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Preconditioners for the Boundary Element Method in Three Dimensions

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1 Introduction

In this paper we consider preconditioners for linear systems arising from the boundary element method (BEM) for solving partial differential equations in $\mathbf{R}^3$. We report on new results recently obtained, partly in joint work with Ernst P. Stephan.

The boundary element method consists in solving an integral equation formulation of a boundary value problem by the Galerkin method. All the integrals are defined on the boundary of the domain under consideration and, thus, only the boundary of the domain needs to be discretized. Therefore, this method is extremely well suited for transmission problems and exterior boundary value problems where unbounded domains occur.

We deal with first kind integral equations that stem from representation formulae for the solutions of the boundary value problems. Typically there appear hypersingular and weakly singular operators that have to be numerically inverted. They are of orders $+1$ and -1, respectively. In this paper we consider Laplace's equation in $\mathbf{R}^3$. Then the hypersingular operator

$$Du(x) := \frac{1}{4\pi}\frac{\partial}{\partial n_x}\int_\Gamma u(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|}\,dS_y$$

and the single layer potential operator

$$Vu(x) := \frac{1}{4\pi}\int_\Gamma \frac{u(y)}{|x-y|}\,dS_y$$

are positive definite. To extend our results to more general problems, e.g., to the Helmholtz equation, one has to deal with indefinite integral operators, the main symbols being the operators D and V. Regarding this generalization we refer to [CW92] for the finite element method and to [ST] for the boundary element method in two dimensions.

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For the preconditioning of linear systems arising from the BEM in two dimensions we refer to [Heu96b, TS96, ST95, HST95].

An outline of the paper is as follows. In §2 we introduce the abstract problem under consideration and the boundary element Galerkin method for solving it. Section 3 is dedicated to the hypersingular operator. In §3 we study a general multilevel method to precondition the linear systems arising from the h-version of the BEM. The obtained condition number is almost bounded independently of the number of levels. For a general 2-level method we get a condition number which behaves logarithmically in the ratio H/h. Here H and h are the mesh sizes of the coarse and the fine level meshes, respectively. For the p-version we consider a preconditioner based on an overlapping decomposition and this results in a bounded condition number. For a direct sum decomposition of the ansatz space we use discretely harmonic basis functions and obtain a condition number which is bounded polylogarithmically in the polynomial degree. In §4 we present an almost optimal bound for the condition number in case of the p-version of the BEM for the single layer potential operator. This preconditioner is based on a general decomposition of the ansatz space. Section 5 reports on some numerical results supporting the theoretical estimates.

2 Boundary Element Method

The weak formulation under consideration is the following:
For given $f \in H^{-\alpha/2}(\Gamma)$, find $u \in \tilde{H}^{\alpha/2}(\Gamma)$ such that

$$\langle Au, \phi\rangle_{L^2(\Gamma)} = \langle f, \phi\rangle_{L^2(\Gamma)} \quad \text{for all } \phi \in \tilde{H}^{\alpha/2}(\Gamma). \tag{1}$$

Here, A is a positive definite operator of order α mapping $\tilde{H}^{\alpha/2}(\Gamma)$ continuously onto $H^{-\alpha/2}(\Gamma)$. In case of the hypersingular operator $\alpha = 1$ and for the weakly singular operator we have $\alpha = -1$. The space $\tilde{H}^{1/2}(\Gamma)$, which is also denoted by $H_{00}^{1/2}(\Gamma)$ in the finite element literature, is the interpolation space half-way between $L^2(\Gamma)$ and $H_0^1(\Gamma)$. The space $H^{-1/2}(\Gamma)$ is the dual space of $\tilde{H}^{1/2}(\Gamma)$ and, vice versa, $\tilde{H}^{-1/2}(\Gamma)$ is the dual space of $H^{1/2}(\Gamma)$ which is the interpolation space half-way between $L^2(\Gamma)$ and $H^1(\Gamma)$. We assume Γ to be a flat rectangular screen in $\mathbf{R}^3$. The extension of our results to arbitrary polyhedral surfaces consisting of rectangular pieces is straight forward.

In the case of $A = D$ eq. (1) models a Neumann problem for the Laplacian in $\mathbb{R}^3 \setminus \bar{\Gamma}$ where the jump across Γ of the normal derivative of the solution is given. When $A = V$ eq. (1) represents the Dirichlet problem where the jump across Γ of the trace of the solution is given.

