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Adaptive Meshes for the Spectral Element Method

Li-Chieh Hsu and Catherine Mavriplis

1 Introduction

The spectral element method [Pat84] is a high order domain decomposition method for the solution of nonlinear time-dependent partial differential equations. The method has been successfully used in the solution of the Navier-Stokes equations for direct simulation of many complex fluid flows e.g., [Kar90, FR94]. Although the method is, in theory, very powerful for complex phenomena such as transitional flows, practical implementation of domain decomposition for optimal resolution of complex features limits its performance. For instance, it is hard to estimate the appropriate number of elements for a specific case. *A priori* selection of regions to be refined or coarsened is difficult especially as the flow becomes more complex and memory limits of the computer are stressed.

In this paper we present an adaptive spectral element method in which decomposition of the domain is automatically determined in order to capture underresolved regions of the domain and to follow regions requiring high resolution as they develop in time. The objective is to provide the best and most efficient solution to a time-dependent nonlinear problem by continually optimizing resource allocation.

Previously [Mav94], the advantages of such an adaptive scheme were shown to be significant: singularities, thin internal or boundary layers may be resolved by automatic detection and refinement. In [Mav94], the relative merits of the refinement options available for spectral elements were demonstrated in one dimension and a simple two-dimensional example was given. In this paper, we offer a more complete description of the two-dimensional adaptive method. The spectral element method provides two modes of resolution refinement: increase in the number of elements (h-refinement) and increase in the polynomial order of the basis functions on each element (p-refinement). In this paper, for simplicity, the refinement options are limited to h-refinement only.

In the following, the spectral element method is briefly reviewed and the adaptive algorithm is described. Example calculations are then presented for two-dimensional heat conduction and Stokes flow problems.

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2 Discretization

Our objective is to simulate complex flows by solving the incompressible Navier-Stokes equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u},$$
$$\nabla \cdot \boldsymbol{u} = 0,$$

where $\boldsymbol{u}$ is the velocity vector, p is the pressure, ρ is the density and ν is the kinematic viscosity. A time splitting scheme [OK80]:

$$\frac{\hat{\boldsymbol{u}} - \boldsymbol{u}^n}{\Delta t} = \sum_{r=0}^{2} \beta_r (-\boldsymbol{u} \cdot \nabla \boldsymbol{u})^{n-r},$$
$$\nabla^2 p^{n+1} = \frac{\rho}{\Delta t} \nabla \cdot \hat{\boldsymbol{u}},$$
$$\frac{\hat{\hat{\boldsymbol{u}}} - \hat{\boldsymbol{u}}}{\Delta t} = -\frac{1}{\rho}\nabla p^{n+1},$$
$$\frac{\boldsymbol{u}^{n+1} - \hat{\hat{\boldsymbol{u}}}}{\Delta t} = \nu \nabla^2 \boldsymbol{u}^{n+1},$$

where $\hat{\boldsymbol{u}}$ and $\hat{\hat{\boldsymbol{u}}}$ are intermediate time step values of velocity between the nth and $n{+}1$st time steps, is used to treat the nonlinear terms explicitly, in this case by third order Adams-Bashforth with coefficients β_r, while treating the diffusion and pressure terms implicitly. The pressure and velocities are governed by Helmholtz problems which are discretized by the spectral element method as follows.

For a model Helmholtz equation

$$(\nabla^2 - \lambda^2)\phi = g,$$

we take the variational form

$$-\int \nabla\phi \cdot \nabla\psi dx - \lambda^2 \int \phi\psi dx = \int g\psi dx \ \ \forall\psi$$

and substitute discrete approximations to all variables following

$$\phi_h^k = \sum_{p=0}^{N}\sum_{q=0}^{M} \phi_{pq}^k h_p(r) h_q(s), \tag{2.1}$$

where the h functions are the Lagrangian interpolants based on the orthogonal set of Legendre polynomials of high degree N or M and r, s are the local coordinates on each element k. Performing Gauss-Lobatto quadrature and summing contributions from adjacent elements we obtain the global matrix equation

$$(A - \lambda^2 B)\phi = Bg,$$

where A is the discrete Laplacian operator and B is the mass matrix. The matrix equation is solved by preconditioned conjugate gradient iteration.

