w) \;=\; \int\limits_{\gamma_k} v_h w_h ds \quad \forall\, v_h, w_h \in V_{k\gamma h} \tag{4.35}$$

where $V_{k\gamma h}$ is the trace of V_{kh} into $\gamma_k = \partial\Omega_k \cap \Omega$, $k = 1, \dots, m$.

Let the matrices

$$\overset{\circ}{H}_k = \begin{pmatrix} \overset{\circ}{H}_{kI} & \overset{\circ}{H}_{kI\gamma} \\ \overset{\circ}{H}_{k\gamma I} & \overset{\circ}{H}_{k\gamma} \end{pmatrix} \tag{4.36}$$

be spectrally equivalent to the matrices $\left(\overset{\circ}{A}_k + M_k\right)^{-1}$, $k = 1, \dots, m$. We can also prove that the matrix

$$\hat{S}_\lambda = \sum_{k=1}^{m} B_{k\gamma} (\overset{\circ}{H}_{k\gamma} + \frac{1}{c} P_{k\gamma}) B_{k\gamma}^T \tag{4.37}$$

is spectrally equivalent to S_λ with constants independent of the value of c.

To construct the interface preconditioner H_λ we shall use the preconditioned Chebyshev iterative procedure [BPS86, Kuz95a]. Let $\hat{H}_\lambda$ be a symmetric positive defined matrix and $\nu_\lambda = \lambda_{\max}/\lambda_{\min}$ where $\lambda_{\max}$ and $\lambda_{\min}$ are the maximal and minimal eigenvalues of $\hat{H}_\lambda \hat{S}_\lambda$, respectively. Then for any $t_\lambda \sim \sqrt{\nu_\lambda}$ the matrix

$$H_\lambda = \left[I_\lambda - \prod_{t=1}^{t_\lambda} \left(I_\lambda - \alpha_t \hat{H}_\lambda \hat{S}_\lambda \right) \right] \hat{S}_\lambda^{-1} \tag{4.38}$$

is spectrally equivalent to the matrix S_λ^{-1} where $\{\alpha_t\}$ is a set of the corresponding Chebyshev parameters.

Let $\hat{B}_\lambda$ be a symmetric positive definite matrix such that $1 \in [\mu_{\min}; \mu_{\max}]$ where $\mu_{\min}$ and $\mu_{\max}$ are the minimal and maximal eigenvalues of the matrix $\hat{B}_\lambda^{-1} \sum\limits_{k=1}^{m} B_{k\gamma}\, \overset{\circ}{H}_{k\gamma}\, B_{k\gamma}^T$, respectively. Then for the choice $\hat{H}_\lambda = \hat{R}_\lambda^{-1}$ where

$$\hat{R}_\lambda = \hat{B}_\lambda + \frac{1}{c} \sum_{k=1}^{m} B_{k\gamma} P_{k\gamma} B_{k\gamma}^T, \tag{4.39}$$

the estimate

$$\nu_\lambda \le \hat{\nu}_\lambda \equiv \mu_{\max}/\mu_{\min} \tag{4.40}$$

holds.

A solution algorithm for a system

$$\hat{R}_\lambda z = g$$

is presented in [Kuz95b, KW95]. It includes a so called "coarse grid" problem based on the projectors $P_{k\gamma}$, $k = 1, \dots, m$.

Arithmetical Complexity for Hierarchical Grids

Assume that grids Ω_{kh} are regular, quasiuniform and hierarchical with the average grid step size $h \sim \sqrt[p]{N}$ where N is the dimension of matrix $\mathcal{A}$.

In this case we can use various V-cycle multilevel preconditioners to define matrix $\overset{\circ}{H}_k$ in (4.33). These preconditioners are spectrally equivalent to the matrices $(\overset{\circ}{A}_k + M_k)^{-1}$, $k = 1, \ldots, m$ and have the optimal order of arithmetical complexity [Osw94, Xu92], i. e. the multiplication with such a preconditioner by a vector costs $O(N)$ arithmetical operations.

Our choice $\overset{\circ}{H}_{k\gamma}$ in (4.37) as the corresponding blocks of V-cycle multilevel preconditioner (BPX or MDS-type) is based on two observations. The first one is obvious: spectral equivalence of $\overset{\circ}{H}_{k\gamma}$ and $\tilde{S}_{\gamma k}^{-1}$ follows directly from the spectral equivalence of H_k and $\left(\overset{\circ}{A}_k + M_k\right)^{-1}$, $k = 1, \ldots, m$. The second observation is rather technical and concerns implementation algorithms for V-cycle multilevel preconditioners: multiplication of $\overset{\circ}{H}_{k\gamma}$ by a vector can be implemented with $O(h^{1-p})$ arithmetical operations. The latter observation has at least one very important consequence: the corresponding matrix $\hat{S}_\lambda$ can be multiplied by a vector with $O(h^{1-p})$ arithmetical operations, i.e. multiplication with $\hat{S}_\lambda$ has the optimal order of arithmetical complexity.