The Galerkin scheme for solving (1) reads as follows:
For a given N-dimensional subspace X_N of $\tilde{H}^{\alpha/2}(\Gamma)$, find $u_N \in X_N$ such that

$$\langle Au_N, \phi\rangle_{L^2(\Gamma)} = \langle f, \phi\rangle_{L^2(\Gamma)} \quad \text{for all } \phi \in X_N. \tag{2}$$

To construct X_N we use a uniform mesh Γ_h on Γ of rectangles of size h.

First let us consider the case $\alpha = 1$. For the h-version of the boundary element method we use piecewise bilinear functions which have the value one at one interior node and vanish at the remaining nodes of Γ_h. Note that the condition $X_N \subset \tilde{H}^{1/2}(\Gamma)$ requires continuous functions which are zero on the boundary of Γ. For the p-version of the boundary element method we use piecewise polynomials of degree p on the mesh Γ_h. As basis functions we take affine images of all combinations of tensor products of piecewise linear functions and of antiderivatives of Legendre polynomials. We note that, to our best knowledge, regarding the efficiency of the implementation there are no algorithms making special use of the Lagrangian interpolation polynomials in the Legendre-Gauss-Lobatto nodes. These functions can be efficiently used in the spectral method which is a special p-version of the finite element method. The efficiency of the spectral method heavily relies on the fact that differential operators have to be discretized. Therefore, in view of approximation properties and the efficiency of the implementation, it is opportune to use the antiderivatives of the Legendre polynomials to construct basis functions for degrees larger than 1.

For the weakly singular operator, i.e. $\alpha = -1$, we just consider the p-version. Then our ansatz spaces X_N are constructed by using affine images of tensor products of Legendre polynomials up to degree p on a mesh Γ_h. We note that $X_N \subset \tilde{H}^{-1/2}(\Gamma)$ does not require continuous functions.

To refer to the parameters h and p of our ansatz space we use the notations

$$X_N = S_p^1(\Gamma_h) \subset \tilde{H}^{1/2}(\Gamma)$$

and

$$X_N = S_p^0(\Gamma_h) \subset \tilde{H}^{-1/2}(\Gamma).$$

In either case, $\alpha = 1$ and $\alpha = -1$, the stiffness matrix in (2), which is also denoted by A, is positive definite since both operators D and V are positive definite. Therefore, the Galerkin method converges quasi-optimally in the energy norm and the method of choice to solve the linear system (2) is the conjugate gradient algorithm. In order to reduce the numbers of iterations which are necessary to solve (2) up to a given accuracy we use preconditioners. By referring to the additive Schwarz frame work, they are defined via decompositions of X_N. To be precise we define the additive Schwarz operator P for a decomposition $X_N = X_1 \cup X_2 \cup \cdots \cup X_k$ by the sum of the projections $P_i:\ X_N \to X_i$, $P = \sum_{i=1}^k P_i$. All the projections are performed with respect to the bilinear form $\langle A\cdot, \cdot\rangle_{L^2(\Gamma)}$. That means we use exact solvers for all the subspaces of X_N. For practical problems they can be replaced by inexact solvers. The additive Schwarz operator P represents the preconditioned stiffness matrix of the linear system and the aim is to find decompositions of X_N which result in small conditions numbers of P. For a survey on additive Schwarz methods we refer to [CM94].

3 Preconditioners for the Hypersingular Operator

h-version

As mentioned above we have to deal with the ansatz space $X_N = S_p^1(\Gamma_h)$ of continuous functions. For the h-version we take the polynomial degree $p = 1$. To define the multilevel preconditioner we consider L mesh sizes $h_1, h_2, \ldots, h_L$ with $h_{l-1} = 2h_l$, $l = 2, \ldots, L$, and $h_L = h$. In a first step we decompose $S_1^1(\Gamma_h)$ into L levels,

$$S_1^1(\Gamma_h) = S_1^1(\Gamma_{h_1}) \cup S_1^1(\Gamma_{h_2}) \cup \cdots \cup S_1^1(\Gamma_{h_L}).$$

This is an overlapping decomposition since, for $1 \leq l \leq m \leq L$, we have $S_1^1(\Gamma_{h_l}) \subset S_1^1(\Gamma_{h_m})$ due to the relations $h_{l-1} = 2h_l$, $l = 2, \ldots, L$. In a second step we totally decompose the subspaces of the different levels except of the coarsest subspace,

$$S_1^1(\Gamma_{h_l}) = S_{1,1}^1(\Gamma_{h_l}) \cup S_{1,2}^1(\Gamma_{h_l}) \cup \cdots \cup S_{1,N_{h_l}}^1(\Gamma_{h_l}).$$