For adaptivity and local refinement, the nonconforming formulation is advantageous since it allows elements to abut in arbitrary manners. For example, a conforming mesh is shown in Fig. 2(a) and a nonconforming mesh is shown in Fig. 3(a). In this case, the matrix equation becomes

$$Q^T(A - \lambda^2 B)Q\phi = Q^T Bg$$

where Q is a transformation matrix which provides an $\mathcal{L}^2$ minimization of the jump in variables on the nonconforming interface [MMP89].

3 The Adaptive Method

The adaptive spectral element method is designed to have low cost and high efficiency in solving complex time-dependent physical problems. The adaptivity is based on error estimators which determine which regions need more resolution. The solution strategy is as follows: compute an initial solution with a suitable initial mesh, estimate errors in the solution locally in each element, modify the mesh according to the error estimators, interpolate the old mesh solutions onto the new elements, and resume the numerical solution process. This solution process is visualized in the flowchart below (Fig. 1).

The error estimators are based on a posteriori estimates of the $\mathcal{L}^2$ and $\mathcal{H}^1$ errors in the spectral representation of the solution on each element [Mav90]. For simplicity, we present the one-dimensional $\mathcal{L}^2$ error estimate on element k, ϵ^k_{est}, as

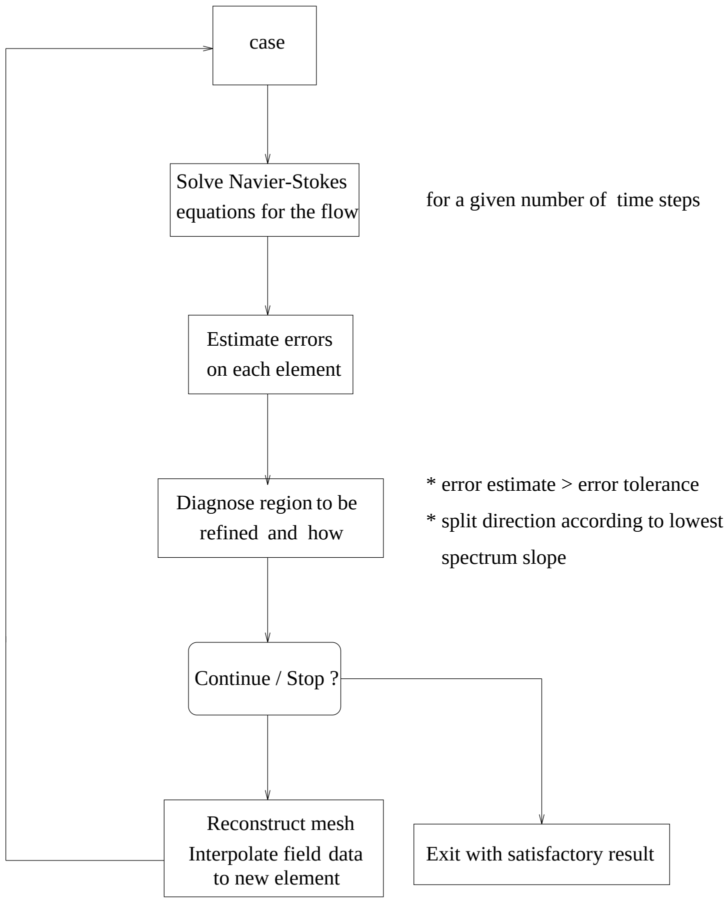
$$\epsilon^k_{est} = \left(\frac{{a^k_N}^2}{\frac{2N+1}{2}} + \int_{N+1}^{\infty} \frac{(a^k(n))^2}{\frac{2n+1}{2}} dn \right)^{\frac{1}{2}}.$$

a^k_n is the one-dimensional spectrum of the numerical solution ϕ^k_h, defined by the elemental spectral discretization (similar to Eqn. 2.1 but in one dimension) rewritten in terms of the Legendre polynomials P_n of order n as

$$\phi^k_h = \sum_{n=0}^{N} a^k_n P_n(r).$$

The $a^k(n)$ function in the error estimate is a model least squares best fit to the last four spectrum points ($n = N-3, N$), which is used to extrapolate the spectrum to infinity in order to estimate the truncation error. In two dimensions, the spectrum is a two-dimensional tensor a^k_{nm}. The two-dimensional elemental error estimate ϵ^k_{est} for element k is made up of the sum of all $\epsilon^k_{m\ est}$ and $\epsilon^k_{n\ est}$ for each $m = 0, M$ and $n = 0, N$, respectively, as well as an extrapolation for $n > N$ **and** $m > M$. In practice, we find that these error estimators are very robust and quite accurate as shown in [Mav90, Mav94] and in the following examples. The value of the estimate ϵ^k_{est} on each element is used to decide whether to adapt or not: if it is greater than a chosen upper error tolerance level, then the particular element k is refined. Similarly, if it is lower than a chosen lower error tolerance level than the element may be coarsened. Coarsening with h-refinement can be difficult to be made robust and, hence, was not implemented in this paper. While local error estimates are used to determine which