It remains to choose preconditioner $\hat{R}_\lambda$, and we do not need an optimal preconditioner because the dimension of S_λ is much smaller than the dimension of A.

In paper [Kuz95a] we proposed to choose $\hat{B}_\lambda$ being equal to a scalar matrix which is a spectrally equivalent to the matrix $\sum\limits_{k=1}^{m} B_{k\gamma} M_{k\gamma}^{-1} B_{k\gamma}^T$. With this choice, obviously

$$\nu_\lambda \leq \text{const} \cdot h^{-2}$$

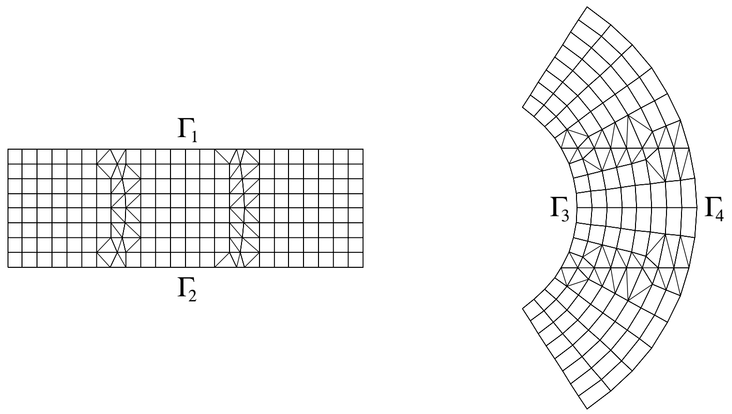
where the constant is independent of h and c, and the multiplication B_λ^{-1} by a vector can be implemented with $O(h^{1-p})$ arithmetical operations.

On the basis of the latter facts we conclude that t_ν should be proportional to h^{-1}, and arithmetical complexity of the corresponding preconditioner H_λ in (4.38) is of the order $O(h^{1-p})$. In some particular cases we can prove [BPS86, Kuz95a] that $t_\nu \sim h^{-1/2}$ and consequently the arithmetical complexity of H_λ is of the order $O(h^{1/2-p})$.

5 Numerical Experiment

The numerical experiments have been performed for two test cases.

Figure 2 Cartesian and polar locally fitted grids



The first test case is presented by the union of a rectangle and a segment of a ring. In the rectangular subdomain we have a rectangular cartesian grid and in the segment we have an orthogonal polar grid. Both grids are fitted to the interface boundary which consists of two straight segments Γ_1 and Γ_2, and two circle's segments Γ_3 and Γ_4. These grids are given in Fig. 2. Here $\partial\Omega_1 \cap \Omega = \Gamma_1 \cup \Gamma_2$ and $\partial\Omega_2 \cap \Omega = \Gamma_3 \cup \Gamma_4$.

Table 1 Cartesian / Polar grids

Cartesian grids in Ω_1	Polar grids in Ω_2	Number of Chebyshev iterations	Number of Lanczos iterations
24×8	16×8	13	68
48×16	32×16	22	72
96×32	64×32	32	75
192×64	128×64	45	77

The second test case is presented by the union of two parallelepipeds. The intersection of them is also a parallelepiped. Both grids are uniform and rectangular ones.

The results of numerical experiments are given in Tables 1 and 2. Two first columns contain information about the grids: product of numbers of nodes for each of the coordinates. The third column contains information about number of iterations t_λ used in construction of the interface preconditioner (4.38). The last column contains the number of iterations which were needed to reduce the $\mathcal{H}$-norm of the initial residual ξ^0 of 10^6 times by the preconditioned Lanczos method.

Table 2 Intersecting parallelepipeds: uniform grids

Grids in Ω_1	Grids in Ω_2	Number of Chebyshev iterations	Number of Lanczos iterations
$16 \times 16 \times 16$	$16 \times 8 \times 8$	24	54
$32 \times 32 \times 16$	$32 \times 16 \times 16$	38	51
$64 \times 64 \times 32$	$64 \times 16 \times 16$	52	51

One can see that t_λ grows up proportionally to $h^{-1/2}$ and that the number of Lanczos iterations is almost constant for both test cases.

Acknowledgement

The results of numerical experiments in Section 5 are kindly given for this paper by S. A. Finogenov and N. Vasiliev. The author is very grateful to both of them. The author would like also to thank P. Le Tallec and a reviewer for helpful discussions and comments.

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