Here each subspace $S_{1,i}^1(\Gamma_{h_l})$ is spanned by exactly one piecewise bilinear basis function on the mesh Γ_{h_l} and N_{h_l} is the dimension of $S_1^1(\Gamma_{h_l})$. The final multilevel decomposition looks like

$$S_1^1(\Gamma_h) = S_1^1(\Gamma_{h_1}) \cup \cup_{l=2}^{L} \left(S_{1,1}^1(\Gamma_{h_l}) \cup \cdots \cup S_{1,N_{h_l}}^1(\Gamma_{h_l}) \right). \tag{3}$$

This means that we use an exact solver for the whole subspace on the coarsest level and that we just use the diagonal preconditioner on all the finer levels. In the 2-level case the coarsest subspace is relatively large and using more levels this subspace becomes smaller. However, in the latter case, the amount of overlapping in the overall decomposition increases. From [Heua] we cite the following result whose proof is a generalization of the theory in [DW91, Zha92, TS96].

Theorem 1 *The additive Schwarz operator corresponding to* (3) *has a condition number which is bounded by*

$$\kappa(P) \leq C h^{-\epsilon}.$$

The constant C is independent of h, the mesh size of the finest level, and of the number of levels L.

We note that the term $h^{-\epsilon}$ in the estimate of the condition number is due to the singularities of the exact solution of our problem at the boundary of the screen Γ. In the case of a closed surface the solution is more regular and the term $h^{-\epsilon}$ does not appear.

To get rid of both, the use of a large coarse subspace and a huge overlapping, we also consider general 2-level methods where one has a coarse mesh Γ_H which is almost independent of the fine mesh Γ_h, the only restriction being the compatibility. The used decomposition is given by

$$S_1^1(\Gamma_h) = S_1^1(\Gamma_H) \cup S_1^1(\Gamma_H \cap \Gamma_h) \cup \cup_{j=1}^{J_H} S_1^1(\Gamma_h \cap \Gamma_j). \tag{4}$$

The space $S_1^1(\Gamma_H)$ consists of the usual continuous piecewise bilinear functions on the mesh Γ_H of size H. $S_1^1(\Gamma_H \cap \Gamma_h)$ is the so-called wirebasket space which is spanned by the piecewise bilinear hat functions of $S_1^1(\Gamma_h)$ which are associated with the nodes of the fine mesh which are on the grid of the coarse mesh. The spaces $S_1^1(\Gamma_h \cap \Gamma_j)$ are spanned by the piecewise bilinear hat functions which are associated with the nodes interior to the restricted meshes $\Gamma_h|_{\Gamma_j}$, $j = 1, \ldots, J_H$. Here, Γ_j, $j = 1, \ldots, J_H$, are the elements of the coarse mesh Γ_H. The result is the following, see [HS].

Theorem 2 *The condition number of the additive Schwarz operator P which is defined by the decomposition* (4) *is bounded by*

$$\kappa(P) \leq C(1 + \log \frac{H}{h})$$

where the constant $C > 0$ is independent of the coarse and fine mesh sizes H and h.

Thus, for this preconditioner, we have bounded condition numbers if the ratio H/h is fixed.

p-version

We consider a fixed rectangular mesh Γ_h and take affine images of tensor products of piecewise linear functions and of antiderivatives of Legendre polynomials as basis functions. For the following overlapping decomposition, which has been investigated for the finite element method in [Pav94], we obtain bounded condition numbers of the corresponding additive Schwarz operator:

$$S_p^1(\Gamma_h) = S_1^1(\Gamma_h) \cup S_p^1(\Gamma_h \cap \Gamma_1') \cup \cdots \cup S_p^1(\Gamma_h \cap \Gamma_{N_h}'). \tag{5}$$

The so-called coarse grid space $S_1^1(\Gamma_h)$ is just the space of the h-version. The remaining spaces are subspaces localized at the neighborhoods of the interior nodes. More precisely $S_p^1(\Gamma_h \cap \Gamma_j')$ is the space of piecewise polynomials of degree p which are globally continuous and which have support contained in the elements adjacent to the node with number j. Therefore, subspaces for adjacent nodes may have common functions and in that case the corresponding blocks of the stiffness matrix overlap.