Figure 1 Adaptive refinement algorithm



regions should be refined, the slopes of the spectrum are used as criteria in choosing the direction of the h-refinement: a lower slope in the r direction indicates that the quality of the solution is poorer in the r direction than in the s direction and hence the element is split in the r direction rather than in the s direction.

Once the new grid has been defined, the current (old) time step solution must be interpolated onto the new topology. The location of the new elements and their Gauss-Lobatto collocation points are determined in the old mesh. Then, through the use of the Lagrangian interpolants from the old mesh, the calculation of the new values of the solution on the Gauss-Lobatto collocation points of the new grid is straightforward. The high order of the spectral element method minimizes errors in this interpolation step of the adaptive process.

There are many factors which affect the efficiency of the adaptive refinement, such as the location of the split position in the split element, the order of the basis functions, the new position for moving a boundary in the case of lowest cost refinement. The frequency of adaptivity and the tolerance for terminating the adaptive process also affect the efficiency of the method. Therefore, an optimal strategy of adaptive process remains a key point to increasing efficiency. Development of such a strategy is under way but is not reported here.

4 Illustrations

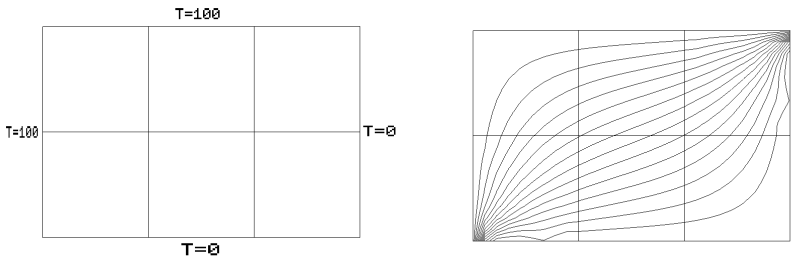
Heat Conduction

As mentioned previously, the Helmholtz/Poisson equation forms the core of our Navier-Stokes solver, hence, we first present a Poisson test problem in order to validate our method. The problem is one of two-dimensional heat conduction in a plate where the left and upper walls are held at temperature of 100 while the two other walls are held at temperature of 0 as shown in Fig. 2(a). The discontinuity of the two competing Dirichlet temperature boundary conditions at the upper right and lower left corners necessitates fine resolution since the spectral representation of a discontinuity leads to poor results. The initial solution was calculated with six equal-sized elements and is shown in Fig. 2. The unphysical kinks in temperature contour lines near the edges of the domain indicate that the numerical solution is poor. The adaptive method effected ten splits in both the upper right and lower left regions for a final solution using 26 elements (shown in Fig. 3(a)). The polynomial order in each element is four. The final solution shown in terms of contours of constant temperature in Fig. 3(b) is much improved. Error estimates (as defined in Section 1.3) have been reduced by approximately two orders of magnitude in all elements with exception for the smallest corner elements which contain the discontinuities.

Driven Cavity Stokes Flow

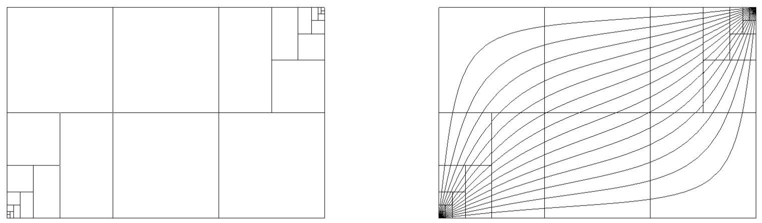
We now turn to a Stokes flow (where the nonlinear terms of the Navier-Stokes equations are negligible for very low Reynolds numbers): two-dimensional laminar flow in a cavity is used to illustrate how the adaptive process performs. This flow serves as a very popular test case (e.g., see references within [PT83]). There are discontinuities in velocity at the upper corners, where $u = 0$ from the no-slip condition on the vertical walls and $u = 1$ from the imposed driving flow on the upper wall. At these points there is a singularity as the vorticity becomes infinite. The driven cavity flow is solved adaptively starting with polynomials of order four on the four equal-sized element grid shown in Fig. 4(a). The Reynolds number for this case is 15. Fig. 5(a) presents the final mesh after a series of adaptation steps. Splitting of elements occurs around the upper corners where the singularities are. In Fig. 5(b), where the streamlines are plotted, we see the solution with 24 elements is well defined in most areas even near the singularities. Indeed, it is significantly improved over the nonadapted initial case shown in Fig. 4(b). The error estimate (as defined in Section 1.3) has been reduced to 10^{-2} in all elements except the smallest corner elements containing the singularities. These solutions are steady results calculated by an unsteady Stokes solver. The $\times$ points in the figures represent locations where the velocities were monitored in time to determine a steady result.