Theorem 3 [Heua] *The condition number of the additive Schwarz operator P which is defined by the decomposition* (5) *is bounded.*

Since the subspaces $S_p^1(\Gamma_h \cap \Gamma_j')$ are rather large for large polynomial degree p one is interested in further splitting the corresponding blocks. Due to the tensor product structure of the basis functions one has a natural decomposition into subspaces of functions which are associated with nodes, edges and elements, separately. However, it is well known that one cannot take the usual nodal hat functions for such a splitting in higher dimensions. This would result in large condition numbers, cf. [BCMP91]. Therefore, in order to use a nonoverlapping decomposition, one has to consider well behaved basis functions, i.e., functions with small energy. As nodal basis functions we take tensor products of the polynomial of degree p which is defined by

$$\|\varphi_0\|_{L^2(-1,1)} = \min_{\varphi \text{ has degree } p} \|\varphi\|_{L^2(-1,1)}, \quad \varphi_0(1) = 1, \ \varphi_0(-1) = 0.$$

The basis functions related to the edges and to the interior of the elements are defined as discrete tensor product solutions in the weak sense of the Laplace equation. For details we refer to [PW96] and [Heub]. The decomposition is as follows:

$$S_p^1(\Gamma_h) = X_0 \oplus X_1 \oplus \cdots \oplus X_{J_h}. \tag{6}$$

Here $X_j = S_p^1(\Gamma_h) \cap \tilde{H}^{1/2}(\Gamma_j)$, $j = 1, \dots, J_h$, where Γ_j is an element of the mesh Γ_h. X_0 is the global space of the remaining functions which are associated with the nodes and the edges of the mesh. This space is called the wirebasket space.

Theorem 4 [Heub] *The condition number of the additive Schwarz operator P defined by the decomposition* (6) *is bounded by*

$$\kappa(P_W) \leq C(1 + \log p)^2.$$

The constant C is independent of the mesh size h and the polynomial degree p.

As shown in [Heub] a similar result holds even for a modified diagonal preconditioner which includes a small block of global functions. Of course, here we also have to use the special discretely harmonic basis functions.

4 Preconditioner for the Weakly Singular Operator

We only study the p-version of the BEM for the single layer potential operator. We use quasi-uniform rectangular meshes of size h on Γ and take discontinuous piecewise polynomials of degree p for the boundary element space $X_N = S_p^0(\Gamma_h)$. We decompose

$$S_p^0(\Gamma_h) = S_p^0(\Gamma_1) \cup \cdots \cup S_p^0(\Gamma_J). \tag{7}$$

The space $S_p^0(\Gamma_j)$ is the restriction of $S_p^0(\Gamma_h)$ onto a subdomain Γ_j where $\bar{\Gamma} = \cup_{j=1}^J \bar{\Gamma}_j$ is a, possibly overlapping, decomposition of Γ. From [Heu96a] we cite the following result.

Theorem 5 *For any $\epsilon > 0$ there exists a constant $C > 0$ such that the condition number of the additive Schwarz operator defined by the decomposition (7) is bounded by*

$$\kappa(P) \leq C p^\epsilon.$$

5 Numerical Results

In this section we present some numerical experiments for the preconditioners defined in the previous sections. We choose the domain Γ to be the square plate $(-1/2, 1/2)^2 \times \{0\}$. For the p-version we use a uniform mesh of 9 elements. Tables 1 and 2 collect some results for the hypersingular operator. Table 1 lists the condition numbers and extremum eigenvalues of the 2-level method for the h-version. As predicted by Theorem 1 the condition numbers are almost bounded. Table 2 shows the results for the p-version. As stated by Theorem 3 the overlapping decomposition produces bounded condition numbers. The numbers for the nonoverlapping decomposition which belongs to the wirebasket preconditioner are just slightly increasing as predicted by Theorem 4. Finally, Table 3 shows the results for the p-version with the single layer potential operator. They are covered by the statement of Theorem 5 and, for the overlapping decomposition, the condition numbers even appear to be bounded. For the nonoverlapping decomposition we simply used the elements as subdomains and for the overlapping decomposition we used patches of 4 elements as subdomains.

REFERENCES

[BCMP91] Babuška I., Craig A., Mandel J., and Pitkäranta J. (1991) Efficient preconditioning for the p-version finite element method in two dimensions. *SIAM J. Numer. Anal.* 28(3): 624–661.

		2-level ASM		
$1/h$	N	κ	$\lambda_{\min}$	$\lambda_{\max}$
4	9	2.58	0.74	1.92
6	25	3.25	0.65	2.10
8	49	3.51	0.61	2.14
10	81	3.62	0.59	2.14
12	121	3.70	0.58	2.14
14	169	3.75	0.58	2.15
16	225	3.78	0.57	2.16

Table 1 h-version of the BEM with the hypersingular operator. Condition numbers and eigenvalues for the 2-level preconditioner.

		overlapping dec.			nonoverlapping dec.		
p	N	κ	$l_{\min}$	$l_{\max}$	κ	$l_{\min}$	$l_{\max}$
1	4	1.12	1.86	2.08	1.00	1.00	1.00
2	25	4.96	0.83	4.10	5.01	0.32	1.61
3	64	4.73	0.87	4.10	6.21	0.26	1.64
4	121	4.64	0.89	4.13	8.06	0.21	1.70
5	196	4.54	0.91	4.13	8.66	0.20	1.71
6	289	4.49	0.92	4.13	9.91	0.18	1.74

Table 2 p-version of the BEM with the hypersingular operator. Condition numbers and eigenvalues for the overlapping and the nonoverlapping decompositions.

[CM94] Chan T. F. and Mathew T. P. (1994) Domain decomposition algorithms. *Acta Numerica* pages 61–143.

[CW92] Cai X.-C. and Widlund O. B. (1992) Domain decomposition algorithms for indefinite elliptic problems. *SIAM J. Sci. Stat. Comput.* 13: 243–258.

[DW91] Dryja M. and Widlund O. B. (1991) Multilevel additive methods for elliptic finite element problems. In Hackbusch W. (ed) *Parallel Algorithms for Partial Differential Equations (Proc. of the Sixth GAMM-Seminar, Kiel, Germany, January 19–21, 1990)*, pages 58–69. Vieweg, Braunschweig, Germany.

[Heua] Heuer N.Additive Schwarz methods for hypersingular integral equations in $\mathbb{R}^3$. Submitted for publication.

[Heub] Heuer N.An iterative substructuring method for the p-version of the boundary element method for hypersingular integral equations in three dimensions. Submitted for publication.

[Heu96a] Heuer N. (1996) Additive Schwarz methods for weakly singular integral equations in $\mathbb{R}^3$ – the p-version. In Hackbusch W. and Wittum G. (eds) *Boundary Elements: Implementation and Analysis of Advanced Algorithms*, volume 54 of *Notes on Numerical Fluid Mechanics*, pages 126–135. Vieweg-Verlag, Braunschweig, Wiesbaden. Proceedings of the 12th GAMM-Seminar, Kiel, January 1996.

[Heu96b] Heuer N. (1996) Efficient algorithms for the p-version of the boundary

		overlapping dec.			nonoverlapping dec.		
p	N	κ	$l_{\min}$	$l_{\max}$	κ	$l_{\min}$	$l_{\max}$
0	9	4.08	0.98	4.00	7.02	0.43	3.02
1	27	4.35	0.92	4.00	8.20	0.38	3.14
2	54	4.21	0.95	4.00	11.40	0.30	3.36
3	90	4.26	0.94	4.00	12.22	0.28	3.39
4	135	4.22	0.95	4.00	14.60	0.24	3.46
5	189	4.24	0.94	4.00	15.26	0.23	3.47
6	252	4.22	0.95	4.00	17.14	0.20	3.50

Table 3 p-version of the BEM with the weakly singular operator. Condition numbers and eigenvalues for overlapping and nonoverlapping decompositions.

element method. *J. Integral Equations Appl.* 8(3). To appear.

[HS] Heuer N. and Stephan E. P.Iterative substructuring for hypersingular integral equations in $\mathbb{R}^3$. Submitted for publication.

[HST95] Heuer N., Stephan E. P., and Tran T. (1995) Multilevel additive Schwarz method for the p- and hp-version boundary element method. Technical Report AMR95/37, The University of New South Wales.

[Pav94] Pavarino L. F. (1994) Additive Schwarz methods for the p-version finite element method. *Numer. Math.* 66: 493–515.

[PW96] Pavarino L. F. and Widlund O. B. (1996) A polylogarithmic bound for an iterative substructuring method for spectral elements in three dimensions. *SIAM J. Numer. Anal.* 33(4): 1303–1335.

[ST] Stephan E. P. and Tran T.Domain decomposition algorithms for indefinite hypersingular integral equations – the h- and p-versions. *SIAM J. Sci. Stat. Comput.* To appear.

[ST95] Stephan E. P. and Tran T. (1995) Additive Schwarz method for the *p*-version boundary element method. Technical Report AMR95/13, The University of New South Wales.

[TS96] Tran T. and Stephan E. P. (1996) Additive Schwarz method for the h-version boundary element method. *Appl. Anal.* 60: 63–84.

[Zha92] Zhang X. (1992) Multilevel Schwarz methods. *Numer. Math.* 63: 521–539.