Figure 2 Heat conduction example before adaptive refinement: (a) elemental mesh (b) temperature contour lines



(a) (b)

Figure 3 Heat conduction example after adaptive refinement: (a) elemental mesh (b) temperature contour lines

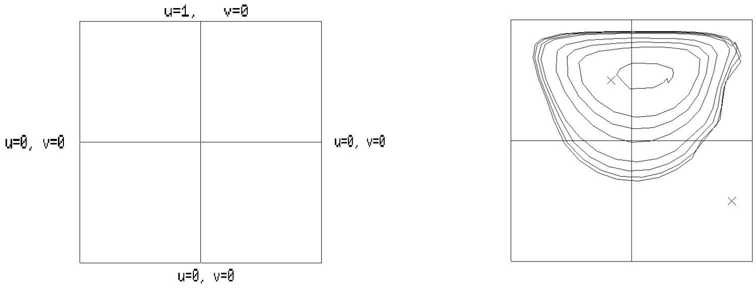


(a) (b)

5 Conclusion

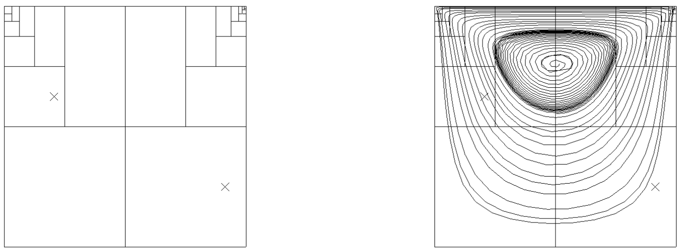
An adaptive spectral element method for the direct simulation of incompressible flows has been developed. The adaptive algorithm effectively diagnoses and refines regions of the flow where complexity of the solution requires increased resolution. The method has been demonstrated on two-dimensional examples in heat conduction and Stokes flows. The refinement has been limited to h-refinement for this paper. In the future, p-refinement will be combined with h-refinement for improved accuracy and efficiency.

Figure 4 Driven cavity flow before adaptive refinement: (a) elemental mesh (b) streamlines



(a) (b)

Figure 5 Driven cavity flow after adaptive refinement: (a) elemental mesh (b) streamlines



(a) (b)

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REFERENCES

[FR94] Fischer P. and Rønquist E. (1994) Spectral element methods for large scale parallel Navier-Stokes calculations. *Computer Methods in Applied Mechanics and Engineering* 116: 69–76.
[Kar90] Karniadakis G. (1989/90) Spectral element simulations of laminar and turbulent flows in complex geometries. *Applied Numerical Mathematics* 6: 85–105.
[Mav90] Mavriplis C. (1990) A posteriori error estimators for adaptive spectral element

techniques. In Wesseling P. (ed) *Notes on Numerical Fluid Mechanics*, volume 29, pages 333–342. Vieweg, Braunschweig.

[Mav94] Mavriplis C. (1994) Adaptive meshes for the spectral element method. *Computer Methods in Applied Mechanics and Engineering* 116: 77–86.

[MMP89] Maday Y., Mavriplis C., and Patera A. (1989) Nonconforming mortar element methods: Application to spectral discretization. In et al T. C. (ed) *Domain Decomposition Methods*. SIAM, Philadelphia.

[OK80] Orszag S. A. and Kells L. C. (1980) Transition to turbulence in plane Poiseuille and plane Couette flow. *Journal of Fluid Mechanics* 96: 159–205.

[Pat84] Patera A. (1984) A spectral element method for fluid dynamics: Laminar flow in a channel expansion. *Journal of Computational Physics* 54(3): 468–488.

[PT83] Peyret R. and Taylor T. (1983) *Computational Methods for Fluid Flow*. Springer, New York.